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FOUR-GENUS AND FOUR-DIMENSIONAL CLASP NUMBER OF A KNOT

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ABSTRACT. For a knot K in the 3-sphere, by using the linking form on the first homology group of the double branched cover of the 3-sphere, we investigate some numerical invariants, 4-genus $g^*(K)$, nonorientable 4-genus $\gamma^*(K)$ and 4-dimensional clasp number $c^*(K)$, defined from the four-dimensional viewpoint. T. Shibuya gave an inequality $g^*(K) \leq c^*(K)$, and asked whether the equality holds or not. From our result in this paper, we find that the equality does not hold in general.

INTRODUCTION

We shall work in piecewise linear and locally flat category. All 4-manifolds and 3-manifolds will be assumed to be oriented.

In [16] T. Shibuya introduced some numerical invariants for classical links from the four-dimensional viewpoint. In this paper, we restrict ourselves to knots and study the relation between the 4-genus (slice genus) and the 4-dimensional clasp number. We also introduce another four-dimensional numerical invariant, the nonorientable 4-genus, for knots.

The 4-*genus* $g^*(K)$ of a knot K in $S^3 = \partial B^4$ is the minimum genus of orientable surfaces in B^4 bounded by K [5]. The *nonorientable* 4-*genus* $\gamma^*(K)$ is the minimum first Betti number of nonorientable surfaces in B^4 bounded by K. For a slice knot, it is defined to be 0 instead of 1. Shibuya [16] defined the 4-*dimensional clasp number* $c^*(K)$ to be the minimum number of the double points of transversely immersed 2-disks in B^4 bounded by K.

Shibuya [16] gave the following inequality

$$g^*(K) \leq c^*(K),$$

and asked whether the equality holds or not. Note that since $g^*(3_1) = c^*(3_1) = 1$, the equality above is best possible. In section 1, we investigate the 4-dimensional clasp number and show that the equality above does not hold in general. For example, we prove $c^*(8_{16}) = 2$ and $g^*(8_{16}) = 1$ (Example 1.5). Here we use the notation of J.W. Alexander and G.B. Briggs [2]. See also [3], [14].

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In section 2, we investigate relations among $\gamma^*(K)$, $g^*(K)$ and $c^*(K)$, and give some upper bounds for $\gamma^*(K)$ in terms of $g^*(K)$ or $c^*(K)$. We also investigate $\gamma^*(K)$ and give a necessary condition for a knot whose nonorientable 4-genus is n.

1. Four-dimensional clasp number

From now on, for a manifold M and a submanifold N of M with codimension two, $D_N(M)$ denotes the double branched cover of M with branched set N.

For an oriented 3-manifold M with finite first homology group, the linking form

$$\lambda : H_1(M;\mathbb{Z}) \times H_1(M;\mathbb{Z}) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

is defined as follows. Let x and y be 1-cycles in M. Suppose that nx bounds a 2-chain c for some $n \in \mathbb{Z}$. Then

$$\lambda([x],[y]) = \frac{c \cdot y}{n} \in \mathbb{Q}/\mathbb{Z},$$

where $c \cdot y$ is the intersection number of c and y.

Let K be a knot in S^3 and $D_K(S^3)$ the double branched cover of S^3 with branched set K. Then a Goeritz matrix U [7] for K is a relation matrix for $H_1(D_K(S^3);\mathbb{Z})$, and the linking form on $H_1(D_K(S^3);\mathbb{Z})$ is given by $\pm U^{-1}$ (the sign depending on the choice of orientation of $D_K(S^3)$) [15], [8].

In this section, we prove the following theorem.

Theorem 1.1. *Let K be a knot in S^3 with $c^*(K) = 1$, and let $D_K(S^3)$ be the double branched cover of S^3 with branch set K. Then the linking form λ on $H_1(D_K(S^3);\mathbb{Z})$ splits into a direct sum $(G_1,\lambda_1) \oplus (G_2,\lambda_2)$ such that*

1. *there is an element $g \in G_1$ with $\lambda_1(g,g) = \pm 2/|G_1|$, and*
2. *there is a subgroup H of G_2 such that $|H|^2 = |G_2|$ and $\lambda_2(h,h') = 0$ for any $h, h' \in H$.*

In the theorem above, λ_2 is called *metabolic*.

