ONE-STEP EXTENSION OF THE BERGMAN SHIFT

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Abstract. In this paper we answer a question of Curto and Fialkow: there exists a quadratically hyponormal weighted shift which is not positively quadratically hyponormal.

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be normal if $T^*T = TT^*$, hyponormal if $T^*T \geq TT^*$, and subnormal if $T = N|_{\mathcal{H}}$, where $N$ is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. If $T$ is subnormal, then $T$ is also hyponormal. Recall that given a bounded sequence of positive numbers $\alpha_0,\alpha_1,\ldots$ (called weights), the (unilateral) weighted shift $W_\alpha$ associated with $\alpha$ is the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for $\ell^2$. It is straightforward to check that $W_\alpha$ can never be normal, and that $W_\alpha$ is hyponormal if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n \geq 0$.

Recall the Bram-Halmos criterion for subnormality, which states that an operator $T \in \mathcal{L}(\mathcal{H})$ is subnormal if and only if

$$\sum_{i,j} (T^ix_j, T^jx_i) \geq 0$$

for all finite collections $x_0, x_1, \ldots, x_k \in \mathcal{H}$ ([1], II.1.9]). Using the Choleski algorithm for operator matrices, it is easy to see that this is equivalent to the positivity of the matrices $(T^iT^j - T^jT^i)_{i,j=1}^k$ for $k = 1, 2, \ldots$. If we denote by $[A,B] := AB - BA$ the commutator of two operators $A$ and $B$, and if we define $T$ to be $k$-hyponormal whenever the $k \times k$ operator matrix $M_k(T) := ([T^iT^j])_{i,j=1}^k$ is positive, then the Bram-Halmos criterion can be rephrased as saying that $T$ is subnormal if and only if $T$ is $k$-hyponormal for every $k \geq 1$ ([7]). Recall ([3], [4]) that $T \in \mathcal{L}(\mathcal{H})$ is weakly $k$-hyponormal if $\sum_{i=0}^k s_iT^i$ is hyponormal for every complex number $s_i$ ($0 \leq i \leq k$). If $k = 2$, then it is said to be quadratically hyponormal. It is known that 2-hyponormal $\Rightarrow$ quadratically hyponormal. In [3, Proposition 7], it is shown that there exists a quadratically hyponormal weighted shift which is not 2-hyponormal.
Let $W_\alpha$ be a hyponormal weighted shift. We write $D(s) := [(W_\alpha + sW_\alpha^2)^*, W_\alpha + sW_\alpha^2]$ for $s \in \mathbb{C}$, and we let

$$D_n(s) := P_n[(W_\alpha + sW_\alpha^2)^*, W_\alpha + sW_\alpha^2]P_n,$$

where $P_n$ is the orthogonal projection onto the subspace generated by $\{e_0, \ldots, e_n\}$. Then $D_n(s)$ is of the form

$$D_n(s) = \begin{pmatrix} q_0 & r_0 & 0 & \cdots & 0 \\ r_0 & q_1 & r_1 & \cdots & 0 \\ 0 & r_1 & q_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & q_{n+1} \\ 0 & 0 & 0 & \cdots & r_{n+1} & q_n \end{pmatrix},$$

where

$$q_n := u_n + |s|^2v_n,$$

$$r_n := s\sqrt{w_n},$$

$$u_n := \alpha_n^2 - \alpha_{n-1}^2,$$

$$v_n := \alpha_n^2\alpha_{n+1}^2 - \alpha_{n-1}^2\alpha_{n-2}^2,$$

$$w_n := \alpha_n^2(\alpha_{n+1}^2 - \alpha_{n-1}^2)^2,$$

and, for notational convenience, $\alpha_{-2} = \alpha_{-1} = 0$. Clearly, $W_\alpha$ is quadratically hyponormal if and only if $D_n(s) \geq 0$ for every $s \in \mathbb{C}$ and every $n \geq 0$. Let $d_n(\cdot) := \det(D_n(\cdot))$. Then $d_n$ satisfies the following 2-step recursive formula:

