PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 12, Pages 3601–3608
S 0002-9939(00)05519-2
Article electronically published on June 7, 2000

PYRAMIDAL VECTORS AND SMOOTH FUNCTIONS ON BANACH SPACES

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(Communicated by Dale Alspach)

ABSTRACT. We prove that if X, Y are Banach spaces such that Y has non-trivial cotype and X has trivial cotype, then smooth functions from X into Y have a kind of "harmonic" behaviour. More precisely, we show that if Ω is a bounded open subset of X and $f : \overline{\Omega} \to Y$ is C^1-smooth with uniformly continuous Fréchet derivative, then $f(\partial\Omega)$ is dense in $f(\overline{\Omega})$. We also give a short proof of a recent result of P. Hájek.

This note is motivated by recent results of P. Hájek ([H1], [H2]) concerning (Fréchet) smooth nonlinear operators on the space c_0.

In [H1], Hájek proved (among other things) that if $f : c_0 \to \mathbb{R}$ is a C^1-smooth map with uniformly continuous derivative on B_{c_0}, then $f'(B_{c_0})$ is a relatively compact subset of l_1. From this, he deduced that if Y is a Banach space with non trivial type and $f : c_0 \to Y$ is C^1-smooth with locally uniformly continuous derivative, then f is locally compact, which means that each point $x \in c_0$ has a neighbourhood V such that $f(V)$ is relatively compact in Y. In [H2], he also proved that the same is true if Y has an unconditional basis and does not contain c_0. These striking results are to be compared with another recent theorem, due to S. M. Bates ([B]), according to which for *any* separable Banach space Y there exists a C^1-smooth surjection from c_0 onto Y; clearly, such a map cannot be locally compact unless Y is finite-dimensional, by the Baire category theorem.

This note is a by-product of several vain attempts to generalize Hájek's local compactness results to all Banach spaces Y not containing c_0.

We show that if X, Y are Banach spaces such that Y has finite cotype and X does not have finite cotype, then smooth functions from X into Y have a kind of "harmonic" behaviour (Theorem 1). We also prove that if Y has finite cotype, then smooth functions from c_0 into Y essentially turn weakly convergent sequences into (norm) Cesaro-convergent sequences (Theorem 2). Both results rest on an elementary finite-dimensional lemma (Lemma 1) involving what we have called *pyramidal vectors* of c_0 (Definition 1). Finally, we give a very short proof of Hájek's basic result for scalar-valued functions, which looks rather different (at least in its form) from the original one. This proof is based on the notion of *strong sequential continuity* (Definition 3), which might be of independent interest.

Let us now fix the notation that will be used throughout this note. The letters X, Y will always designate (real) Banach spaces. If Z is a normed space, we denote

Received by the editors February 19, 1999.
2000 *Mathematics Subject Classification.* Primary 46B20.

by B_Z the closed unit ball of Z. For any set $B \subseteq Z$, we denote by $C^1_u(B,Y)$ the set of all maps $f : Z \to Y$ which are C^1-smooth on some neighbourhood of B with uniformly continuous derivative on B. If ω is a modulus of continuity, we put $C^{1,\omega}(B,Y) = \{f \in C^1_u(B,Y);\ \forall u,v \in B\ ||f'(u) - f'(v)|| \leq \omega(||u-v||)\}$; and if $M > 0$, we let $C^{1,\omega,M}(B,Y) = \{f \in C^{1,\omega}(B,Y); ||f'(x)|| \leq M \text{ on } B\}$. Finally, we denote by $(e_i)_{i\geq 0}$ the canonical basis of c_0, and by c_{00} the linear span of the e_i's; and if N is a positive integer, we put $l^N_\infty = \text{span}\{e_i; 0 \leq i \leq N-1\}$.

