

POINTWISE CONVERGENCE OF A CLASS OF NON-ORTHOGONAL WAVELET EXPANSIONS

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ABSTRACT. Non-orthogonal wavelet expansions associated with a class of mother wavelets is considered. This class of wavelets comprises mother wavelets that are not necessarily integrable over the whole real line, such as Shannon's wavelet. The pointwise convergence of these wavelet expansions is investigated. It is shown that, unlike other wavelet expansions, the ones under consideration do not necessarily converge pointwise to the functions at points of continuity, unless a more stringent condition, such as bounded variation, is imposed.

1. INTRODUCTION

Wavelet expansions have been the focus of many research papers in the last few years. One of the reasons for their popularity is that they provide a more efficient representation of functions than other orthogonal expansions. The orthogonal wavelet expansion of a function in $L^2(\mathcal{R})$ is known to converge to it in the sense of $L^2(\mathcal{R})$, but the pointwise convergence is a little bit more subtle. Y. Meyer [4, Ch. 2] was among the first to study the convergence of orthogonal wavelet expansions. He showed that if the mother wavelet is r -regular, the orthogonal wavelet expansion of a function will converge to it in the sense of $L^p(\mathcal{R})$, $1 \leq p < \infty$, and in the sense of some Sobolov spaces as well.

A function $f(x)$, $x \in \mathcal{R}^d$, $d \geq 1$, is said to be r -regular (in the sense of Meyer) if

$$|D^\alpha f(x)| \leq \frac{C_{\alpha,m}}{(1+|x|)^m},$$

for all α with $|\alpha| \leq r$ and $m = 0, 1, 2, \dots$, where $C_{\alpha,m}$ are constants. Here $\alpha = (\alpha_1, \dots, \alpha_d)$ is a multi-index with α_i ($i = 1, \dots, d$) being a non-negative integer and $|\alpha| = \sum_{i=1}^d \alpha_i$, and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}.$$

Before we state some important results on pointwise convergence of wavelet expansions, let us recall the following definition.

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Definition 1. A multiresolution analysis of \mathcal{R}^d is a decomposition of $L^2(\mathcal{R}^d)$ into a nested sequence of closed subspaces $\{V_i\}_{i=-\infty}^\infty$ such that

- 1) $\dots \subset V_{-1} \subset V_{-2} \subset V_0 \subset V_1 \subset V_2 \subset \dots$,
- 2) $\bigcup_{i=-\infty}^\infty V_i$ is dense in $L^2(\mathcal{R}^d)$,
- 3) $\bigcap_{i=-\infty}^\infty V_i = \{0\}$,
- 4) $f(x) \in V_i$ if and only if $f(2x) \in V_{i+1}$, for all i ,
- 5) V_0 is closed under integer translation, i.e., $f(x) \in V_0$ implies that $f(x - k) \in V_0$, for all $k = (k_1, \dots, k_d)$ with k_i being a nonnegative integer,
- 6) there exists a function $\phi(x) \in V_0$, called the scaling function or the father wavelet, such that $\{\phi_k(x) = \phi(x - k)\}$ is an orthonormal basis for V_0 .

A multiresolution analysis is said to be r -regular if its scaling function is r -regular.

Let W_j denote the orthogonal complement of V_j in V_{j+1} . It is known that there is a family $\{\psi^\lambda(x)\}_{\lambda \in \Lambda}$ (whose cardinality depends on the dimension d) of basic mother wavelets such that $\{\psi_{m,n}^\lambda(x) = 2^{md/2} \psi^\lambda(2^m x - n)\}$ is an orthonormal basis of W_m , for fixed m , where m is an integer, n is a multi-integer, and $\lambda \in \Lambda$, and is an orthonormal basis of $L^2(\mathcal{R}^d)$, for all m and n . In one dimension the cardinality of Λ is one; hence there is one basic mother wavelet ψ generating an orthonormal basis of $L^2(\mathcal{R})$.