$$d_0 = q_0, \quad d_1 = q_0q_1 - |r_0|^2, \quad d_{n+2} = q_{n+2}d_{n+1} - |r_{n+1}|^2d_n,$$

if we let $t := |s|^2$, we observe that $d_n$ is a polynomial in $t$ of degree $n + 1$, and if we write $d_n = \sum_{i=0}^{n+1} c(n, i)t^i$, then the Maclaurin coefficients $c(n, i)$ satisfy a double-indexed recursive formula; namely

$$c(n+2, i) = u_{n+2}c(n+1, i) + v_{n+2}c(n+1, i-1) - w_{n+1}c(n, i-1),$$

$$c(n, 0) = u_0 \cdots u_n, \quad c(n, n+1) = v_0 \cdots v_n, \quad c(1, 1) = u_1v_0 + v_1u_0 - w_0$$

$(n \geq 0, i \geq 1)$.

We begin with:

**Definition 1** ([4, 5, 6]). Let $\alpha : \alpha_0, \alpha_1, \cdots$ be a weight sequence, let $W_\alpha$ be the corresponding weighted shift, and let $c(n, i)$ be the Maclaurin coefficients of the polynomial $d_n$. We say that $W_\alpha$ is **positively quadratically hyponormal** if $c(n, i) \geq 0$ for all $n, i \geq 0$ with $0 \leq i \leq n + 1$, and $c(n, n + 1) > 0$ for all $n \geq 0$.

Clearly, positively quadratically hyponormal $\implies$ quadratically hyponormal. In 1994, Curto and Fialkow ([4 Problem 4.7]) asked if the converse is true: if $W_\alpha$ is a quadratically hyponormal weighted shift, does it follow that $W_\alpha$ is positively quadratically hyponormal? In this paper we answer it negatively.

If the weight sequence $\alpha = \{\alpha_n\}_{n=0}^\infty$ is given by

$$\alpha_n = \sqrt{\frac{n+1}{n+2}} \quad (n \geq 0),$$

then the corresponding weighted shift is called the **Bergman shift**. It is well known that the Bergman shift is subnormal.
The following is an one-step extension of the Bergman shift.

**Theorem 2.** For \( x > 0 \), let \( T_x \) be the weighted shift whose weight sequence is given by

\[
\alpha_0 = \sqrt{x}, \quad \alpha_n = \sqrt{\frac{n}{n+1}} \quad (n \geq 1).
\]

Then we have:

(a) \( T_x \) is positively quadratically hyponormal \( \iff 0 < x \leq \frac{22}{17} \).

(b) \( 0 < x \leq \frac{71}{154} \implies T_x \) is quadratically hyponormal.

(c) \( T_x \) is not quadratically hyponormal for \( x = \frac{1}{2} \).

(d) \( T_x \) is 2-hyponormal \( \iff 0 < x \leq \frac{7}{15} \).

(e) \( T_x \) is never subnormal for any \( x > 0 \).

**Proof.** (a) We use an idea of Curto ([3, Proposition 7]). Suppose \( T_x \) is hyponormal and hence \( 0 < x \leq \frac{12}{25} \). Write \( d_n(t) = \sum_{i=0}^{n+1} c(n, i) t^i \). From (0.2) we can check directly that

\[
\begin{align*}
(c(0,0)) &= x, \\
(c(0,1)) &= \frac{1}{2}x, \\
(c(1,0)) &= x(\frac{1}{2} - x), \\
(c(1,1)) &= x(\frac{1}{4} - \frac{1}{2}x), \\
(c(1,2)) &= \frac{1}{4}x,
\end{align*}
\]