Definition 1. Let K be a positive integer. We say that $a \in c_0$ is a *K-pyramidal vector* if one can write $a = \sum_{i=1}^{r} \lambda_i \mathbf{1}_{A_i}$, where $r \leq K$, $(A_1, \dots A_r)$ is a decreasing sequence of (nonempty) finite intervals of $\mathbb{N}$ such that $\min A_i < \min A_{i+1}$ if $i \leq r-1$, and $\lambda_1, \dots, \lambda_r \in]0;1/K]$.

Recall that a Banach space Z is said to have *cotype q* ($2 \leq q \leq \infty$) if there is a numerical constant C such that $||\mathbf{z}||_{l_q(Z)} \leq C\, ||\sum \varepsilon_i z_i||_{L_q(Z)}$ for all finite sequences $\mathbf{z} = (z_1, \dots z_n) \subseteq Z$ (where (ε_i) is the sequence of Rademacher functions).

Lemma 1. *Assume that Y has finite cotype q. Let ω be a modulus of continuity and let $M > 0$. Given a positive integer K and $\varepsilon > 0$, there exists a positive integer N satisfying the following property: for any open set $V \subseteq l^N_\infty$ such that $0 \in V \subseteq B_{l^N_\infty}$ and for any $f \in C^{1,\omega,M}(\overline{V}, Y)$, one can find a K-pyramidal vector a such that*

$$a \in \partial V \ \text{ and } \ ||f(a) - f(0)|| < \varepsilon + \omega(1/K).$$

Proof. We will use the hypothesis on Y in the following way: *for any bounded linear operator $T : c_0 \to Y$ and any positive number α, the number of integers i such that $||Te_i|| \geq \alpha$ does not exceed $C_q^q \dfrac{||T||^q}{\alpha^q}$, where C_q is the cotype constant of Y.* This is easy to check.

Fix a positive integer K and $\varepsilon > 0$.

We choose a sequence of positive integers $(N_0, \dots, N_K)$ such that for $i \leq K-1$, N_i is "much greater" than N_{i+1}. More precisely, the N_i's are selected in such a way that

$$\frac{N_0 - C\, N_1^q/\varepsilon^q}{1 + C\, N_1^q/\varepsilon^q} \geq 1 + N_1\,,\ \dots,\ \frac{N_{K-1} - C\, N_K^q/\varepsilon^q}{1 + C\, N_K^q/\varepsilon^q} \geq 1 + N_K\,,$$

where $C = C_q^q M^q$. Finally, we put $N = N_0$.

Now, fix an open set $V \subseteq l^N_\infty$ such that $0 \in V \subseteq B_{l^N_\infty}$ and a function $f \in C^{1,\omega,M}(\overline{V}, Y)$.

Let us say that a decreasing sequence $(A_0, \dots A_p)$ $(0 \leq p \leq K)$ of subintervals of $[0;N[$ is *admissible* if the following properties are satisfied:

(i) $A_0 = [0;N[$ and each A_i has cardinality N_i.

(ii) For all $i \geq 1$ $[a_{i-1}; a_i] \subseteq V$, where $a_0 = 0$ and $a_l = \dfrac{1}{K} \sum_{j=1}^{l} \mathbf{1}_{A_j}$ if $l \geq 1$.

(iii) If $i \geq 1$, then $||f'(a_{i-1}).\, e_l|| < \dfrac{\varepsilon}{N_i}$ for all $l \in A_i$.

Notice that $(A_0) = ([0;N[)$ is admissible, and that there is no admissible sequence of length $K+1$, because $V \subseteq B_{l^N_\infty}$.

Let $(A_0, \dots, A_p)$ be an admissible sequence of maximal length. Then $p \leq K-1$; hence, by the choice of the sequence $(N_0, \dots, N_K)$, it is possible to find an interval

$A_{p+1} \subseteq A_p$ of cardinality N_{p+1} such that $\min A_{p+1} > \min A_p$ and

$$\forall l \in A_{p+1} \quad ||f'(a_p).\, e_l|| < \varepsilon / N_{p+1} \ .$$

Since $(A_0, \dots A_{p+1})$ cannot be admissible, this implies that the segment $I = [a_p; a_p + \frac{1}{K}\, \mathbf{1}_{A_{p+1}}]$ is not contained in V; and since $a_p \in V$, I must intersect ∂V.