From the definition, it follows by induction that

$$(1) \quad L^2(\mathcal{R}^d) = \bigoplus_{i=-\infty}^\infty W_i,$$

where \oplus denotes the direct sum and

$$(2) \quad L^2(\mathcal{R}^d) = V_0 \bigoplus_{i=0}^\infty W_i.$$

It also follows that in one dimension $\{\phi_{m,n}(x) = 2^{m/2} \phi(2^m x - n)\}_{n=-\infty}^\infty$ is an orthonormal basis for V_m , for each m . From (1) and (2), it follows that

$$(3) \quad f(x) = \sum_{j;k;\lambda} a_{j,k}^\lambda \psi_{j,k}^\lambda(x),$$

where $a_{j,k}^\lambda$ are the L^2 expansion coefficients of f and

$$(4) \quad f(x) = \sum_k b_k \phi_k(x) + \sum_{j \geq 0; k; \lambda} a_{j,k}^\lambda \psi_{j,k}^\lambda(x),$$

where b_k and $a_{j,k}^\lambda$ are the L^2 expansion coefficients of f . The convergence of the above series is in the sense of L^2 . In one dimension, we have

$$f(x) = \sum_k b_{m,k} \phi_{m,k} + \sum_{j \geq m; k} a_{j,k} \psi_{j,k}(x) = f_m(x) + r_m(x).$$

The function f_m is in the space V_m and in fact is the projection of f onto V_m .

Assuming that the scaling function ϕ of the multiresolution analysis is r -regular, G. Walter [6] proved that the orthogonal wavelet expansion of a function $f \in L^2 \cap L^1$ converges to f pointwise at every point of continuity of f and uniformly on compact

subsets of any interval (a, b) on which f is continuous. Later, he relaxed this condition and assumed that ϕ satisfies the condition

$$|\phi(x)| \leq \frac{C}{(1 + |x|)^3}.$$

In [2, 3], S. Kelly, M. Kon, and L. Raphael improved Walter’s results by proving pointwise convergence of orthogonal wavelet expansions not only under less stringent conditions, but also by extending them to n dimensions. Crucial to their proofs is the following definition.

Definition 2. A function $f(x)$ on \mathcal{R}^d is radial if it depends on $|x|$ only. A real-valued radial function is radial decreasing if $f(x) \leq f(y)$ whenever $|y| \leq |x|$. A function $f(x)$ is in the class \mathcal{RB} if it is absolutely bounded by an L^1 radial decreasing function $\eta(x)$, i.e., with $\eta(x_1) = \eta(x_2)$ whenever $|x_1| = |x_2|$ and with $\eta(x_1) \leq \eta(x_2)$ whenever $|x_1| \geq |x_2|$, and $\eta(x) \in L^1(\mathcal{R}^d)$.

They showed, among other things, that if ϕ, ψ^λ are in the class \mathcal{RB} for all λ , then the series (3) and (4) of any function $f \in L^p(\mathcal{R}^d)$ converge to f pointwise almost everywhere. Moreover, if only $\psi^\lambda \ln(2 + |x|) \in \mathcal{RB}$, then the wavelet expansion (3) of any function $f \in L^p(\mathcal{R}^d)$ converges to f pointwise almost everywhere.

In all the work cited above on pointwise convergence it is essential that the summation kernel of the wavelet series

$$(5) \quad P_m(x, y) = \sum_{j < m; k; \lambda} \psi_{j,k}^\lambda(x) \bar{\psi}_{j,k}^\lambda(y) = \sum_k \phi_{m,k}(x) \bar{\phi}_{m,k}(y)$$

be absolutely bounded by a dilation of an L^1 - radially decreasing convolution kernel, i.e., by $H(2^m|x - y|)$, where $H \in \mathcal{RB}$. There are, however, some mother wavelets that do not satisfy these conditions. Chief among them is Shannon’s wavelet, which is given by

