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BETWEEN THE LINDELÖF PROPERTY AND COUNTABLE TIGHTNESS

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Abstract. We consider a class of compact spaces K for which the space $P(K)$ of probability Radon measures on K has countable tightness in the *weak** topology. We show that that class contains those compact zero-dimensional spaces for which $C(K)$ is weakly Lindelöf, and, under MA + ¬CH, all compact spaces K with $C(K)$ having property (C) of Corson.

1. Introduction

For an arbitrary set X we denote by $[X]^{\leq\omega}$ the family of all at most countable subsets of X. Recall that a topological space X is said to have *countable tightness* at point $x \in X$ if for every set $A \subseteq X$ with $x \in \overline{A}$ there is $I \in [A]^{\leq\omega}$ such that $x \in \overline{I}$; in such a case we write $t(x, X) = \omega$. We say that the space X has countable tightness and write $t(X) = \omega$ if $t(x, X) = \omega$ for every $x \in X$.

Let X be a completely regular Hausdorff topological space, and let $C_p(X)$ be the space of continuous functions on X endowed with the topology of pointwise convergence. According to a theorem due to Arhangel'skiĭ and Pytkeev (see Theorem II.1.1 of [1]), $t(C_p(X)) = \omega$ if and only if X^n is Lindelöf for every n. That result led to the following problem:

Assume that X is Lindelöf; is it true that every compact subspace of $C_p(X)$ has countable tightness?

This interesting and apparently difficult question has been answered in positive by Arhangel'skiĭ [1], Theorem IV.11.14, assuming the Proper Forcing Axiom. It seems that no further solutions are known (cf. [3]). It is even not known if one can show in ZFC that the space $\beta\omega$ cannot be embedded into $C_p(X)$, with X being a Lindelöf space (see Arhangel'skiĭ [2], Problem 52; cf. Cascales, Manjabacas and Vera [4]).

One can also consider the following version of the problem above: Let E be a Banach space which is Lindelöf in its weak topology; is it true that the unit ball in E^* has countable tightness in its weak* topology? This is partially related to a problem posed by Corson, if the product of weak topologies on $E \times E$ is Lindelöf provided E is weakly Lindelöf (cf. [13]). In such a setting, it is perhaps more natural to consider property (C) of Corson rather than weak Lindelöfness itself (see [17]).

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The present paper deals with an even more concrete question: for which compact spaces K, does the space $P(K)$ (of probability Radon measures on K) have countable tightness in its *weak** topology? We show below that the tightness of $P(K)$ is countable whenever $C(K)$ is weakly Lindelöf, under an additional assumption that K is zero dimensional. Moreover, using a recent result of Fremlin [9], we show that assuming Martin's axiom and the negation of CH, $P(K)$ has countable tightness provided $C(K)$ has property (C) of Corson. Our theorems (partially) extend a result due to Pol [18].

2. Some properties of $C(K)$ and $P(K)$

In the sequel, K always stands for a compact Hausdorff topological space. For a given space K, we denote by $C(K)$ the Banach space of continuous functions on K. As usual, the conjugate space $C(K)^*$ is identified with $M(K)$, the space of signed Radon measures on K of finite variation. We denote by $P(K)$ the set of probability Radon measures on K. We always consider $P(K)$ with its *weak** topology.

Note that the tightness of $P(K)$ is the same as the tightness of the unit ball $M_1(K)$ in $M(K)$. It is clear that $t(P(K)) \leq t(M_1(K))$. On the other hand, denoting $\{(a,b) : |a| + |b| \leq 1\}$ by D, we have a continuous surjection $\theta : P(K) \times P(K) \times D \longrightarrow M_1(K)$ given by $\theta(\mu, \nu, a, b) = a\mu - b\nu$, so, using 3.12.8 (f),(a) of [7] we get $t(M_1(K)) \leq t(P(K))$.