Let $\lambda = \min\{t \geq 0;\ a_p + t\, \mathbf{1}_{A_{p+1}} \in \partial V\}$, and put $a = a_{p+1} = a_p + \lambda\, \mathbf{1}_{A_{p+1}}$.

Since $\lambda \in]0; 1/K]$, a is a K-pyramidal vector, and of course $a \in \partial V$.

Moreover, if we put $h_i = a_i - a_{i-1}$ $(1 \leq i \leq p+1)$, then $||f'(a_{i-1}).\, h_i|| < \varepsilon / K$ and $||h_i|| \leq 1/K$ for all i, whence

$$\begin{aligned} ||f(a_i) - f(a_{i-1})|| &\leq ||f'(a_{i-1}).\, h_i|| + ||h_i||\, \omega(||h_i||) \\ &< \varepsilon/K + 1/K\, \omega(1/K)\ (1 \leq i \leq p+1). \end{aligned}$$

Therefore, we get

$$\begin{aligned} ||f(a) - f(0)|| &\leq \sum_{i=1}^{p+1} ||f(a_i) - f(a_{i-1})|| \\ &< \varepsilon + \omega(1/K) \ . \end{aligned}$$

Remark. Let us give a geometrical interpretation of the above proof. For simplicity, assume that $V = B_{l_\infty^N}$. The points a_i satisfy

$$\|a_{i+1} - a_i\| = \|h_{i+1}\| = 1/K \qquad \text{and} \qquad \|a_i\| = i/K,$$

so they form a path joining 0 to the unit sphere of c_0. They are constructed in such a way that the norm of a_i is attained at each point of A_i, so that a_i lies on an edge of $i/K.\, B_{c_0}$ (because $|A_i| \geq 2$). Therefore, the norm of c_0 is rough at the point a_i. Hence it is possible to select a direction h_{i+1} very close to the kernel of $f'(a_i)$ (so that $f(a_i + h_{i+1}) - f(a_i)$ is very small) such that $\|a_i + h_{i+1}\| - \|a_i\|$ is large (actually $\|a_i + h_{i+1}\| - \|a_i\| = \|h_{i+1}\| = 1/K$). Moreover, in order to iterate the construction, the direction h_{i+1} should be selected in such a way that $a_{i+1} = a_i + h_{i+1}$ lies on an edge of $(i+1)/K.\, B_{c_0}$, which explains the choice of the sequence $(N_0, \dots, N_K)$. Summing up, we obtain that $f(a_K) - f(a_0)$ is small and $\|a_K\| - \|a_0\| = 1$ is "very" large.

Theorem 1. *Assume that Y has finite cotype and that X does not have finite cotype, and let Ω be a bounded open subset of X. Then, for any function $f \in C_u^1(\overline{\Omega}, Y)$, $f(\partial\Omega)$ is dense in $f(\overline{\Omega})$.*

Proof. By the Maurey-Pisier theorem, X contains l_∞^n's uniformly. Thus, it should be clear that Theorem 1 follows easily from Lemma 1. We give the details anyway.

Let $f \in C_u^1(\overline{\Omega}, Y)$, and denote by ω be the modulus of uniform continuity of f' on $\overline{\Omega}$. Choose $C > 0$ such that $\forall x \in \Omega\ x + C\, B_X \supseteq \Omega$ and let $M = 2C \sup\{||f'(x)||;\ x \in \overline{\Omega}\}$. Finally, let $\varepsilon > 0$, choose a positive integer K such that $\omega(1/K) < \varepsilon/2$, and let N be an integer satisfying the conclusion of Lemma 1 for ω, M, K and $\varepsilon/2$.

