EQUILIBRIUM MEASURE OF A PRODUCT SUBSET OF $\mathbb{C}^n$

ZBIGNIEW BŁOCKI

(Communicated by Steven R. Bell)

Abstract. In this note we show that an equilibrium measure of a product of two subsets of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, is a product of their equilibrium measures. We also obtain a formula for $(dd^c \max\{u, v\})^p$, where $u, v$ are locally bounded plurisubharmonic functions and $2 \leq p \leq n$.

Introduction

Let $E$ be a bounded subset of $\mathbb{C}^n$. The function

$$V_E := \sup\{u \in PSH(\mathbb{C}^n) : u|_E \leq 0, \sup_{z \in \mathbb{C}^n} (u(z) - \log^+ |z|) < \infty\}$$

is called a global extremal function (or the Siciak extremal function) of $E$. It is known that $V_E$, the upper regularization of $V_E$, is plurisubharmonic in $\mathbb{C}^n$ if and only if $E$ is not pluripolar. In such a case, by [BT1], $(dd^c V_E)^n$ is a well defined nonnegative Borel measure and it is called an equilibrium measure of $E$. We refer to [Kl] for a detailed exposition of this topic.

In this note we shall show

**Theorem 1.** Let $E$ and $F$ be nonpluripolar bounded subsets of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Then

$$V_{E \times F}^* = \max\{V_E^*, V_F^*\}$$

and

$$(dd^c V_{E \times F}^*)^{n+m} = (dd^c V_E^*)^n \land (dd^c V_F^*)^m.$$**

Note that here we treat $V_E^*$ (resp. $V_F^*$) as a function of $\mathbb{C}^{n+m}$ independent of the last $m$ (respectively first $n$) variables.

The formula (1) was proved by Siciak (see [Si] for $E, F$ compact (see also [Ze] for a proof using the theory of the complex Monge-Ampère operator). For $n = m = 1$ the proof of (2) can be found in [BT2].

If $E \subset D$, where $D$ is a bounded domain in $\mathbb{C}^n$, then the function

$$u_{E, D} := \sup\{v \in PSH(D) : v \leq 0, v|_E \leq -1\}$$

is called a relative extremal function of $E$. Combining our methods of the proof of Theorem 1 with a result from [EP] we can also obtain

Received by the editors February 18, 1999.

2000 Mathematics Subject Classification. Primary 32U15; Secondary 32W20.

This work was partially supported by KBN Grant #2 PO3A 003 13.
Theorem 2. Let $D$ be a bounded domain in $\mathbb{C}^n$ and $G$ a bounded domain in $\mathbb{C}^m$. Then for arbitrary subsets $E \subset D$, $F \subset G$ we have
\begin{equation}
 u^*_{E \times F, D \times G} = \max\{u^*_{E, D}, u^*_{F, G}\}
\end{equation}
and
\[(dd^c u^*_{E \times F, D \times G})^{n+m} = (dd^c u^*_{E, D})^n \land (dd^c u^*_{F, G})^m.\]

The relative Monge-Ampère capacity of $E \subset D$ is defined by
\[c(E, D) := \sup \left\{ \int_E (dd^c u)^n : u \in PSH(D), -1 \leq u \leq 0 \right\},\]
provided that $E$ is Borel. If $E \subset D$ is arbitrary, then, as usual, we can define
\[c^*(E, D) := \inf_{U \subset E, U \text{ open}} c(U, D),\]
\[c_*(E, D) := \sup_{K \subset E, K \text{ compact}} c(K, D).\]

By [BT1], if $E \subset D$ and $D$ is hyperconvex (that is $(u^*_{E, D})_* = 0$ on $\partial D$), then
\[c^*(E, D) = \int_D (dd^c u^*_{E, D})^n.\]

Moreover, $c^*(E, D) = c(E, D) = c_*(E, D)$ if $E$ is Borel. Theorem 2 thus gives

Theorem 3. Assume that $D$ and $G$ are bounded hyperconvex domains in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Then for $E \subset D$, $F \subset G$ we have
\[c^*(E \times F, D \times G) = c^*(E, D)c^*(F, G).\]

I would like to thank N. Levenberg for inspiring discussions and E. Poletsky for his help in the proof of Lemma 8 below.