$$\psi(x - 1/2) = \frac{\sin 2\pi x - \sin \pi x}{\pi x},$$

and for which the associated scaling function ϕ is given by $\phi(x) = \sin \pi x / \pi x$. The summation kernels in the subspaces V_0 and W_0 are easily seen to be

$$(6) \quad \frac{\sin \pi(x - y)}{\pi(x - y)} = \sum_{k=-\infty}^{\infty} \frac{\sin \pi(x - k)}{\pi(x - k)} \frac{\sin \pi(y - k)}{\pi(y - k)},$$

and

$$(7) \quad \frac{\sin 2\pi(x - y) - \sin \pi(x - y)}{\pi(x - k)} = \sum_{k=-\infty}^{\infty} \frac{\sin 2\pi(x - k) - \sin \pi(x - k)}{\pi(x - k)} \frac{\sin 2\pi(y - k) - \sin \pi(y - k)}{\pi(y - k)}.$$

Equation (6) is well known, but (7), to the best of our knowledge, is less known. From (6) and (7) it is clear that neither kernel is in $L^1(\mathcal{R})$; hence neither one can be absolutely bounded by an $L^1(\mathcal{R})$ radially decreasing function.

Of course, it is easy to study directly the pointwise convergence of the Shannon wavelet series, but this is a very special case and of no great interest. Instead, we shall study the pointwise convergence of wavelet expansions associated with a class of mother wavelets that contains Shannon’s wavelet as a special case. We shall assume that neither the scaling function of the multiresolution analysis, ϕ ,

nor the mother wavelet ψ is in L^1 , and we shall not assume that they generate an orthonormal basis of V_0 or of W_0 . This means that unlike the wavelet expansions studied in [2, 3, 6], ours are not necessarily orthonormal. Therefore, we replace condition (6) of Definition 1 by the weaker condition that $\{\phi(x - k)\}$ is a Riesz basis of V_0 .

The foregoing discussion may give a false impression that our results are stronger than those of [2, 3, 6], while in fact they are neither stronger nor weaker, but complementary. On the one hand, we do not assume that ϕ or ψ is integrable or generates an orthonormal basis, but on the other hand we assume a stronger condition on ϕ , namely, that its Fourier transform $\hat{\phi}$ has compact support. By not requiring that ϕ or ψ be absolutely integrable, the partial sums of the wavelet series behave differently from those studied in [2, 3, 6]. The continuity of a function f is no longer sufficient to ensure that its wavelet series expansion will converge to it pointwise. A more stringent condition, such as bounded variation, is needed.

2. THE MAIN RESULT

We adopt the following standard notation. The Fourier transform of a function $f(x)$ is defined as

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{i\omega x} dx,$$

so that the inverse transform is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{-i\omega x} d\omega,$$

whenever the integrals exist. A summation of the form \sum_k is understood to mean $\sum_{k=-\infty}^{\infty}$. The characteristic function of a set A will be denoted by $\chi_A(x)$, i.e., $\chi_A(x) = 1$ if $x \in A$ and zero otherwise. The space of all locally integrable functions on \mathcal{R} will be denoted by $L^1_{loc}(\mathcal{R})$.

Definition 3. Let \mathcal{A} be the set of all scaling functions $\phi \in L^2(\mathcal{R})$ that satisfy the following conditions:

- 1) $\hat{\phi}(\omega)$ has support in some interval $[-a, a]$, $0 < a \leq \pi$, and $|\phi(x)| \leq c/(|x|+1)$;
- 2) there exist two positive numbers A and B such that

$$0 < A \leq \sum_k |\hat{\phi}(\omega + 2\pi k)|^2 \leq B < \infty;$$

- 3) $\phi(x)$ satisfies the dilation equation

$$\phi(x) = \sum_k a_k \phi(2x - k),$$

for some sequence $\{a_k\}$ with $\sum_k |a_k|^2 < \infty$;

- 4) $\hat{\phi}(\omega)$ is bounded and continuous near zero with $\hat{\phi}(0) \neq 0$;
- 5) the support of $\hat{\phi}$, which will be denoted by E , is of the form

$$E = \bigcup_{i=1}^N [a_i, b_i] \cup O,$$

where O is a set of measure zero.