Since X contains l_∞^n's uniformly, we can choose a linear embedding $T : l_\infty^N \to X$ such that $||T|| \leq 2C$ and $||T^{-1}|| \leq 1/C$.

Now, let $x_0 \in \Omega$ and put $V = \{u \in l_\infty^N;\ x_0 + Tu \in \Omega\}$.

The set V is an open neighbourhood of 0 in l_∞^N, and it is contained in $B_{l_\infty^N}$ because $||T^{-1}|| \leq 1/C$. Moreover, the function $\tilde{f}$ defined by $\tilde{f}(u) = f(x_0 + Tu)$ is in $C^{1,\omega,M}(\overline{V}, Y)$. Therefore, by the choice of N, there exists a point $a \in \partial V$ such that $||\tilde{f}(a)-\tilde{f}(0)||<\varepsilon$. Then $x=x_0+Ta$ belongs to $\partial\Omega$, and $||f(x)-f(x_0)||<\varepsilon$. This concludes the proof.

Definition 2. Let N be a positive integer, and let $\mathcal{S}_N$ be the permutation group of $\{0;\dots;N-1\}$. A function $f: c_0\to Y$ is said to be $\mathcal{S}_N$*-invariant* if for each $\sigma\in\mathcal{S}_N$ and all $x\in c_0$, one has $f(x_\sigma)=f(x)$, where $x_\sigma=(x_{\sigma(0)},\dots,x_{\sigma(N-1)},x_N,x_{N+1},\dots)$.

Lemma 2. *Assume that Y has finite cotype. Let ω be a modulus of continuity, and let $M>0$. Finally, let K be a positive integer and let $\varepsilon>0$.*

a) *Given a finite set $F\subseteq B_{c_{00}}$, there exists a positive integer N satisfying the following property: for every $f\in C^{1,\omega,M}(B_{c_0},Y)$, one can find a normalized K-pyramidal vector a whose support is contained in $[0;N[$ and disjoint from $\bigcup_{x\in F}\operatorname{supp}x$, and such that*

$$\forall x\in F\quad ||f(x+a)-f(x)||<\varepsilon+\omega(1/K).$$

b) *Let $F=\{0;e_0\}$. If K is large enough and if N is chosen as in* **a)**, *then $||f(e_0)-f(0)||<3\varepsilon$ for each $\mathcal{S}_N$-invariant $f\in C^{1,\omega,M}(B_{c_0},Y)$.*

Proof. **a)** Let $F=\{x_1;\dots;x_m\}$ be a finite subset of $B_{c_{00}}$, and choose an integer L such that $\bigcup_{i=1}^{m}\operatorname{supp}x_i\subseteq[0;L]$. Let also $\tilde{Y}=l_\infty^m(Y)$. Finally, let T be the "right-shift" operator on c_0, defined by $T\left(\sum\alpha_i e_i\right)=\sum\alpha_i e_{i+1}$.

Then, for any $f\in C^{1,\omega,M}(B_{c_0},Y)$, the function $\tilde{f}$ defined by

$$\tilde{f}(u)=\left(f(x_1+T^{L+1}u),\dots,f(x_m+T^{L+1}u)\right)$$

is in $C^{1,\omega,M}(B_{c_0},\tilde{Y})$. Therefore, part **a)** follows from Lemma 1 applied to $\tilde{Y}$.

b) Choose an integer N satisfying the conclusion of **a)** for $F=\{0;e_0\}$, and let $f\in C^{1,\omega,M}(B_{c_0},Y)$ be $\mathcal{S}_N$-invariant.