Proofs

If $\Omega$ is an open subset of $\mathbb{C}^n$ and $1 \leq p \leq n$, then by [BT1] the mapping
\[ (u_1, \ldots, u_p) \mapsto (dd^c u_1 \wedge \cdots \wedge dd^c u_p)^p \]
is well defined on the set $(PSH \cap L^\infty_{loc}(\Omega))^p$ and its values are nonnegative currents of bidegree $(p, p)$. Moreover, (4) is symmetric and continuous with respect to decreasing sequences. First, we shall prove

Theorem 4. Let $u, v$ be locally bounded pluriharmonic functions. Then, if $2 \leq p \leq n$, we have
\[(dd^c \max\{u, v\})^p = (dd^c \max\{u, v\})^p \wedge \sum_{k=0}^{p-1} (dd^c u)^k \wedge (dd^c v)^{p-1-k} - \sum_{k=1}^{p-1} (dd^c u)_k^k \wedge (dd^c v)^{p-k}.\]

Proof. We leave it as an exercise to the reader to show that a simple inductive argument reduces the proof to the case $p = 2$. By the continuity of (4) under decreasing sequences we may also assume that $u, v$ are smooth.

Let $\chi : \mathbb{R} \to [0, +\infty)$ be smooth and such that $\chi(x) = 0$ if $x \leq -1$, $\chi(x) = x$ if $x \geq 1$ and $0 \leq \chi' \leq 1$, $\chi'' \geq 0$ everywhere. Define
\[\psi_j := v + \frac{1}{j} \chi(j(u - v)).\]
Denote for simplicity $w = \max\{u, v\}$ and $\alpha = u - v$. We can easily check that $\psi_j \downarrow w$ as $j \uparrow \infty$. An easy computation gives

\begin{equation}
\frac{dd^c(\chi(j\alpha)/j)}{j} = \chi'(j\alpha)dd^c\alpha + j\chi''(j\alpha)\alpha \wedge d^c\alpha.
\end{equation}

Therefore

\begin{equation}
\frac{dd^c\psi_j}{j} = \chi'(j\alpha)dd^c\psi_j + (1 - \chi'(j\alpha))dd^c\psi_j + j\chi''(j\alpha)\alpha \wedge d^c\alpha
\end{equation}

and, in particular, $\psi_j$ is plurisubharmonic.

From the definition of $\psi_j$ we obtain

\begin{equation}
(dd^c\psi_j)^2 = (dd^c\psi_j)^2 + 2dd^c(\chi(j\alpha)/j) \wedge dd^c\psi_j + (dd^c(\chi(j\alpha)/j))^2.
\end{equation}

We have weak convergences

\begin{equation}
\begin{aligned}
(dd^c\psi_j)^2 &\rightarrow (dd^c\psi)^2, \\
(dd^c(\chi(j\alpha)/j)) &\rightarrow (dd^c(\chi(\alpha)))) \wedge dd^c\psi
\end{aligned}
\end{equation}

so it remains to analyze the third term of the right-hand side of (6). Using (5) and the fact that $(\alpha \wedge d^c\alpha)^2 = 0$, we compute

\begin{align*}
(dd^c(\chi(j\alpha)/j))^2 &= (\chi'(j\alpha))^2(dd^c\alpha)^2 + 2j\chi'(j\alpha)\chi''(j\alpha)\alpha \wedge d^c\alpha \wedge dd^c\alpha \\
&= d(\chi'(j\alpha))^2d^c\alpha \wedge dd^c\alpha \\
&= dd^c(\gamma(j\alpha)/j) \wedge dd^c\alpha,
\end{align*}