The class of mother wavelets associated with the class \mathcal{A} will be called the class of Shannon-type wavelets and will be denoted by \mathcal{A}_W .

In addition to Shannon’s wavelet, the class \mathcal{A}_W contains infinitely many functions; among them, for example, is the function

$$\psi(x - 1/2) = \frac{\sin 2\pi t + \sin \pi t - 2 \sin(3\pi t/2)}{\pi t},$$

which is associated with the scaling function $\phi(t) = [2 \sin(\pi t/2) - \sin \pi t]/\pi t$, or

$$\hat{\phi}(\omega) = \begin{cases} 1, & |\omega| < \pi/2, \\ -1, & \pi/2 < |\omega| < \pi \end{cases}$$

(for more examples of wavelets in the class \mathcal{A}_W , see [7]).

Let us define V_0 as the closed linear span of the set $\{\phi(x - k)\}$ and likewise V_m as the closed linear span of $\{\phi_{m,k} = 2^{m/2}\phi(2^m x - k)\}$. The space V_0 is the same as the space B_E of functions bandlimited to E since any $f \in B_E$ can be written in the form $\hat{f} = \hat{g}\hat{\phi}$, where \hat{g} is the 2π -periodic extension of $\hat{f}/\hat{\phi}$. Condition (2) implies that $\{\phi(x - k)\}$ is a Riesz basis of V_0 . Condition (3) implies that the spaces V_m have the nested property (1) of Definition 1. Conditions (2) and (3) ensure that the set $\phi_{m,k}$ is a Riesz basis of V_m for each m and that $\bigcap_m V_m = \{0\}$. Condition (4) ensures that property (2) of Definition 1 holds; see [1, p. 141–142].

Therefore, Conditions (1)–(4) guarantee that we have a multiresolution analysis and a mother wavelet $\psi(x)$. Without loss of generality, we assume that $\hat{\phi}(0) = 1$. Condition (5) implies that if

$$\begin{aligned} q(x) &= \frac{1}{\sqrt{2\pi}} \int_E e^{-ix\omega} d\omega = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^N \int_{a_k}^{b_k} e^{-ix\omega} d\omega \\ &= \frac{1}{\sqrt{2\pi}} \sum_{k=1}^N \frac{e^{-ia_k x} - e^{-ib_k x}}{ix}, \end{aligned}$$

then

$$(8) \quad \lambda q(\lambda x) = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^N \frac{e^{-ia_k \lambda x} - e^{-ib_k \lambda x}}{ix},$$

which is clearly bounded for all x and λ . Since $\{\phi(x - k)\}$ is a Riesz basis of V_0 , it has a biorthonormal basis $\{\phi_k^*(x)\}$ such that for any $f \in V_0$ we have

$$f(x) = \sum_k \langle f, \phi_k^* \rangle \phi_{0,k}(x),$$

in the sense of $L^2(\mathcal{R})$, where $\phi_{0,k}(x) = \phi(x - k)$. Similarly, for any $f \in V_m$ and fixed m , we have

$$f(x) = \sum_k \langle f, \phi_{m,k}^* \rangle \phi_{m,k}(x),$$

where $\{\phi_{m,k}^*(x)\}_{k=-\infty}^{\infty}$ is the biorthonormal basis of $\{\phi_{m,k}(x)\}_{k=-\infty}^{\infty}$. From the relation

$$\langle \phi_n, \phi_k^* \rangle = \delta_{k,n} = \int_{-\infty}^{\infty} \phi(x - n) \overline{\phi_k^*(x)} dx$$

we obtain by a change of variable that

$$\langle \phi_n, \phi_k^* \rangle = \delta_{k,n} = 2^m \int_{-\infty}^{\infty} \phi(2^m x - n) \overline{\phi_k^*(2^m x)} dx.$$

It follows that $\{2^{m/2}\phi_k^*(2^m x)\}_{k=-\infty}^{\infty}$ is the biorthonormal basis of $\{\phi_{m,k}(x)\}_{k=-\infty}^{\infty}$ for each m .