By **a)**, one can find a normalized K-pyramidal vector $a=\frac{1}{K}\sum_{i=1}^{K}\mathbf{1}_{A_i}$ supported by $]0;N[$ such that

$$||f(a)-f(0)||,\ ||f(e_0+a)-f(e_0)||<\varepsilon+\omega(1/K)\ .$$

Let $i_0=\min A_K$, and put $h=\frac{1}{K}\sum_{i=1}^{K}e_{l_i}$, where $l_i=\min A_i-1$ $(1\le i\le K)$. Then $||h||=1/K$ and $a,\,a+h\in B_{c_0}$; hence $||f(a+h)-f(a)||\le M/K$. But $a+h=(a+e_0)_\sigma$, where $\sigma\in\mathcal{S}_N$ is the cycle $(0,\dots,i_0)$. Therefore (by $\mathcal{S}_N$-invariance) $f(a+h)=f(a+e_0)$, whence $||f(a+e_0)-f(a)||\le M/K$. By the choice of a, it follows that

$$\begin{aligned}||f(e_0)-f(0)||&\le||f(e_0)-f(e_0+a)||+||f(e_0+a)-f(a)||+||f(a)-f(0)||\\&\le M/K+2\left(\epsilon+\omega(1/K)\right).\end{aligned}$$

This proves **b)**.

Theorem 2. *Assume that Y has finite cotype.*

a) *If $f\in C_u^1(B_{c_0},Y)$, then the sequence $\left(f(e_i)\right)$ converges to $f(0)$ in the Cesaro sense.*

b) *More precisely, given a modulus of continuity ω and positive numbers M, ε, there exists a positive integer N such that* $\left\| \frac{1}{n} \sum_{i=0}^{n-1} f(e_i) - f(0) \right\| < \varepsilon$ *for all $f \in C^{1,\omega,M}(B_{c_0}, Y)$ and all $n \geq N$.*

Proof. Let ω be a modulus of continuity, and let $M > 0$. If $f \in C^{1,\omega,M}(B_{c_0}, Y)$, then, for any positive integer n, the function $\tilde{f}$ defined by $\tilde{f}(x) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} f(x_\sigma)$ still belongs to $C^{1,\omega,M}(B_{c_0}, Y)$, and it is also $\mathcal{S}_n$-invariant; moreover, $\tilde{f}(0) = f(0)$ and $\tilde{f}(e_0) = \frac{1}{n} \sum_{i=0}^{n-1} f(e_i)$. Thus, Theorem 2 follows from Lemma 2.

Corollary. *Assume that Y has finite cotype, and let $f : c_0 \to Y$ be a C^1-smooth function such that f' is uniformly continuous on bounded sets. Then any sequence $(x_i) \subseteq c_0$ weakly converging to some $x \in c_0$ has a subsequence (x'_i) such that $f(x'_i) \to f(x)$ in the Cesaro sense.*

Remark. Clearly, the conclusion of Theorem 2 **b)** holds for any subsequence of (e_i), with the *same* integer N.

It is very likely that Theorem 2 is far from being best possible. An "optimal" statement could be the following: *if Y does not contain c_o and $f \in C^1_u(B_{c_0}, Y)$, then f turns weak-Cauchy sequences from B_{c_0} into norm convergent sequences in Y.*

Notice that if Y has an unconditional basis, then the above statement is indeed true, as shown in [H2].

In the same spirit, given a pair of Banach spaces (X, Y) and a function $f \in C^1(B_X, Y)$, one may consider the following two properties:

(1) *f turns Cauchy sequences (from B_X) for the "weak" topology generated by $\mathcal{L}(X, Y)$ into (norm) convergent sequences.*

(2) *$f'(B_X)$ is a relatively compact subset of $\mathcal{L}(X, Y)$.*

It follows at once from the mean-value theorem that property **(2)** is stronger than **(1)**. Moreover, it is observed in [H2] that when $Y = \mathbb{R}$, **(1)** are **(2)** are equivalent provided X does not contain l_1 and f' is uniformly continuous on B_X. Finally, if $Y = \mathbb{R}$ and $X = c_0$, then both properties are true; this is the main result of [H1].

In the remainder of this note, we give a short proof of a slightly weaker form of this last result (Theorem 3 below).