and

\[
\begin{align*}
(c(2,0)) &= \frac{x}{16} (\frac{1}{2} - x), \\
(c(2,1)) &= \frac{x}{6} (\frac{1}{2} - x), \\
(c(2,2)) &= \frac{x}{12} (1 - x), \\
(c(2,3)) &= \frac{x}{3} (1 - x), \\
(c(3,0)) &= \frac{x}{24} (\frac{1}{2} - x), \\
(c(3,1)) &= \frac{x}{16} (\frac{1}{2} - x), \\
(c(3,2)) &= \frac{x}{12} (\frac{1}{2} - x), \\
(c(3,3)) &= \frac{x}{6} (16 - 21x), \\
(c(3,4)) &= \frac{x}{3} (1 - x),
\end{align*}
\]

and

\[
\begin{align*}
(c(4,0)) &= \frac{x}{1440} (\frac{1}{2} - x), \\
(c(4,1)) &= \frac{x}{1050} (\frac{1}{2} - x), \\
(c(4,2)) &= \frac{x}{720} (\frac{1}{2} - x), \\
(c(4,3)) &= \frac{x}{60} (22 - 47x), \\
(c(4,4)) &= x (\frac{1}{270} - \frac{7}{1440}x), \\
(c(4,5)) &= \frac{x}{240} (1 - x).
\end{align*}
\]

Observe

\[ c(n, i) \geq 0 \quad \text{for all } 0 \leq n, i \leq 3 \text{ with } 0 \leq i \leq n + 1. \]

Note that

\[ c(n, 0) = u_0 \cdots u_n \geq 0 \quad \text{for all } n \geq 0 \]

and

\[ c(n, 1) = u_0 \cdots u_{n-1} \alpha_n^2 (\alpha_{n+1}^2 - \alpha_{n-1}^2) \geq 0 \quad (n \geq 2). \]
Proof of Claim I. Since for $n \geq 2$,
\[
v_{n+2}c(n+1,1) - w_{n+1}c(n,1) \\
= v_{n+2}u_0 \cdots u_n \alpha_{n+1}^2 (\alpha_{n+2}^2 - \alpha_n^2) - w_{n+1}u_0 \cdots u_{n-1} \alpha_{n+1}^2 (\alpha_{n+2}^2 - \alpha_{n-1}^2) \\
(2.4) \\
= u_0 \cdots u_{n-1} \frac{24}{(n+1)^2(n+2)^2(n+3)^2(n+4)} \geq 0,
\]
it follows that if $c(n+1,2) \geq 0$, then for $n \geq 2$,
\[
c(n+2,2) = u_{n+2}c(n+1,2) + v_{n+2}c(n+1,1) - w_{n+1}c(n,1) \geq 0.
\]
But since $c(n,2) \geq 0$ for $n = 1, 2, 3$, the above argument gives that $c(n,2) \geq 0$ for all $n \geq 1$. This proves Claim I.

Claim II. For $n \geq 4$, $i \geq 4$,
\[
c(n,i) = v_n c(n-1,i-1).
\]
Proof of Claim II. First of all we prove that for $n \geq 4$, $i \geq 1$,
\[
c(n,i) = v_n c(n-1,i-1) + u_n \cdots u_4 h_i \quad \text{with} \quad h_i := u_3c(2,i) - w_2c(1,i-1).
\]
A simple calculation shows that $u_{n+1}v_n = w_n$ for all $n \geq 3$. Thus if $n = 4$,
\[
c(4,i) = u_4c(3,i) + v_4c(3,i-1) - w_3c(2,i-1) \\
= v_4c(3,i-1) + u_4 \left( u_3c(2,i) + v_3c(2,i-1) - w_2c(1,i-1) \right) - w_3c(2,i-1) \\
= v_4c(3,i-1) + u_4 \left( u_3c(2,i) - w_2c(1,i-1) \right) + (u_4v_3 - w_3)c(2,i-1) \\
= v_4c(3,i-1) + u_4h_i
\]
and a similar calculation works for the inductive step. Now
\[
h_1 = \frac{1}{36}x(x - \frac{1}{2}), \quad h_2 = \frac{1}{72}x(x - \frac{1}{2}), \quad h_3 = -\frac{1}{144}x^2, \quad \text{and} \quad h_i = 0 \quad \text{for} \quad i \geq 4,
\]
which together with (2.5) proves Claim II.