We recall the following version of the second mean-value theorem: If f is increasing and h is continuous on $J = [a, b]$, then there exists $c \in J$ such that

$$\int_a^b f(x)h(x) dx = f(a) \int_a^c h(x) dx + f(b) \int_c^b h(x) dx.$$

Now we can state and prove our main theorem.

Theorem 1. Let $\phi \in \mathcal{A}$, $\phi_k^*(x) = O(k^{-\alpha})$ as $k \rightarrow \infty$ for some $\alpha > 0$, and $f \in L^2(\mathcal{R})$. Assume that either

- 1) $f \in L^1(\mathcal{R})$ and of bounded variation on every bounded interval, or
- 2) $f \in L^1_{loc}(\mathcal{R})$, of bounded variation in neighborhoods of $\pm\infty$ and tends to zero as $|x| \rightarrow \infty$,

then the wavelet series

$$f_m(x) = \sum_k \langle f, \phi_{m,k}^* \rangle \phi_{m,k}(x)$$

converges to $f(x)$ as $m \rightarrow \infty$ at every point of continuity of $f(x)$.

The result is best possible in the sense that the continuity of a function at a point is not sufficient to ensure the convergence of its wavelet expansion.

Proof. From Eq. (5), it is clear that the partial sum $\sum_{j < m,k} a_{j,k} \psi_{j,k}(x)$ of the wavelet expansion of f is the same as the projection $f_m(x)$ of f on V_m . Therefore, it suffices to study the convergence of $f_m(x)$ as $m \rightarrow \infty$.

The space V_0 is a reproducing-kernel Hilbert space since point evaluation is continuous:

$$|f(x)| = \left| \frac{1}{\sqrt{2\pi}} \int_E \hat{f}(\omega) e^{-ix\omega} d\omega \right| \leq C \|\hat{f}\| = C \|f\|.$$

Therefore, it has a reproducing kernel $k(t, x)$, which is easily seen to be a convolution kernel. For, if $f \in V_0$, then

$$(9) \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_E \hat{f}(\omega) e^{-ix\omega} d\omega = \int_{-\infty}^{\infty} f(t) k(t, x) dt,$$

where

$$k(t, x) = \frac{1}{2\pi} \int_E e^{i\omega(t-x)} d\omega = k(t-x).$$

The last integral in (9) is absolutely convergent by the Cauchy-Schwarz inequality because both f and k are in $L^2(\mathcal{R})$. The reproducing kernel has also the representation $k(t, x) = \sum_n \phi_n^*(t) \phi(x-n)$, where the series converges in the sense of L^2 . But from the hypothesis, the series also converges absolutely and uniformly for all x and t to a function $q_0(t, x)$. Thus, by standard arguments, $q_0(t, x) = k(t-x)$ almost everywhere. One can also show directly that if $f \in V_0$, then

$$\begin{aligned} \int_{-\infty}^{\infty} f(t) q_0(t, x) dt &= \sum_n \left(\int_{-\infty}^{\infty} f(t) \phi_n^*(t) dt \right) \phi(x-n) \\ &= \sum_n \langle f, \phi_n^* \rangle \phi(x-n) = f(x), \end{aligned}$$

since strong convergence implies weak convergence and, in addition, strong convergence in a reproducing-kernel Hilbert space implies pointwise convergence, as well as uniform convergence on any set on which $k(x, x)$ is uniformly bounded.