For the sake of readability, we will impose "global" smoothness conditions on the functions we are dealing with. Accordingly, we shall say that a function $f : c_0 \to Y$ is *smooth* if f is C^1-smooth and f' is uniformly continuous on bounded sets.

Definition 3. Let $(G, +)$ be an abelian topological group, and let B be a subset of G. We say that a function $f : G \to Y$ is *strongly sequentially continuous in B* if for every sequence $(x_n) \subseteq B$ and every sequence (h_i) converging to 0 in G, one has

$$\lim_{i\to\infty} \left(\liminf_{n\to\infty} ||f(x_n + h_i) - f(x_n)|| \right) = 0.$$

It is easily checked that the definition of strong sequential continuity can be reformulated as follows: *a function $f : G \to Y$ is strongly sequentially continuous* *in a set B if and only if, for each sequence (h_i) converging to* 0 *in G, the sequence of functions (f_n) defined by $f_n(x) = \inf\{||f(x+h_i) - f(x)||;\ i \leq n\}$ converges to* 0 *uniformly on B.*

This definition may look a bit artificial. It is "justified" by the following lemma.

Lemma 3. *Let G be an abelian topological group and let $f : G \to Y$. Assume that f is strongly sequentially continuous in some set $B \subseteq G$. Then the following statements hold.*

a) *f turns Cauchy sequences from B into (norm) convergent sequences in Y.*

b) *If, in addition, every sequence from B admits a Cauchy subsequence, then $f_{|B}$ is uniformly sequentially continuous on B.*

Proof. **a)** By contradiction, assume that there exists a Cauchy sequence $(x_i) \subseteq B$ such that $(f(x_i))$ is not convergent. Then we can find a positive number ε and two subsequences (y_n), (z_m) of (x_i) such that $\forall n, m \geq 0\ \ ||f(y_n) - f(z_m)|| \geq \varepsilon$; this is obvious if the set $\{f(x_i);\ i \geq 0\}$ is not relatively compact in Y (because in this case, $(f(x_i))$ admits an ε-separated subsequence), and obvious as well if it is, because in that case, $(f(x_i))$ has at least two cluster points.

Now, for all $i, n, m \geq 0$, one has

$$||f(y_n) - f[y_n + (z_m - x_i)]|| + ||f[z_m + (y_n - x_i)] - f(z_m)|| \geq \varepsilon.$$

Thus, by Ramsey's theorem for triples of integers, we may assume that either $\forall i, n, m \in \mathbb{N},\ m < i < n,\ \|f(y_n) - f[y_n + (z_m - x_i)]\| \geq \varepsilon/2$, or $\forall i, n, m \in \mathbb{N}$, $m < i < n$, $\|f[z_m + (y_n - x_i)] - f(z_m)\| \geq \varepsilon/2$. In the first case, we get in particular $\liminf_{n\to\infty} \|f(y_n) - f[y_n + (z_{i-1} - x_i)]\| \geq \varepsilon/2$ for all $i \geq 1$, which is impossible because $(z_{i-1} - x_i) \to 0$ as $i \to \infty$ and f is strongly sequentially continuous in B. In the second case, we get $\|f[z_0 + (y_{i+1} - x_i)] - f(z_0)\| \geq \varepsilon/2$ for all $i \geq 1$, which is again impossible because f is sequentially continuous at z_0. This proves **a)**.

Part **b)** is a straightforward consequence of **a)**.

The following remarks will not be used in the proof of Theorem 3.

Remarks. **1.** Clearly, strong sequential continuity in $B \subseteq G$ implies sequential continuity at each point of B. Moreover, it is not difficult to check that if $f : \mathbb{R} \to \mathbb{R}$ is strongly sequentially continuous in $\mathbb{R}$, then $\liminf_{|x|\to\infty} |f(x)/x| < +\infty$. Thus, strong sequential continuity in $\mathbb{R}$ is a stronger property than usual continuity.