Claim III. If $c(n,3) \geq 0$, then $c(n+1,3) \geq 0$ for $n \geq 4$.
Proof of Claim III. Since for $n \geq 4$,
\[
v_{n+1}c(n,2) - w_n c(n-1,2) \\
= v_{n+1} \left( u_n c(n-1,2) + v_n c(n-1,1) - w_{n-1}c(n-2,1) \right) - w_n c(n-1,2) \\
= (v_{n+1}u_n - w_n) c(n-1,2) + v_{n+1} \left( v_n c(n-1,1) - w_{n-1}c(n-2,1) \right) \\
= c(n-1,2) \cdot \frac{4}{n(n+1)^2(n+2)^2(n+3)} + v_{n+1}g_n \geq 0,
\]
where $g_n := v_n c(n-1,1) - w_{n-1}c(n-2,1) \geq 0$ by (2.4). Therefore if $c(n,3) \geq 0$,
then
\[ c(n+1, 3) = u_{n+1}c(n, 3) + v_{n+1}c(n, 2) - w_n c(n-1, 2) \geq 0, \]
which proves Claim III.

It now follows from (2.1), (2.2), (2.3), Claim I and Claim II that \( c(n, i) \geq 0 \) for all \( n, i \geq 0 \) with \( 0 \leq i \leq n + 1 \) if and only if \( c(n, 3) \geq 0 \) for all \( n \geq 4 \). Therefore by Claim III,
\[ c(n, i) \geq 0 \quad \text{for all } n, i \geq 0 \iff c(4, 3) \geq 0 \iff x \leq \frac{22}{47}. \]

This proves statement (a).

(b) In view of (a), it suffices to show that if \( \frac{22}{47} < x \leq \frac{71}{151} \), then \( T_x \) is quadratically hyponormal. Thus suppose \( \frac{22}{47} < x \leq \frac{71}{151} \). Then we have:

(i) \( c(n, i) \geq 0 \) for all \( 0 \leq n, i \leq 4 \) with \( 0 \leq i \leq n + 1 \) except for \( c(4, 3) \);
(ii) (2.2), (2.3), Claim I, Claim II, and Claim III all in the proof of (a) hold.

Observe that
\[
c(5, 3) = u_5 c(4, 3) + v_5 c(4, 2) - w_3 c(3, 2) \\
= \frac{1}{30} x \left( \frac{11}{4320} - \frac{47}{8640} x \right) + \frac{1}{4200} x \left( \frac{1}{2} - x \right) - \frac{11}{64800} x \left( \frac{1}{2} - x \right) \\
= \frac{1}{604800} x (72 - 151 x).
\]

Thus \( c(5, 3) \geq 0 \) since \( x \leq \frac{71}{71} \). Therefore by Claim III,
\[ c(n, 3) \geq 0 \quad \text{for } n \geq 5 \quad \text{and} \quad 0 < x \leq \frac{71}{151}. \]

Thus by Claim II, we have
\[
(2.6) \quad c(n, i) \geq 0 \quad \text{for all } n, i \geq 0 \quad \text{except for } c(n, n-1) \quad (n \geq 4).
\]

Note that \( c(4, 3) < 0 \). Again by Claim II,
\[
(2.7) \quad c(n, n-1) < 0 \quad \text{for all } n \geq 4.
\]

Note that \( d_n(t) \geq 0 \) for \( n = 0, 1, 2, 3 \). Observe by Claim II that if \( n \geq 6 \), then
\[
c(n, n-2)t^{n-2} + c(n, n-1)t^{n-1} + c(n, n)t^n \\
= v_n \cdots v_6 t^{n-5} \left( c(5, 3)t^3 + c(5, 4) t^4 + c(5, 5) t^5 \right).
\]