where $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ is such that $\gamma' = (\chi')^2$. In fact, if $\gamma$ is chosen so that $\gamma(-1) = 0$, then $\gamma(jx) \downarrow \max\{0, x\}$ as $j \uparrow \infty$ and

\begin{align*}
(dd^c(\chi(j\alpha)/j))^2 &\rightarrow (dd^c(\chi(\alpha)))) \wedge dd^c\alpha
\end{align*}

weakly. Combining this with (6) and (7) we conclude

\begin{align*}
(dd^c\psi_j)^2 &= (dd^c\psi_j)^2 + 2dd^c(\chi(\alpha)) \wedge dd^c\psi_j + dd^c(\chi(\alpha))^2 \wedge dd^c\psi \\
&= dd^c\psi \wedge (dd^c\psi \wedge dd^c\psi) - dd^c\psi \wedge dd^c\psi
\end{align*}

which completes the proof of Theorem 4. \hfill \square

From Theorem 4 we can immediately get the following two consequences:

**Corollary 5.** If $u$ is locally bounded, plurisubharmonic and $h$ is pluriharmonic, then

\begin{equation}
(dd^c \max\{u, h\})^p = dd^c \max\{u, h\} \wedge (dd^c u)^{p-1}.
\end{equation}

\hfill \square

**Corollary 6.** Suppose $u, v$ are locally bounded plurisubharmonic functions with $(dd^c u)^p = 0$ and $(dd^c v)^q = 0$, where $1 \leq p, q \leq n$ and $p + q \leq n$. Then $(dd^c \max\{u, v\})^{p+q} = 0$. \hfill \square

The main part of the proof of (2) will be contained in

**Theorem 7.** Let $D$ be open in $\mathbb{C}^n$ and $G$ open in $\mathbb{C}^m$. Assume that $u, v$ are nonnegative plurisubharmonic functions in $D$ and $G$, respectively, such that

\begin{align*}
\int_{\{u > 0\}}(dd^c u)^n = 0 \quad \text{and} \quad \int_{\{v > 0\}}(dd^c v)^m = 0.
\end{align*}

Then, treating $u, v$ as functions on $D \times G$, we have

\begin{align*}
(dd^c \max\{u, v\})^{n+m} = (dd^c u)^n \wedge (dd^c v)^m.
\end{align*}
Proof. Let $w, \chi$ and $\psi_j$ be defined in the same way as in the proof of Theorem 4. By Theorem 4 and since $(dd^c v)^{n+1} = 0$, $(dd^c v)^{m+1} = 0$, we have

$$
(dd^c w)^{n+m} = dd^c w \wedge [(dd^c u)^{n-1} \wedge (dd^c v)^m + (dd^c u)^n \wedge (dd^c v)^{m-1}] - (dd^c u)^n \wedge (dd^c v)^m.
$$

(8)

Using the hypothesis on $u, v$ we may compute

$$
(dd^c \psi_j \wedge (dd^c u)^{n-1} \wedge (dd^c v)^m)
$$

$$
= \left[\chi'(0)(dd^c u)^n + j \chi''(j u) du \wedge (dd^c u)^{n-1}\right] \wedge (dd^c v)^m
$$

$$
= dd^c(\chi(j u)/j) \wedge (dd^c u)^{n-1} \wedge (dd^c v)^m.
$$

Since $\chi(j u)/j \downarrow u$ as $j \uparrow \infty$, it follows that

$$
(dd^c w \wedge (dd^c u)^{n-1} \wedge (dd^c v)^m = (dd^c u)^n \wedge (dd^c v)^m
$$

and, similarly,

$$
(dd^c w \wedge (dd^c u)^n \wedge (dd^c v)^{m-1} = (dd^c u)^n \wedge (dd^c v)^m.
$$

This, together with (8), finishes the proof.

For the proof of Theorem 1 we need a lemma which is an extension of a result from [Sa].