Let E be a Banach space equipped with its weak topology. Corson [5] introduced the following convex analogue of the Lindelöf property: Say that E has property (C) if for every family $\mathcal{C}$ of closed convex subsets of E we have $\bigcap \mathcal{C} \neq \emptyset$ provided every countable subfamily of $\mathcal{C}$ has nonempty intersection.

Clearly, E has property (C) whenever E is weakly Lindelöf (recall that closed= weakly closed for convex sets). The converse does not hold: Pol [17] showed that if K is the "two arrows space", then the space $C(K)$ has property (C) but is not weakly Lindelöf.

Pol [17] showed that E has property (C) if and only if the *weak** topology on E^* satisfies a certain condition of "convex countable tightness". If $E = C(K)$, that result reads as follows (cf. Pol [18], Lemma 3.2).

Theorem 2.1 (R. Pol)**.** *For a compact space K the following are equivalent:*

(i) *the space $C(K)$ has property (C);*

(ii) *whenever $\mu \in \overline{\mathfrak{M}}$, where $\mathfrak{M} \subseteq P(K)$, then there is $\mathfrak{N} \in [\mathfrak{M}]^{\leq\omega}$ such that $\mu \in \overline{\mathrm{conv}\,\mathfrak{N}}$.*

It is not clear whether condition (ii) from the theorem above is equivalent to saying that $P(K)$ has countable tightness (for arbitrary compact space K). This problem was investigated by Pol [18], who in particular showed the following.

Theorem 2.2 (R. Pol)**.** *Assume that K is a compact space such that every $\mu \in P(K)$ is countably determined. Then the space $C(K)$ has property (C) if and only if $t(P(K)) = \omega$.*

A measure $\mu \in P(K)$ is called *countably determined* if there is a countable family $\mathcal{F}$ of compact subsets of K such that $\mu(B) = \sup\{\mu(F) : F \subseteq B, F \in \mathcal{F}\}$ for every open set B. For arbitrary compact space K the following implications hold (cf. Pol [17, 18]):

$$(*) \qquad t(P(K)) = \omega \Longrightarrow C(K) \text{ has property (C)} \Longrightarrow t(K) = \omega.$$

Indeed, the first implication follows from Theorem 2.1 while the second may be checked in a straightforward way (given $A \subseteq K$, $x \in \overline{A}$, consider sets $\{g \in C(K) : g_{|B} = 0,\ g(x) = 1\}$, where $B \in [A]^{\le\omega}$). Moreover, if $C(K)$ has property (C) then every measure $\mu \in P(K)$ has a separable support. Indeed, otherwise there is a measure $\mu \in P(K)$ that vanishes on all separable closed subset of K, and considering the family $(C_x)_{x\in K}$, where

$$C_x = \{g \in C(K) : g(x) = 0,\ \mu(g) \ge 1\},$$

we see that $C(K)$ fails to have property (C).

Haydon [10] and Kunen [11] presented closely related results that are relevant here. They, in particular, showed that assuming the continuum hypothesis there exist a first-countable compact space K and a measure $\mu \in P(K)$ such that $L_1(\mu)$ is nonseparable and $\mu(M) = 0$ whenever $M \subseteq K$ is separable. In fact the resulting space K is Corson compact; see also [12] and [16] for similar constructions carried out under weaker axioms. As noticed in [17], for such a space K we have $t(K) = \omega$ while $C(K)$ fails to have property (C), and thus $P(K)$ has uncountable tightness.

It seems unknown whether the implication

$$t(K) = \omega \Longrightarrow C(K) \text{ has property (C)}$$

is relatively consistent. We shall show in the next section that, assuming Martin's axiom and the negation of the continuum hypothesis, if $C(K)$ has property (C), then $t(P(K)) = \omega$.

We enclose here the result not directly related to our main subject. Note that property (C) of a Banach space E may be weakened to saying that E is *realcompact* in its weak topology (that is, every maximal filter in the family of weak zero subsets of E, closed under countable intersection, is fixed); see Corson [5] and Edgar [6] for further information. The argument below closely follows the proof of Theorem 3.1 from [14]; see also Plebanek [15].