Thus if \( c(5, 3)t^3 + c(5, 4) t^4 + c(5, 5) t^5 \geq 0 \) for every \( t \geq 0 \), then \( d_n(t) \geq 0 \) for every \( n \geq 6 \) and every \( t \geq 0 \) because other Maclaurin coefficients are nonnegative. Thus it will suffice to show that if \( \frac{22}{47} < x \leq \frac{71}{151} \), then
\[
(2.8) \quad c(n, n-2)t^{n-2} + c(n, n-1)t^{n-1} + c(n, n)t^n \geq 0 \quad \text{for } n = 4, 5
\]
(this also implies that \( d_n(t) \geq 0 \) for \( n = 4, 5 \)).
Claim IV. If \( \frac{22}{47} < x \leq \frac{71}{151} \), then
\[ c(4,2)t^2 + c(4,3)t^3 + c(4,4)t^4 \geq 0 \quad \text{for every } t \geq 0. \]

Proof of Claim IV. There are two cases to consider.

Case 1 (\( 0 < t \leq \frac{7}{24} \)). From (0.2) we have
\[ u_5c(4,3) + v_5c(4,2) = c(5,3) + w_4c(3,2). \]
Since by (2.6), \( c(5,3) + w_4c(3,2) \geq 0 \) we have
\[ u_5c(4,3) + v_5c(4,2) \geq 0, \]
so that
\[ c(4,2) + \frac{7}{24}c(4,3) \geq 0. \]
Since \( c(4,2) \geq 0 \) and \( c(4,3) < 0 \), it follows that if \( 0 < t \leq \frac{7}{24} \), then \( c(4,2) + c(4,3)t \geq 0. \) Since \( c(4,4) \geq 0 \) we have that \( c(4,2)t^2 + c(4,3)t^3 + c(4,4)t^4 \geq 0. \)

Case 2 (\( t \geq \frac{7}{24} \)). From (0.2) we have
\[ u_5c(4,4) + v_5c(4,3) = c(5,4) + w_4c(3,3). \]
A straightforward calculation shows that if \( x \leq \frac{71}{151} \), then
\[ c(5,4) + w_4c(3,3) = \frac{11}{37800}x \left( 1 - \frac{47}{22}x \right) + \frac{1}{180}x \left( \frac{1}{45} - \frac{7}{240}x \right) \]
\[ = \frac{47}{113400}x \left( 1 - \frac{71}{376}x \right) \geq 0. \]
Thus \( u_5c(4,4) + v_5c(4,3) \geq 0 \), so that \( c(4,3) + \frac{7}{24}c(4,4) \geq 0 \), which implies that if \( t \geq \frac{7}{24} \), then \( c(4,3) + c(4,4)t \geq 0. \) Since \( c(4,2) \geq 0 \) we have that \( c(4,2)t^2 + c(4,3)t^3 + c(4,4)t^4 \geq 0. \)

Claim V. If \( \frac{22}{47} < x \leq \frac{71}{151} \), then
\[ c(5,3)t^3 + c(5,4)t^4 + c(5,5)t^5 \geq 0 \quad \text{for every } t \geq 0. \]

Proof of Claim V. From (0.2) we have
\[ (2.9) \begin{cases} u_6c(5,4) + v_6c(5,3) = c(6,4) + w_5c(4,3), \\ u_6c(5,5) + v_6c(5,4) = c(6,5) + w_5c(4,4). \end{cases} \]
By the same argument as in Claim IV, it suffices to show that if \( x \leq \frac{71}{151} \), then
\[ (2.10) \quad u_6c(5,4) + v_6c(5,3) \geq 0 \quad \text{and} \quad u_6c(5,5) + v_6c(5,4) \geq 0. \]
Indeed, a straightforward calculation shows that if \( x \leq \frac{71}{151} \), then
\[ c(6,4) + w_5c(4,3) = x \left( \frac{1}{100800} - \frac{151}{7257600}x \right) + \frac{2}{735}x \left( \frac{11}{4320} - \frac{47}{8640}x \right) \]
\[ = \frac{107}{6350400}x \left( 1 - \frac{1809}{856}x \right) \geq 0. \]