**Lemma 8.** Let $E, F, D, G$ be as in Theorem 2. For $\varepsilon > 0$ set

$$
E_\varepsilon := \{V_E^* < \varepsilon\}, \quad F_\varepsilon := \{V_F^* < \varepsilon\},
$$

$$
\overline{E}_\varepsilon := \{u_{E,D}^* < -1 + \varepsilon\}, \quad \overline{F}_\varepsilon := \{u_{E,G}^* < -1 + \varepsilon\}.
$$

Then

(9)

$$
V_{E_\varepsilon}^* \uparrow V_E^*, \quad V_F^* \uparrow V_F^*, \quad V_{E \times F_\varepsilon}^* \uparrow V_{E \times F}^*,
$$

(10)

$$
u_{E_\varepsilon,D}^* \uparrow u_{E,D}^*, \quad u_{F_\varepsilon,G}^* \uparrow u_{F,G}^*, \quad u_{E \times F_\varepsilon,D \times G}^* \uparrow u_{E \times F,D \times G}^*,
$$

as $\varepsilon \downarrow 0$, and every convergence is uniform.

**Proof.** The set $E \setminus E_\varepsilon = E \cap \{V_E^* \geq \varepsilon\}$ is pluripolar by Bedford-Taylor’s theorem on negligible sets (see [B-T]). It follows that

$$
V_E^* - \varepsilon \leq V_{E_\varepsilon}^* \leq V_{E_\varepsilon}^* \leq V_E^*
$$

which gives the first two convergences of (9). In order to show the third one, observe that

(11)

$$
\max\{V_E, V_F\} \leq V_{E \times F} \leq V_E + V_F.
$$

Indeed, the first inequality in (11) follows easily from the definition of extremal function. Fixing one of the variables $(z, w) \in \mathbb{C}^n \times \mathbb{C}^m$, we see that the second inequality in (11) is satisfied, first on the cross $(E \times \mathbb{C}^m) \cup (\mathbb{C}^n \times F)$, and then everywhere.

By (11) $V_{E \times F} \leq 2\varepsilon$ on $E_\varepsilon \times F_\varepsilon$. On the other hand, by (11) the set $(E \times F) \setminus (E_\varepsilon \times F_\varepsilon)$ is contained in $(E \times F) \cap \{V_{E \times F}^* \geq \varepsilon\}$ and is thus pluripolar. Therefore

$$
V_{E \times F}^* - 2\varepsilon \leq V_{E_\varepsilon \times F_\varepsilon} = V_{E \times F_\varepsilon}^* \leq V_{E \times F}^*.
$$

and this gives (9).
Similarly as in (11) we can show:
\[ \max\{u_{E,D}, u_{F,G}\} \leq u_{E \times F, D \times G} \leq -u_{E,D}u_{F,G}. \]
Now the proof of (10) is parallel to that of (9).

**Proof of Theorem 1.** If \( E, F \) are compact and \( L\)-regular (that is, \( V_E \) and \( V_F \) are continuous), then (1) was shown in [Si] and (2) follows immediately from Theorem 7. For \( E, F \) open we can find sequences of compact, \( L\)-regular sets with \( E_j \uparrow E \) and \( F_j \uparrow F \). Then \( V_{E_j} \downarrow V_E, V_{F_j} \downarrow V_F \) and \( V_{E_j \times F_j} \downarrow V_{E \times F} \) as \( j \to \infty \). This gives (1) and (2) for open sets. The general case can now be deduced from Lemma 8.

**Proof of Theorem 2.** The proof of (3) for open subsets can be found in [EP]. Now the proof is the same as the proof of Theorem 1.

**Remark.** Although (3) is stated in [EP] for arbitrary subsets \( E, F \), the way from open subsets to the general case is not so straightforward as the authors claim—one needs Lemma 8.

**References**


Institute of Mathematics, Jagiellonian University, Reymonta 4, 30-059 Kraków, Poland

E-mail address: blocki@im.uj.edu.pl