Remark 1.2. We can regard the theorem above as a 4-dimensional version of a result of W.B.R. Lickorish on the unknotting number [9], which states that if the unknotting number of a knot K is 1, then the linking form λ on $H_1(D_K(S^3);\mathbb{Z})$ is of the form as λ_1 in Theorem 1.1. Here we need λ_2 as a '4-dimensional part'.

In order to prove Theorem 1.1, we need the following two lemmas.

Lemma 1.3 (Gilmer [6, Lemma 1])**.** *Let M be a rational homology 3-sphere and λ the linking form on $H_1(M;\mathbb{Z})$. If M bounds a 4-manifold W with an intersection matrix V on $H_2(W;\mathbb{Z})$, then λ splits into a direct sum $(G_1,\lambda_1) \oplus (G_2,\lambda_2)$ such that*

1. $\det(V) = \pm|G_1|$ *and λ_1 is represented by the matrix $-V^{-1}$, and*
2. *λ_2 is metabolic.*

Here λ_1 is said to be represented by $-V^{-1}$ if there are some generators $u_1, u_2, \dots, u_n$ of G_1 such that $\lambda_1(u_i,u_j)$ is equal to the (i,j)-entry of $-V^{-1}$ for any i,j. □

Lemma 1.4. *Let W be a compact 4-manifold with $H_1(W;\mathbb{Z}) = 0$ and $\partial W \cong S^3$, and let F be a properly embedded, compact, possibly nonorientable surface in W with $\partial F \cong S^1$. If F represents zero in $H_2(W,\partial W;\mathbb{Z}_2)$, then $\beta_2(D_F(W)) = 2\beta_2(W) + \beta_1(F)$, where β_i is the i-th rational Betti number.*

Proof. Let (W', F') be the pair of the closed 4-manifold and the closed surface that is obtained from two copies of (W, F) by gluing their common boundaries. Note that F' represents zero in $H_2(W'; \mathbb{Z}_2)$. It follows from [13, 3.2 and 7.2] that $\beta_2(D_{F'}(W')) = 4\beta_2(W) + 2\beta_1(F)$. Since $\partial(D_F(W))$ is a rational homology 3-sphere, from the Mayer-Vietris exact sequence we have $\beta_2(D_{F'}(W')) = 2\beta_2(D_F(W))$. Hence we have $\beta_2(D_F(W)) = 2\beta_2(W) + \beta_1(F)$. □

Proof of Theorem 1.1. Suppose that $c^*(K) = 1$. Then there is an immersed 2-disk in a 4-ball with exactly one double point and with boundary K. By removing an open, small neighborhood of the double point, we obtain a disk with two holes Y properly embedded in $S^3 \times I$ such that $Y \cap (S^3 \times \{0\}) = K$ and $Y \cap (S^3 \times \{1\})$ is a Hopf link L. Since a Hopf link bounds an annulus A in the boundary of a 4-ball B_0^4, we have an orientable surface $F = Y \cup A$ with genus 1 in the 4-ball $B^4 = (S^3 \times I) \cup B_0^4$. Thus we see $D_F(B^4) = D_Y(S^3 \times I) \cup D_A(B_0^4)$ with $\partial(D_F(B^4)) = D_K(S^3)$. Note that $\partial(D_Y(S^3 \times I)) = D_K(S^3) \cup \mathbb{R}P^3$ since $D_L(S^3)$ is the 3-dimensional real projective space $\mathbb{R}P^3$. We may assume that $D_A(B_0^4)$ is a 4-ball with 2-handle h^2 attached along the trivial knot in its boundary with framing ± 2 according to the linking number of L (for example, see [1]). By Lemma 1.4 we have that $\beta_2(D_F(B^4)) = 2$. Hence there is an intersection matrix V with respect to some basis α_1 and α_2 of $H_2(D_F(B^4); \mathbb{Z})/(\text{torsion part})$. Set

$$V = \begin{pmatrix} a & c \\ c & b \end{pmatrix}.$$

Lemma 1.3 states that the linking form λ splits into a direct sum $(G_1, \lambda_1) \oplus (G_2, \lambda_2)$ such that

1. $ab - c^2 = \pm|G_1|$ and λ_1 is represented by the matrix $-V^{-1}$, and
2. λ_2 is metabolic.