The projection of $f \in L^2(\mathcal{R})$ on V_0 is

$$f_0(x) = \int_{-\infty}^{\infty} f(t)q_0(t, x) dt,$$

and similarly by Property 1) of Definition 1, we have that the projection of $f \in L^2(\mathcal{R})$ on the space V_m is given by

$$f_m(x) = \int_{-\infty}^{\infty} f(t)q_m(t, x) dt,$$

where

$$q_m(t, x) = \sum_n \phi_{m,n}^*(t)\phi_{m,n}(x) = 2^m \sum_n \phi_n^*(2^m t)\phi(2^m x - n).$$

Thus, $q_m(t, x) = 2^m q_0(2^m t, 2^m x)$. Setting $\lambda = 2^m$, we obtain

$$(10) \quad f_m(x) = f_\lambda(x) = \lambda \int_{-\infty}^{\infty} f(t)q_0(\lambda t, \lambda x) dt.$$

Furthermore, we have ([5, p. 25])

$$\chi_E(\omega) = \int_{-\infty}^{\infty} q_0(z)e^{-i\omega z} dz,$$

which implies, in view of Condition 4 of Definition 3, that

$$(11) \quad \lambda \int_{-\infty}^{\infty} q_0(\lambda(t-x)) dt = \int_{-\infty}^{\infty} q_0(z)dz = \chi_E(0) = 1.$$

From Equations (10) and (11), we obtain

$$\begin{aligned} f_\lambda(x) - f(x) &= \lambda \int_{-\infty}^{\infty} [f(t) - f(x)] q_0(\lambda(t-x)) dt \\ &= \lambda \int_{-\infty}^{\infty} [f(x+\eta) - f(x)] q_0(\lambda\eta) d\eta = I_1(x, \lambda) + I_2(x, \lambda), \end{aligned}$$

where

$$I_1(x, \lambda) = \int_{-\infty}^0, \quad \text{and} \quad I_2(x, \lambda) = \int_0^{\infty}.$$

We shall only show that $\lim_{\lambda \rightarrow \infty} I_2(x, \lambda) = 0$ since the proof for I_1 is similar.

Let $I_2 = I_3 + I_4 + I_5 - I_6$, where

$$I_3 = \lambda \int_0^\delta [f(x+\eta) - f(x)] q_0(\lambda\eta) d\eta,$$

$$I_4 = \lambda \int_\delta^A [f(x+\eta) - f(x)] q_0(\lambda\eta) d\eta,$$

$$I_5 = \lambda \int_A^\infty f(x+\eta)q_0(\lambda\eta) d\eta, \quad \text{and} \quad I_6 = \lambda \int_A^\infty f(x)q_0(\lambda\eta) d\eta,$$

where δ and A are to be determined later on. Let $\epsilon > 0$ be given. Since

$$I_6 = \lambda f(x) \int_A^\infty q_0(\lambda\eta) d\eta = f(x) \int_{\lambda A}^\infty q_0(y) dy,$$

and

$$(12) \quad \int_0^\infty q_0(y) dy < \infty,$$

it follows that by choosing λ large enough, we can make $|I_6| \leq \epsilon/6$. From the relation

$$q_0(y) = \frac{1}{2\pi} \int_E e^{i\omega y} d\omega,$$

it follows that

$$\lambda \int_a^b q_0(\lambda y) dy = \frac{1}{2i\pi} \int_{-\infty}^{\infty} e^{ia\omega\lambda} \left\{ \chi_E(\omega) \frac{e^{i\omega\lambda(b-a)} - 1}{\omega} \right\} d\omega,$$

for any finite numbers a, b . But since the function in the curly brackets is integrable, it follows that $|\lambda \int_a^b q_0(\lambda y) dy|$, or equivalently, $|\int_{a\lambda}^{b\lambda} q_0(z) dz|$ is bounded by, say L , for all λ . Moreover, by (12), we have

$$(13) \quad \lambda \int_a^b q_0(\lambda y) dy = \int_{a\lambda}^{b\lambda} q_0(z) dz \rightarrow 0$$

as $\lambda \rightarrow \infty$ for any $b > a > 0$.

