NOTE ON A LITTLEWOOD-PALEY INEQUALITY

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Abstract. We show that a recent result of Littlewood-Paley type, due to the author, is essentially best-possible.

In a recent paper [W], the author proved Littlewood-Paley inequalities for certain finite linear sums \( f = \sum_I \lambda_I \phi_I(x) \), defined on \( \mathbb{R}^d \). The summation here is indexed over \( D \), the family of dyadic cubes in \( \mathbb{R}^d \); the \( \lambda_I \) are complex numbers. The functions \( \phi_I(x) \), assumed to be smooth, belong to a family \( F \) that is “almost-orthogonal” and satisfies a mild decay condition. Precisely, we assume that there is an \( M > d/2 \) such that for all \( x \in \mathbb{R}^d \) and all \( I \in D \),

\[
|\phi_I(x)| + \ell(I) |\nabla \phi_I(x)| \leq |I|^{-1/2} (1 + |x - x_I|/\ell(I))^{-M}.
\]

The notation is more or less standard: \( x_I \) denotes \( I \)'s center, \( \ell(I) \) is its sidelength, and \( |I| \) is its Lebesgue measure (per tradition, we shall use \( |E| \) to mean the Lebesgue measure of any measurable set \( E \)). We assume in addition that, for any finite linear sum \( \sum_I \gamma_I \phi_I(x) \),

\[
\int_{\mathbb{R}^d} |\sum_I \gamma_I \phi_I(x)|^2 \, dx \leq \sum_I |\gamma_I|^2.
\]

Recall that a non-negative \( \sigma \in L^1_{\text{loc}}(\mathbb{R}^d) \) is said to be an \( A_\infty \) weight if there are positive constants \( a \) and \( b \) such that, for all cubes \( Q \subset \mathbb{R}^d \) and measurable subsets \( E \subset Q \),

\[
\frac{\int_E \sigma}{\int_Q \sigma} \leq a \left( \frac{|E|}{|Q|} \right)^b.
\]

The main result from [W] is:

**Theorem 1.** Let \( F \) be a family satisfying (1) and (2). Let \( \rho > d \). For every \( \sigma \in A_\infty \) as above and \( 0 < \rho < \infty \), there is a constant \( C = C(M, d, \rho, a, b, \rho) \) such that, for all finite linear sums \( f = \sum_I \lambda_I \phi_I(x) \) from \( F \),

\[
\int_{\mathbb{R}^d} |f|^\rho \sigma \, dx \leq \int_{\mathbb{R}^d} \left( \sum_I \frac{|\lambda_I|^2}{|I|} (1 + |x - x_I|/\ell(I))^{-2M + \rho} \right)^{\rho/2} \sigma \, dx.
\]
The thing to watch is the exponent $-2M + \rho$. If there were only one term in the sum, we could do no better than $-2M$. The author finds it remarkable that, with such weak hypotheses on the family $\mathcal{F}$, one can get a result that seems so close to best-possible.

In this note we show that Theorem 1 is, in fact, within $\epsilon$ of best-possible.

**Theorem 2.** Theorem 1 fails for $p = 2$ and all $p < d$.

**Proof of Theorem 2.** First we will give our original proof, which uses well-known facts about Bochner-Riesz kernels. The argument has the disadvantage of only working for $d > 1$. At the end we will indicate how to extend the proof to $d = 1$. The extension is easy, but it uses somewhat lesser-known facts about more general oscillatory kernels, due to Jurkat and Sampson [JS].

For $\delta > 0$, let $K^\delta$ be the function whose Fourier transform is $(1 - |\xi|^2)^\delta$. From [St] (pages 338 and 390), we have:

$$|K^\delta(x)| + |\nabla K^\delta(x)| \leq C(1 + |x|)^{-(\delta + (d+1)/2)}.$$  

Asymptotic estimates for $K^\delta$ imply that, as long as $\delta < (d - 1)/2$, the convolution operator $K^\delta * f$ cannot be bounded on $L^p(\mathbb{R}^d)$ unless

$$\frac{2d}{d + 1 + 2\delta} < p < \frac{2d}{d - 1 - 2\delta},$$

in particular, for such $\delta$, convolution with $K^\delta$ must be unbounded on $L^p(\mathbb{R}^d)$ for large values of $p$.

We now prove a simple lemma.

**Lemma.** Let $\{I_j\}_j$ be the collection of dyadic cubes with sidelength equal to 1. Suppose that, for each $j$, we have a function $h_j$ satisfying:

$$\text{supp } h_j \subset I_j,$$

$$\int |h_j|^2 \, dx \leq 1.$$

For each $j$, set $\phi_j = K^\delta * h_j$. Then, with $M = \delta + (d + 1)/2$, we have

$$|\phi_j(x)| + |\nabla \phi_j(x)| \leq C(1 + |x - x_{I_j}|)^{-M};$$

and, for all finite linear sums $\sum_j \lambda_j \phi_j$,

$$\int \left| \sum_j \lambda_j \phi_j \right|^2 \, dx \leq \sum_j |\lambda_j|^2.$$  

**Proof of the Lemma.** Inequality (3) follows from the bounds on $K^\delta$ given above. Inequality (4) follows because

$$\int \left| \sum_j \lambda_j h_j \right|^2 \, dx \leq \sum_j |\lambda_j|^2$$

and $\|K^\delta\|_\infty \leq 1$.