Theorem 2.3. *If K is a compact space of countable tightness, then the Banach space $C(K)$ is realcompact in its weak topology.*

Proof. We shall check that if $z \in C(K)^{**}$ is a functional which is $weak^*$ continuous on every $weak^*$ separable subspace of $C(K)^*$, then $z \in C(K)$. According to a result due to Corson [5], this characterizes realcompactness of the weak topology on $C(K)$.

Denote by $\delta_x \in P(K)$ the Dirac measure at $x \in K$. Define a function φ on K by the formula $\varphi(x) = z(\delta_x)$. Note that φ is continuous on every separable subspace of M of K, since z is $weak^*$ continuous on the subspace spanned by $(\delta_x)_{x\in M}$. Since $t(K) = \omega$, it follows that $\varphi \in C(K)$. Now it suffices to check that $z(\mu) = \mu(\varphi)$ for every $\mu \in P(K)$; in other words we are to prove that $w = z - \varphi = 0$.

Suppose the contrary; we may then assume that there is $\mu \in P(K)$ with $w(\mu) > 0$. Given a Borel set $B \subseteq K$, denote by μ_B the restriction of a measure μ to B. Using the Radon-Nikodym theorem to the signed μ-continuous measure $B \to w(\mu_B)$ we get a μ-measurable function g on K with $w(\mu_B) = \int_B g\,\mathrm{d}\mu$ for every Borel set B. Take $c > 0$ and a closed set $M \subseteq \{g \ge c\}$ so that $\mu(M) > 0$. Now the measure $\nu = \frac{1}{\mu(M)} \cdot \mu_M$ is such that $w(\nu_B) \ge c\,\nu(B)$ for all Borel sets B. We may assume that M is self-supporting, that is $\mu(M \cap V) > 0$ whenever $V \subseteq K$ is an open set with $V \cap M \ne \emptyset$.

By a result due to Shapirovskiĭ [19], there is a point $x \in M$ which has countable π-character in M. That means that there is a sequence $(U_n)_{n\in\omega}$ of open subsets of M such that for every neighbourhood of x contains U_n for some n. Letting $\nu_n = \frac{1}{\nu(U_n)}\nu_{U_n}$ and $\mathfrak{N} = (\nu_n)_{n\in\omega}$, one can easily check that $\delta_x \in \overline{\mathfrak{N}}$. Now we have $w(\nu_n) \geq c$, $w(\delta_x) = 0$, and this is a contradiction, since the functional w is *weak** continuous on $\overline{\mathfrak{N}}$.

3. On countable tightness of $P(K)$

We shall see in this section that countable tightness of $P(K)$ is somewhat related to separability of measures from $P(K)$. We say that a measure $\mu \in P(K)$ is *separable* if $L_1(\mu)$ is separable as a Banach space. Clearly, every countably determined measure is separable.

For the sake of the proof of our first lemma we recall the following property of tightness: If K is a compact space and $Z \subseteq K$ is a closed $\mathcal{G}_\delta$ set, then $t(x, Z) = t(x, K)$ for every $x \in Z$ (see [7], 3.12.8(c)).

Lemma 3.1. *Assume that K is a compact space and $\mu \in P(K)$ is separable. If $t(\mu, P(K)) > \omega$, then the Banach space $C(K)$ does not have property (C).*

Proof. We find and fix a countable family $\mathcal{F} \subseteq C(K)$ which is dense in $L_1(\mu)$. Applying the above remark to μ and the zero set

$$\mathfrak{Z} = \bigcap_{f\in\mathcal{F}} \{\nu \in P(K) : \ \nu(f) = \mu(f)\},$$

we have $t(\mu, \mathfrak{Z}) > \omega$. Thus we can find an uncountable family $\mathfrak{M} \subseteq \mathfrak{Z}$ such that $\mu \in \overline{\mathfrak{M}}$ while $\mu \notin \overline{\mathfrak{N}}$ whenever $\mathfrak{N} \in [\mathfrak{M}]^{\leq\omega}$.