If $f \in L^1(\mathcal{R})$, then in view of (8) we can choose A large enough so that

$$|I_5| \leq C \int_A^\infty |f(x + \eta)| d\eta \leq \epsilon/6.$$

Because f is assumed to be of bounded variation, we can assume, without loss of generality, that it is monotone increasing, and set $g(\eta) = f(x + \eta) - f(x)$. Clearly, g is monotone increasing with $g(0) = 0$. And since q_0 is continuous, being the Fourier transform of an integrable function χ_E , it follows from the second mean-value theorem

$$I_3 = g(\delta) \int_{c\lambda}^{\delta\lambda} q_0(y) dy,$$

for some c with $0 < c < \delta$. By choosing δ small enough so that $|g(\delta)| \leq \epsilon/(3L)$, we obtain that $|I_3| \leq \epsilon/3$.

Having determined δ and A we can now estimate I_4 . By applying the second mean-value theorem once more, we obtain

$$(14) \quad I_4 = \lambda \left\{ [f(x + \delta) - f(x)] \int_\delta^c q_0(\lambda\eta) d\eta + [f(x + A) - f(x)] \int_c^A q_0(\lambda\eta) d\eta \right\},$$

for some $\delta < c < A$. Thus, by taking the limit of (14) as $\lambda \rightarrow \infty$, we obtain by (13) that $I_4 \rightarrow 0$ as $\lambda \rightarrow \infty$, and this completes the proof for case 1.

In case 2, $f \in L^1_{loc}(\mathcal{R})$ but is of bounded variation in a neighborhood of $\pm\infty$, and tends to zero as $|x| \rightarrow \infty$. Therefore, we write $I_2 = I_3 + I_8$, where I_3 is defined as before and

$$I_8 = \lambda \int_\delta^\infty [f(x + \eta) - f(x)] q_0(\lambda\eta) d\eta,$$

which we may write as

$$I_8 = \lim_{N \rightarrow \infty} I_8^N = \lim_{N \rightarrow \infty} \lambda \int_\delta^N [f(x + \eta) - f(x)] q_0(\lambda\eta) d\eta.$$

Showing that I_3 can be made arbitrarily small is the same as before. As for I_8 , we have by the the second mean-value theorem once more that

$$I_8^N = \lambda \left\{ [f(x + \delta) - f(x)] \int_\delta^B q_0(\lambda\eta) d\eta + [f(x + N) - f(x)] \int_B^N q_0(\lambda\eta) d\eta \right\}$$

for some $\delta < B < N$. Thus

$$I_8^N = [f(x + \delta) - f(x)] \int_{\lambda\delta}^{\lambda B} q_0(y) dy + [f(x + N) - f(x)] \int_{\lambda B}^{\lambda N} q_0(y) dy.$$

Taking the limits as $N \rightarrow \infty$ yields

$$I_8 = [f(x + \delta) - f(x)] \int_{\lambda\delta}^{\lambda B} q_0(y) dy - f(x) \int_{\lambda B}^{\infty} q_0(y) dy,$$

which, after taking the limits as $\lambda \rightarrow \infty$, yields that $I_8 \rightarrow 0$, and this completes the proof of part (2).

To show that the continuity of a function at a point is not sufficient to ensure the pointwise convergence of its wavelet expansion, one needs only to observe that for Shannon's wavelet

$$f_m(x) = f_\lambda(x) = \int_{-\infty}^{\infty} f(t) \frac{\sin \lambda\pi(x-t)}{\pi(x-t)} dt,$$

which is Fourier's integral formula. It is well known that the mere continuity of a function at a point is not sufficient for the convergence of that integral; see [5]. \square

If ϕ generates an orthogonal basis, then the condition $\phi_k^*(x) = O(k^{-\alpha})$ is not needed since in this case $\phi_k(x) = \phi_k^*(x) = O(k^{-1})$.

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