2. It is plain that any uniformly sequentially continuous function $f : G \to Y$ is strongly sequentially continuous in G. On the other hand, Lemma 3 implies that if $f : \mathbb{R} \to Y$ is strongly sequentially continuous in some bounded set $B \subseteq \mathbb{R}$, then $f_{|B}$ is uniformly continuous on B. More generally, if $G = (X, w)$, where X is a Banach space not containing l_1, then strong sequential continuity in bounded sets is equivalent to sequential uniform continuity on bounded sets, by Rosenthal's l_1-theorem.

3. We are unable to determine whether strong sequential continuity in $\mathbb{R}$ is equivalent to uniform continuity.

4. The following example shows that even in very simple groups, strong sequential continuity does not imply uniform continuity.

Let $\mathbb{D}$ be the group of dyadic real numbers $\Big(\mathbb{D} = \{k/2^p;\ k \in \mathbb{Z}, p \in \mathbb{N}\}\Big)$, and let $f : \mathbb{D} \to \mathbb{R}$ be the even function defined on $\mathbb{D} \cap [n, n+1]$ $(n \in \mathbb{N})$ by

$f(x) = \sin(\pi x)\sin(2^n \pi x)$. It is easy to check that f is not uniformly continuous on $\mathbb{D}$. Yet, we claim that f is strongly sequentially continuous in $\mathbb{D}$.

To show this, let us fix $\varepsilon > 0$ and a sequence $(h_i) \subseteq \mathbb{D}$ converging to 0. It is enough to prove that for large enough n, $\sup\limits_{x\in\mathbb{D}_+} \inf\limits_{i\le n} |f(x+h_i) - f(x)| \le 2\pi\varepsilon$ (where $\mathbb{D}_+ = \mathbb{D} \cap [0; +\infty[$).

First, we choose n_0 such that $|h_{n_0}| < \varepsilon/2$, and we write $h_{n_0} = k_0/2^{p_0}$ ($p_0 \in \mathbb{N}$, $k_0 \in \mathbb{Z}$).

Since f is uniformly continuous around $[0; p_0+1]$, there exists an integer n_1 such that

$$\forall x \in \mathbb{D} \cap [0; p_0+1] \quad |f(x+h_{n_1}) - f(x)| \le \varepsilon\,.$$

Next, if $x \in [n - \varepsilon/2, n + \varepsilon/2]$ for some $n \in \mathbb{N}$, then $x + h_{n_0} \in [n - \varepsilon, n + \varepsilon]$; so

$$|f(x+h_{n_0}) - f(x)| \;\le\; |f(x+h_{n_0})| + |f(x)| \;\le\; 2\pi\varepsilon$$

for all $x \in \mathbb{D}_+ \cap \Big(\bigcup\limits_{n\in\mathbb{N}} [n-\varepsilon/2, n+\varepsilon/2]\Big)$.

Finally, if $x \in \mathbb{D}_+ \setminus \Big([0; p_0+1] \cup \bigcup\limits_{n\in\mathbb{N}} [n-\varepsilon/2, n+\varepsilon/2]\Big)$, then there exists $n \ge p_0+1$ such that both x and $x + h_{n_0}$ lie in $[n, n+1]$. Since $n \ge p_0 + 1$, the function $t \mapsto \sin(2^n \pi t)$ is h_{n_0} periodic; hence

$$f(x+h_{n_0}) - f(x) = \sin(2^n\pi x)\left[\sin\pi(x+h_{n_0}) - \sin(\pi x)\right].$$

Thus

$$|f(x+h_{n_0}) - f(x)| \;\le\; |\sin\pi(x+h_{n_0}) - \sin(\pi x)| \;\le\; \pi\varepsilon/2$$

for all $x \in \mathbb{D}_+ \setminus \Big([0; p_0+1] \cup \bigcup\limits_{n\in\mathbb{N}} [n-\varepsilon/2, n+\varepsilon/2]\Big)$.