Now we will find an element $g \in G_1$ with $\lambda_1(g, g) = \pm 2/|G_1|$. Let β be an element in $H_2(D_A(B_0^4); \mathbb{Z})$ represented by a 2-sphere that consists of the core of 2-handle h^2 and a 2-disk bounded by attaching a sphere of h^2. Note that the self-intersection number is ± 2. From the Mayer-Vietoris exact sequence we have the natural injection $j : H_2(D_A(B_0^4); \mathbb{Z}) \longrightarrow H_2(D_F(B^4); \mathbb{Z})$ since $H_2(\mathbb{R}P^3; \mathbb{Z}) = 0$. Therefore $j(\beta)$ is an element $x\alpha_1 + y\alpha_2$ in $H_2(D_F(B^4); \mathbb{Z})/(\text{torsion part})$ and $(x, y)V\,{}^t(x, y) = \pm 2$. Hence for some generators u_1 and u_2 of G_1, there is an element $g = yu_1 - xu_2$ such that $\lambda_1(g, g) = (y, -x)(-V^{-1})\,{}^t(y, -x) = \mp 2/(ab - c^2)$. This completes the proof. □

The following example implies that the equality of Shibuya's inequality $g^*(K) \leq c^*(K)$ [16] does not hold in general.

Example 1.5. $c^*(8_{16}) = 2$ and $g^*(8_{16}) = 1$.

Proof. Since we can easily see that $c^*(8_{16}) \leq 2$ and $g^*(8_{16}) = 1$, we shall prove that $c^*(8_{16}) \geq 2$. A Goeritz matrix for 8_{16} is

$$\begin{pmatrix} 3 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{pmatrix}.$$

So we see that $H_1(D_{8_{16}}(S^3); \mathbb{Z}) \cong \mathbb{Z}/35\mathbb{Z}$ and that the linking form λ is defined by $\lambda(g, g) = \pm 11/35$ for some generator g. Note that λ has no metabolic part because 35 is square free. Suppose that $c^*(8_{16}) \leq 1$. From Theorem 1.1, there is some integer n such that $\lambda(ng, ng) = \pm 2/35$. Thus we have $\pm 2/35 = \lambda(ng, ng) = n^2\lambda(g, g) = 11n^2/35$ in $\mathbb{Q}/\mathbb{Z}$. This implies that

$$n^2 \equiv 16 \times 11n^2 \equiv 16 \times (\pm 2) \equiv \mp 3 \pmod{35}.$$

Hence we have $n^2 \equiv \pm 3 \pmod 7$ and $n^2 \equiv \pm 3 \pmod 5$. This is a contradiction. □

2. Various four-genera

The *nonorientable genus* $\gamma(K)$ of a knot K is the minimum first Betti number of nonorientable surfaces bounded by K [4], [12]. (In [4], [12] the nonorientable genus is called the *crosscap number* and is denoted by $C(K)$.) For the trivial knot, it is defined to be 0 instead of 1. From the definitions, the following proposition is clear.

Proposition 2.1. *For any knot K, the following inequality holds*:

$$\gamma^*(K) \le \gamma(K).$$ □

Since we can construct a nonorientable surface from an orientable surface by adding a half-twisted band, we have inequality similar to [4], [12].

Proposition 2.2. *For any knot K, the following inequality holds*:

$$\gamma^*(K) \le 2g^*(K) + 1.$$ □

For a knot K, we define $\Gamma^*(K)$ to be $\min\{2g^*(K), \gamma^*(K)\}$. By this definition, an inequality $\Gamma^*(K) \le \gamma^*(K)$ clearly holds. For a relation with the 4-dimensional clasp number, we have

Proposition 2.3. *For any knot K, the following inequality holds*:

1. $\Gamma^*(K) \le \begin{cases} c^*(K) & \text{if } c^*(K) \text{ is even,} \\ c^*(K)+1 & \text{otherwise.} \end{cases}$
2. $\gamma^*(K) \le \begin{cases} c^*(K) & \text{if } c^*(K) \text{ is even and } c^*(K) \ne 2, \\ c^*(K)+1 & \text{otherwise.} \end{cases}$

The inequalities in Propositions 2.1 and 2.3 are best possible. See Remark 2.9 (1), (2).