Given functions $g_1, \dots , g_k \in C(K)$, and $\varepsilon > 0$, we let

$$V(g_1, \dots , g_k, \varepsilon) = \{\nu \in P(K) : \ (\forall i \leq k)|\nu(g_i) - \mu(g_i)| < \varepsilon\},$$

which is a *weak**-open neighbourhood of $\mu \in P(K)$.

Note that we can find a positive ε such that for every $\mathfrak{N} \in [\mathfrak{M}]^{\leq\omega}$ there are a number n and functions $g_1, \dots , g_n \in C(K)$ such that

$$(*) \qquad \mathfrak{N} \cap V(g_1, \dots , g_n, 3\varepsilon) = \emptyset.$$

This follows from the fact that the closure of no countable family in $\mathfrak{M}$ contains μ.

For any measure $\nu \in P(K)$ the set

$$Z(\nu) = \{g \in C(K) : \ \nu(g) \geq 2\varepsilon, \ \mu(g) \leq \varepsilon\}$$

is convex and closed in $C(K)$. Since $\mu \in \overline{\mathfrak{M}}$, we have $\bigcap_{\nu\in\mathfrak{M}} Z(\nu) = \emptyset$. We shall check that $C(K)$ does not have property (C) by showing that the set $\bigcap_{\nu\in\mathfrak{N}} Z(\nu)$ is nonempty whenever $\mathfrak{N} \in [\mathfrak{M}]^{\leq\omega}$.

For a fixed countable $\mathfrak{N} \subseteq \mathfrak{M}$, we find $g_1, \dots , g_n \in C(K)$ satisfying $(*)$. Letting $\delta = \varepsilon/n$, we take functions $f_i \in \mathcal{F}$ such that $\int |f_i - g_i| \, d\mu < \delta$ for $i = 1, 2, \dots , n$. Now consider a function $h = \sum_{i=1}^{n} |f_i - g_i|$.

We have $\mu(h) \leq \delta n = \varepsilon$. On the other hand, for any $\nu \in \mathfrak{N}$ there is $j \leq n$ such that $|\nu(g_j) - \mu(g_j)| \geq 3\varepsilon$, and hence

$$\begin{aligned} \nu(h) \geq |\nu(g_j) - \nu(f_j)| &\geq |\nu(g_j) - \mu(g_j)| - |\nu(f_j) - \mu(g_j)| \\ &\geq 3\varepsilon - |\mu(f_j) - \mu(g_j)| \geq 2\varepsilon. \end{aligned}$$

We have shown that $h \in \bigcap_{\nu\in\mathfrak{N}} Z(\nu)$, and this finishes the proof.

The above lemma and Theorem 2.1 yield the following extension of Theorem 2.2.

Theorem 3.2. *Assume that K is a compact space such that every $\mu \in P(K)$ is separable. Then the space $C(K)$ has property (C) if and only if $t(P(K)) = \omega$.*

As we mentioned before under CH there are first-countable compact spaces carrying nonseparable Radon measures. The situation is quite different under Martin's axiom (MA) and ¬CH. The following result due to Fremlin [9] has solved so-called Haydon's problem (cf. [12], [16]).

Theorem 3.3 (D. Fremlin)**.** *Assume MA and ¬CH. The following are equivalent for a compact space K:*
(i) *there is a nonseparable measure $\mu \in P(K)$;*
(ii) *there is a continuous surjection from K onto $[0,1]^{\omega_1}$.*
In particular, if $t(K) = \omega$, then every measure $\mu \in P(K)$ is separable.

Theorem 3.4. *Under MA and ¬CH, the following two conditions are equivalent for* every *compact space K:*
(i) *the space $P(K)$ has countable tightness;*
(ii) *the Banach space $C(K)$ has property (C).*

Proof. If $C(K)$ has property (C), then K has countable tightness (see the previous section). Hence every Radon measure on K is separable by Theorem 3.3, and $t(P(K)) = \omega$ by Theorem 3.2.