Therefore, if $n \ge \mathrm{Max}(n_0, n_1)$, we have, for all $x \in \mathbb{D}_+$,

$$\inf_{i\le n} |f(x+h_i) - f(x)| \;\le\; \mathrm{Max}(\varepsilon, \pi\varepsilon/2, 2\pi\varepsilon) \;\le\; 2\pi\varepsilon\,.$$

After this detour, we can now state and prove the following result:

Theorem 3 (Hájek)**.** *Let $f : c_0 \to \mathbb{R}$ be a smooth function. Then f is uniformly continuous on bounded sets when c_0 is equipped with its weak topology.*

Proof. The weak topology is metrizable on any bounded subset of c_0, and each bounded sequence in c_0 admits a weak-Cauchy subsequence; hence, by Lemma 3, we may content ourselves with proving that f is strongly sequentially continuous in every bounded subset of $G = (c_0, w)$. Therefore, we have to show that if $(h_i) \subseteq c_0$ is weakly null, then $\inf\{||f(x+h_i) - f(x)||;\, i \le n\} \to 0$ uniformly on bounded sets.

Let us fix a weakly null sequence (h_i) and a bounded set $B \subseteq c_0$.

By extracting a subsequence if necessary, we may assume that there exists a bounded linear operator $T : c_0 \to c_0$ such that $T(e_i) = h_i$ for all i.

Let ω_0 be the modulus of uniform continuity of f' on $B + TB_{c_0}$, and let $M_0 = \sup\{||f'(w)||;\, w \in B + TB_{c_0}\}$. Then, for every $x \in B$, the function f_x defined by $f_x(u) = f(x+Tu)$ is in $C^{1,\omega,M}(B_{c_0}, \mathbb{R})$ where $\omega = ||T||.\,\omega_0$ and $M = ||T||.\,M$;

hence, by Theorem 2 (and the remark following it),

$$\lim_{n\to\infty}\left(\sup\Big\{\Big|\frac{1}{n}\sum_{i\in F}[f(x+h_i)-f(x)]\Big|;\ F\subseteq\mathbb{N},\,|F|=n\Big\}\right)=0 \text{ uniformly on } B\,.$$

Since f is real-valued, this concludes the proof.

Corollary (Hájek)**.** *If $f : c_0 \to \mathbb{R}$ is smooth, then f' is a compact map, which means that for every bounded set $B \subseteq c_0$, $f'(B)$ is relatively compact in l_1.*

Proof. We could apply the result mentioned above about the equivalence of properties **(1)** and **(2)**, but we give a direct proof for completeness.

Let $f : c_0 \to \mathbb{R}$ be a smooth function, and assume that for some bounded $B \subseteq c_0$, $f'(B)$ is not relatively compact in l_1. Then one can find a positive number ε, a sequence $(x_i) \subseteq B$ and a sequence $(h_i) \subseteq B_{c_0}$, such that the h_i's are disjointly supported and $|f'(x_i).\,h_i| \geq \varepsilon$ for all i.

Let ω be the modulus of (uniform) continuity of f' on $B + B_{c_0}$, and fix $\alpha \in]0;1]$. Then, for each $i \geq 0$, one can write

$$\begin{aligned}|f(x_i+\alpha h_i)-f(x_i)| &\geq |f'(x_i).\,(\alpha h_i)| - ||\alpha h_i||\,\omega(||\alpha h_i||)\\ &\geq \alpha\,(\varepsilon-\omega(\alpha)).\end{aligned}$$

Since (h_i) is weakly null, this contradicts Theorem 3 if α is small enough.

To conclude this note, let us mention still another innocent question.

Theorem 3 implies that any smooth function $f : c_0 \to \mathbb{R}$ can be (uniquely) extended to a function $\tilde{f} : l_\infty \to \mathbb{R}$ which is w^*-continuous on bounded sets. It would be interesting to know if such an extension inherits any smoothness property from f.

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