Proof. First we consider the case that $c^*(K)$ is even. Set $c^*(K) = 2m$. Then there is an immersed 2-disk Δ in a 4-ball B^4 with $2m$ double points $p_1, p_2, ..., p_m, q_1, q_2, ..., q_m$ and with boundary K, and there are mutually disjoint arcs $a_1, a_2, ..., a_m$ in Δ such that each a_i joins p_i to q_i. Let D be a domain 2-disk of Δ and p_{ij} (resp. q_{ij}) ($j = 1, 2$) the preimages of p_i (resp. q_i) ($i = 1, 2, ..., m$). We may assume that the preimage of a_i joins p_{i1} to q_{i1}. Let $D(p_i)$ and $D(q_i)$ be 2-disks in Δ that are images of small neighborhoods of p_{i2} and q_{i2} in D respectively. It is not hard to see that there are mutually disjoint embeddings $h_i : D^2 \times I \longrightarrow B^4$ $(i = 1, 2, ..., m)$ such that $h_i(\{0\} \times I) = a_i$, $h_i(D^2 \times \partial I) = D(p_i) \cup D(q_i)$ and $h(D^2 \times I) \cap \Delta = a_i \cup D(p_i) \cup D(q_i)$. Thus we have an embedded surface $(\Delta - \bigcup_i^m h_i((D^2 - \{0\}) \times \partial I)) \cup (\bigcup_i^m h_i(\partial D^2 \times I))$ with the first Betti number $2m$ and its boundary K. This implies that $\Gamma^*(K) \le c^*(K)$.

Note that this embedded surface is orientable if and only if the signs of each pair p_i and q_i are opposite for any i $(= 1, 2, ..., m)$. If $c^*(K) \ge 4$, then we can choose a pair of double points in Δ whose signs are the same. Hence we have $\gamma^*(K) \le c^*(K)$ for $c^*(K) \ge 4$. On the other hand, if $c^*(K) = 0$, then by the definition of $\gamma^*(K)$, $\gamma^*(K) = c^*(K) = 0$. If $c^*(K) = 2$, then by Proposition 2.2 and the inequality $\Gamma^*(K) \le c^*(K)$, we have $\gamma^*(K) \le \Gamma^*(K) + 1 \le c^*(K) + 1$.

Next we consider the case that $c^*(K)$ is odd. By arguments similar to that in the case above, we have an immersed surface F in a 4-ball B^4 with one double point p and $\partial F = K$ such that the first Betti number of the preimage of F is $c^*(K) - 1$. Take a closed, small neighborhood N of p. Note that $\partial(F \cap N)$ is a Hopf link. By removing $F \cap N$ from F and attaching an annulus to it without compatible orientation, we obtain an embedded, nonorientable surface in B^4 with the first Betti number $c^*(K)+1$. Hence we have $\Gamma^*(K) \leq \gamma^*(K) \leq c^*(K)+1$. □

From Proposition 2.3 and the definition of $\Gamma^*(K)$, we have the following corollary.

Corollary 2.4. *For a knot K, if $g^*(K) = c^*(K) \geq 1$, then $\Gamma^*(K) = \gamma^*(K)$.* □

Combining Lemmas 1.3 and 1.4, we get the following theorem.

Theorem 2.5. *Let W be a compact 4-manifold with $H_1(W;\mathbb{Z}) = 0$ and $\partial W \cong S^3$. Let K be a knot in ∂W and $D_K(S^3)$ the double branched cover of S^3 with branch set K. If K bounds a compact, possibly nonorientale surface F in W that represents zero in $H_2(W, \partial W; \mathbb{Z}_2)$, then the linking form λ on $H_1(D_K(S^3);\mathbb{Z})$ splits into a direct sum $(G_1, \lambda_1) \oplus (G_2, \lambda_2)$ such that*

1. *there is a $(2\beta_2(W)+\beta_1(F)) \times (2\beta_2(W)+\beta_1(F))$-matrix V such that* $\det V = \pm|G_1|$ *and λ_1 is represented by $-V^{-1}$, and*
2. *λ_2 is metabolic.* □

From the theorem above, we have the following two corollaries. The second corollary is a '4-dimensional version' of a result of Lickorish [10, Theorem] concerning the nonorientable genus.