We will now suppose that Theorem 1 holds for some $\rho < d$, and see what happens.
Let $f \in L^2(\mathbb{R}^d)$ have compact support. For each $j$, set $\gamma_j = (\int_{I_j} |f|^2)^{1/2}$. Define:

$$h_j = \begin{cases} f \chi_{I_j} / \gamma_j, & \text{if } \gamma_j > 0, \\ 0, & \text{otherwise}, \end{cases}$$

and set $\phi_j = K^\delta * h_j$, as in the Lemma, where we choose $\delta > 0$ to satisfy:

$$\frac{\rho - 1}{2} < \delta < \frac{d - 1}{2}. \tag{5}$$

Inequality (5) implies that $2M - \rho > d$ and that convolution with $K^\delta$ is unbounded on $L^p$ for large $p$. Let us fix a $p$ for which this is the case.

Note that $\{\phi_j\}_j$ is a finite collection, and that

$$\sum_j \gamma_j \phi_j = K^\delta * f.$$

Let $s > 1$ be the dual exponent to $p/2$, and let $1 < r < s$. Since we have already used $M$ to mean an exponent, we shall denote the Hardy-Littlewood maximal operator by $T$. Set $T_r g = (T(|g|^r))^{1/r}$. It is a standard fact that, if $g \neq 0$, then $T_r g \in A_\infty$, with $A_\infty$ parameters only depending on $r$ and $d$. In fact, $T_r g$ belongs to the family $A_1$: There is a constant $C = C(r, d)$ such that $T(T_r g) \leq CT_r g$ almost everywhere. One more fact we shall need: There is a constant $C = C(r, s, d)$ such that

$$\|T_r g\|_s \leq C\|g\|_s$$

for all $g \in L^r$.

If Theorem 1 held for the given $\rho$, then, for all $f$ as described and all $g \in L^s$, we would have:

$$\int |K^\delta * f|^2 g \, dx \leq \int |K^\delta * f|^2 T_r g(x) \, dx$$

$$\leq C \sum_j |\gamma_j|^2 \int T_r g(x) (1 + |x - x_{I_j}|)^{-2M+\rho} \, dx$$

$$\leq C \sum_j |\gamma_j|^2 T(T_r g)(x_{I_j})$$

$$\leq C \sum_j |\gamma_j|^2 \int_{I_j} T_r g \, dx$$

$$\leq C \int (\sum_j \gamma_j^2 \chi_{I_j}) T_r g \, dx$$

$$\leq C \left( \int \left( \sum_j \gamma_j^2 \chi_{I_j} \right)^{p/2} \, dx \right)^{2/p} \|T_r g\|_s$$

$$\leq C \left( \int \left( \sum_j \gamma_j^p \chi_{I_j} \, dx \right)^{2/p} \right)^{2/p} \|g\|_s$$

$$\leq C \|f\|_p^2 \|g\|_s,$$

which would imply that the convolution with $K^\delta$ was bounded on $L^p(\mathbb{R}^d)$. This proves Theorem 2 when $d > 1$. 
To handle $d = 1$, we replace $K^\delta$ with a kernel $K_{\alpha,\beta}$, defined by:

$$K_{\alpha,\beta}(t) = (1 + |t|)^{-\beta} e^{i|t|\alpha}.$$ 

Here $\alpha$ and $\beta$ are positive numbers which will be chosen presently (depending on $\rho$).

Notice that, if $\alpha$ and $\beta$ are both less than 1, then convolution with $K_{\alpha,\beta}$ must be unbounded on $L^p(\mathbb{R})$ when $p < \beta^{-1}$ (look at $K_{\alpha,\beta} \ast \chi_B$, where $B$ is a small interval centered at the origin); and, therefore, it must be unbounded on $L^p(\mathbb{R})$ for large $p$'s. Notice also that, for such $\alpha$ and $\beta$, we have:

$$|K_{\alpha,\beta}(t)| + |K'_{\alpha,\beta}(t)| \leq C(\alpha, \beta)(1 + |t|)^{-\beta}$$

for all $t$. (Strictly speaking, $K_{\alpha,\beta}$ does not satisfy (6) at $t = 0$, but this can be fixed by convolving $K_{\alpha,\beta}$ with a suitable mollifier: what is crucial here is the Lipschitz smoothness.)

We use the following result from [JS] (page 410):

**Theorem 3.** Let $\alpha$, $\beta$ be positive, $\alpha \neq 1$, $\beta < 1$, and $(\alpha/2) + \beta \geq 1$. Then convolution with $K_{\alpha,\beta}$ is bounded on $L^p(\mathbb{R})$ when

$$\frac{\alpha}{\alpha + \beta - 1} \leq p \leq \frac{\alpha}{1 - \beta}.$$ 

In particular, the convolution will be bounded on $L^2$.

It is easy to see that the one-dimensional analogue of the “simple lemma” goes through verbatim with $K_{\alpha,\beta}$ in place of $K^\delta$, and with $M \equiv \beta$. Now let $0 < \rho < 1$. Pick $0 < \beta < 1$, so close to 1 that $2\beta - \rho > 1$, and so there exists an $\alpha$, $0 < \alpha < 1$, such that $(\alpha/2) + \beta \geq 1$. Fix these $\alpha$ and $\beta$.

If Theorem 1 held for this $\rho$, then, by a virtual repetition of our earlier argument, convolution with $K_{\alpha,\beta}$ would be bounded on $L^p(\mathbb{R})$ for large $p$'s.

Theorem 2 is proved.

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**References**


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