Lemma 3.5. *Let K be a compact zero-dimensional space. If $C(K)$ is Lindelöf in its weak topology, then every measure $\mu \in P(K)$ is separable.*

Proof. Suppose that there is a nonseparable Radon measure on K. We can then find a measure $\mu \in P(K)$ which is homogenous of uncountable Maharam type. By the Maharam theorem (see [8]), the measure algebra of μ contains a subalgebra isomorphic to the measure algebra of the usual product measure on 2^{ω_1}. It follows that one can find a family $(B_\alpha)_{\alpha<\omega_1}$ of Borel subsets of K with $\mu(B_\alpha) = 1/2$, which are independent with respect to μ.

We fix a positive constant $\varepsilon \le 1/16$. Next we find for every $\alpha < \omega_1$ a closed and open set $V_\alpha \subseteq K$, such that $\mu(B_\alpha \triangle V_\alpha) < \varepsilon$. Since

$$V_\alpha \cap V_\beta^c \supseteq B_\alpha \cap B_\beta^c \setminus [(B_\alpha \triangle V_\alpha) \cup (B_\beta^c \triangle V_\beta^c)],$$

we have $\mu(V_\alpha \cap V_\beta^c) \ge 1/4 - 2\varepsilon$ whenever $\alpha \ne \beta$.

We let $\mathcal{F}$ be the weak closure in $C(K)$ of the family of continuous functions $(\chi_{V_\alpha})_{\alpha<\omega_1}$. Note that every $f \in \mathcal{F}$ may be written as $f = \chi_F$, where $F \subseteq K$ is closed and open (and $|\mu(F) - 1/2| \le \varepsilon$). We define

$$U(F) = \{g \in C(K) : \int_{F^c} g \, \mathrm{d}\mu < \varepsilon\},$$

for every $\chi_F \in \mathcal{F}$. Obviously, $U(F)$ is a weak open neighbourhood of χ_F. Thus $(U(F))_{\chi_F \in \mathcal{F}}$ is an open cover of $\mathcal{F}$; we shall prove that $\mathcal{F}$ (and hence $C(K)$) is not Lindelöf, by checking that no countable family of sets of the form $U(F)$ covers $\mathcal{F}$.

Consider a sequence $(F_k)_{k\in\omega}$ of sets, the characteristic functions of which lie in $\mathcal{F}$. For every k there is $\alpha_k < \omega_1$ such that $\mu(F_k \setminus V_{\alpha_k}) < \varepsilon$. Indeed, the set

$$\{g \in C(K) : \int_{F_k} g \, \mathrm{d}\mu > \mu(F_k) - \varepsilon\}$$

is an open neighbourhood of χ_{F_k} so for some $\alpha_k < \omega_1$ we have

$$\int_{F_k} \chi_{V_{\alpha_k}} \, \mathrm{d}\mu > \mu(F_k) - \varepsilon,$$

which gives $\mu(F_k \cap V_{\alpha_k}) > \mu(F_k) - \varepsilon$, and $\mu(F_k \setminus V_{\alpha_k}) < \varepsilon$.

Now take any $\eta \in \omega_1 \setminus \{\alpha_k : k \in \omega\}$. We have

$$\mu(V_\eta \setminus F_k) \geq \mu(V_\eta \setminus V_{\alpha_k}) - \mu(F_k \setminus V_{\alpha_k}) \geq 1/4 - 2\varepsilon - \varepsilon \geq \varepsilon.$$

Hence $\int_{F_k^c} \chi_{V_\eta} \, \mathrm{d}\mu \geq \varepsilon$ for every k, which means that $\chi_{V_\eta} \notin \bigcup_{k\in\omega} U(F_k)$, and the proof is complete.

Lemma 3.1 and Lemma 3.5 give immediately the following.

Theorem 3.6. *Let K be a compact zero-dimensional space. If the Banach space $C(K)$ is Lindelöf in its weak topology, then $t(P(K)) = \omega$.*

We do not know if the above theorem holds true for an arbitrary compact space K (not necessarily zero dimensional).

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