Corollary 2.6. *Let K be a knot in S^3 with $\Gamma^*(K) \leq n$. Then the linking form λ on $H_1(D_K(S^3);\mathbb{Z})$ splits into a direct sum $(G_1, \lambda_1) \oplus (G_2, \lambda_2)$ such that*

1. *there is an $n \times n$-matrix V such that* $\det V = \pm|G_1|$ *and λ_1 is represented by $-V^{-1}$, and*
2. *λ_2 is metabolic.* □

Corollary 2.7. *Let K be a knot in S^3 with $\Gamma^*(K) \leq 1$. Then the linking form λ on $H_1(D_K(S^3);\mathbb{Z})$ splits into a direct sum $(G_1, \lambda_1) \oplus (G_2, \lambda_2)$ such that*

1. *λ_1 is represented by the 1×1-matrix $\pm(1/|G_1|)$, and*
2. *λ_2 is metabolic.* □

From the corollary above, we have

Example 2.8. $\Gamma^*(4_1) = \gamma^*(4_1) = 2$.

It is already known that $\gamma^*(4_1) = 2$ [17], [18].

Proof. Since we can easily see that $\Gamma^*(4_1) \leq \gamma^*(4_1) \leq 2$, we shall prove that $\Gamma^*(4_1) \geq 2$. A Goeritz matrix for 4_1 is

$$\begin{pmatrix} -3 & 1 \\ 1 & -2 \end{pmatrix}.$$

So we see that $H_1(D_{4_1}(S^3);\mathbb{Z}) \cong \mathbb{Z}/5\mathbb{Z}$ and that the linking form λ is defined by $\lambda(g,g) = \pm 2/5$ for some generator g. Note that λ has no metabolic part because 5 is square-free. Suppose that $\Gamma^*(4_1) \leq 1$. From Corollary 2.7, there is some integer n such that $\lambda(ng, ng) = \pm 1/5$. Thus we have $\pm 1/5 = \lambda(ng, ng) = n^2\lambda(g,g) = 2n^2/5$ in $\mathbb{Q}/\mathbb{Z}$. But this is a contradiction since an easy calculation shows this cannot occur. □

The following remark, Remark 2.9 (1), (2), implies that the inequalities in Propositions 2.1 and 2.3 are best possible. Remark 2.9 (3) gives a nontrivial example that implies the inequality $\Gamma^*(K) \leq c^*(K)$ (for $c^*(K)$ is even) in Proposition 2.3 is best possible.

Remark 2.9. (1) For a slice knot K, we have $\Gamma^*(K) = \gamma^*(K) = c^*(K) = 0$.

(2) Since $\gamma(4_1) \leq 2$ and $c^*(4_1) \leq 1$, by Example 2.8, we have $\Gamma^*(4_1) = \gamma^*(4_1) = \gamma(4_1) = c^*(4_1) + 1 = 2$.

(3) For a knot K, K. Murasugi [11] gave the inequality $|\sigma(K)|/2 \leq g^*(K)$, where $\sigma(K)$ is the signature of K. Since $|\sigma(3_1\#3_1)| = 4$ and $g^*(3_1\#3_1) \leq c^*(3_1\#3_1) \leq 2$, we have $g^*(3_1\#3_1) = c^*(3_1\#3_1) = 2$. In [18] the second-named author showed that $\gamma^*(3_1\#3_1) \geq 2$. Hence by Proposition 2.3 and Corollary 2.4, we have $\gamma^*(3_1\#3_1) = \Gamma^*(3_1\#3_1) = c^*(3_1\#3_1) = 2$.

Unfortunately, the authors do not know whether the inequality in Proposition 2.2 is best possible or not. Note that the authors in [12] showed that an inequality $\gamma(K) \leq 2g(K) + 1$ is best possible. So we have the following conjecture.

Conjecture 2.10. *There is a knot K such that*

$$\gamma^*(K) = 2g^*(K) + 1.$$

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