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and
\[ c(6, 5) + w_5c(4, 4) = x \left( \frac{11}{453600} - \frac{47}{907200}x \right) + \frac{2}{735}x \left( \frac{1}{270} - \frac{7}{1440}x \right) \]
\[ = \frac{109}{3175200}x \left( 1 - \frac{413}{218}x \right) \geq 0, \]
which together with (2.9) proves (2.10) and hence Claim V. Now Claim IV and Claim V prove (2.8); therefore if \( 0 < x \leq \frac{71}{151} \), then \( T_x \) is quadratically hyponormal.

(c) Observe that if \( x = \frac{1}{2} \), then
\[ d_4(t) = \frac{1}{1080}t^5 + \frac{11}{17280}t^4 - \frac{1}{11520}t^3, \]
so that
\[ \lim_{t \to 0^+} \frac{d_4(t)}{t^3} = -\frac{1}{11520} < 0, \]
which implies that \( T_x \) is not quadratically hyponormal.

(d) Remember ([3, Corollary 5]) that if \( W_\alpha \) is the weighted shift with weights \( \alpha = \{\alpha_n\}_{n=0}^\infty \), then \( W_\alpha \) is 2-hyponormal if and only if
\[ \alpha_{n+1}^2(\alpha_{n+2}^2 - \alpha_n^2)^2 \leq (\alpha_{n+1}^2 - \alpha_n^2)(\alpha_{n+2}^2\alpha_{n+3}^2 - \alpha_n^2\alpha_{n+1}^2) \quad (n \geq 0). \]
Thus \( T_x \) is 2-hyponormal if and only if
\[ \alpha_1^2(\alpha_2^2 - x^2)^2 \leq (\alpha_1^2 - x)(\alpha_2^2\alpha_3^2 - x \alpha_1^2); \]
that is,
\[ \frac{1}{2}(2x - x)^2 \leq \frac{1}{2} - x(\frac{1}{2} - \frac{1}{2}x), \]
or equivalently, \( 0 < x \leq \frac{5}{3} \).

(e) Let \( W_\alpha \) be the weighted shift with weights \( \alpha = \{\alpha_n\}_{n=0}^\infty \) and let \( \beta_n := \alpha_0^2 \cdots \alpha_{n-1}^2 \) for \( n \geq 1 \). Then C. Berger’s characterization of subnormality for unilateral weighted shifts (cf. [2, III.8.16]) states that \( W_\alpha \) is subnormal if and only if there exists a Borel probability measure \( \mu \) supported on \( [0, ||W_\alpha||^2] \), with \( ||W_\alpha||^2 \in \text{supp} \mu \) such that
\[ \beta_n = \int_0^{||W_\alpha||^2} t^n \, d\mu(t) \quad \text{for all } n \geq 1. \]
By an argument of Curto ([3, Proposition 8]), if \( W_\alpha \) is a weighted shift whose restriction to \( \{e_1, e_2, \cdots\} \) is subnormal, with associated measure \( \mu \), then \( W_\alpha \) is subnormal if and only if
(i) \( \frac{1}{n} \in L^1(\mu); \)
(ii) \( \alpha_0^2 \leq \left( \frac{1}{n} \right)^{-1}. \)
A straightforward calculation shows that the Bergman shift has measure \( d\mu = dt \). Indeed, \( \beta_n = \frac{1}{n+1} = \int_0^1 t^n \, d\mu(t) \) has a solution \( d\mu = dt \). Thus \( \frac{1}{n} \) is not integrable with respect to \( \mu \). This implies that \( T_x \) is never subnormal for any \( x > 0 \).

**Corollary 3.** Let \( T_x \) be as in Theorem 2. If \( \frac{22}{47} < x \leq \frac{71}{151} \), then \( T_x \) is quadratically hyponormal and not positively quadratically hyponormal.
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