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APPROXIMATING DISCRETE VALUATION RINGS BY REGULAR LOCAL RINGS

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ABSTRACT. Let k be a field of characteristic zero and let $(V, \mathbf{n})$ be a discrete rank-one valuation domain containing k with $V/\mathbf{n} = k$. Assume that the fraction field L of V has finite transcendence degree s over k. For every positive integer $d \leq s$, we prove that V can be realized as a directed union of regular local k-subalgebras of V of dimension d.

1. INTRODUCTION

Suppose $(R, \mathbf{m})$ is a local Noetherian domain with fraction field F and $(V, \mathbf{n})$ is a valuation domain that *birationally dominates* R in the sense that R is a subring of V, V has fraction field F and $\mathbf{n} \cap R = \mathbf{m}$. Classically, in the case where R is essentially finitely generated over a field, *local uniformization of R along V*, in algebraic terms, means the existence of a regular local domain S between R and F, such that S is dominated by V and is essentially finitely generated over R. If R is a regular local ring and P is a prime ideal of R, *embedded local uniformization* translates to the existence of S as above having a prime ideal Q with $Q \cap R = P$ and S/Q a regular local domain.

The consideration of embedded local uniformization yields a representation of the valuation domain V as a directed union of regular local rings. If $V/\mathbf{n}$ is algebraic over $R/\mathbf{m}$, then classical methods give a representation of V as a directed union of regular local rings all having dimension equal to the dimension of R [A, Lemma 12]. In this article we prove that certain rank-one discrete valuation rings (DVRs) can be represented as a directed union of regular local domains of dimension d for every positive integer d less than or equal to the dimension of R. We use a technique inspired by Nagata [N1] and developed in our earlier work for the construction of new Noetherian domains [HRW1], [HRW2], [HRW3].

The classical approach for obtaining embedded local uniformization, introduced by Zariski in the 1940's [Z1], uses local quadratic transforms of R along V. The

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first local quadratic transform of R along V is defined to be $R_1 = R[\mathbf{m}/a]_{\mathbf{m}_1}$, where $a \in \mathbf{m}$ is such that $\mathbf{m}V = aV$ and $\mathbf{m}_1 := \mathbf{n} \cap R[\mathbf{m}/a]$. R_1 is also called the *dilatation of R by the ideal* $\mathbf{m}$ *along* V [N2, page 141]. More generally, if $I \subseteq \mathbf{m}$ is a nonzero ideal of R, the *dilatation of R by I along V* is the ring $R[I/a]_{\mathbf{m}_1}$, where $a \in I$ is such that $IV = aV$ and $\mathbf{m}_1 = \mathbf{n} \cap R[I/a]$. It is straightforward to check that R_1 is uniquely determined by R, V and the ideal I [N2, page 141]. For each positive integer i, we define the $(i+1)$th *quadratic transform* R_{i+1} *of R along* V inductively: R_{i+1} is the first local quadratic transform of R_i along V. Then $R_{i+j} = (R_i)_j$ for all $i, j \geq 0$.

Using monoidal transforms (a special type of dilatation), S. Cutkosky in [C1] and [C2] has recently done interesting work on birational factorization and local uniformization.

1.1 Background remarks.

(1) If R is a regular local ring, it is well known that a quadratic transform R_1 of R is again a regular local ring [N2, (38.1)].

(2) If R is a pseudogeometric regular local domain of dimension 2 and V is a valuation domain that birationally dominates R, a classical result of Zariski and Abhyankar asserts that $V = \bigcup_{n=1}^{\infty} R_n$ [A, Lemma 12].

(3) Suppose R is a regular local ring of dimension $d \geq 3$. For certain valuation rings V that birationally dominate R, the union $\bigcup_{n=1}^{\infty} R_n$ of the local quadratic transforms of R along V is strictly smaller than V [S, (4.13)]. In many cases Shannon in [S, (4.5), page 308] proves that V is a directed union of iterated monoidal transforms of R, where a *monoidal transform of R* is a dilatation of R by a prime ideal P for which R/P is again regular.

(4) If $(V, \mathbf{n})$ is a DVR that birationally dominates a regular local ring R, then $V = \bigcup_{n=1}^{\infty} R_n$,[1] where R_n is the nth quadratic transform of R along V [A, page 336], [Z2, page 27-28].

A proof of item (4) goes as follows: a nonzero element η of V has the form $\eta = b/c$, where $b, c \in R$. If $(b,c)V = V$, then $b/c \in V$ implies $cV = V$. Since V dominates R, it follows that $cR = R$, so $b/c \in R$ in this case. If $\eta = b/c$, with $b, c \in R$ and $(b,c)V = \mathbf{n}^k$, we prove by induction that $\eta \in R_k$. We have already done the case where $k = 0$. Assume for every regular local domain $(S, \mathbf{p})$ birationally dominated by V, and every nonzero element $\beta/\gamma \in V$ with $\beta, \gamma \in S$, $(\beta, \gamma)V = \mathbf{n}^j$ and $0 \leq j < k$ we have $\beta/\gamma \in S_j$, where S_j is the jth iterated local quadratic transform of S along V. Suppose $\beta, \gamma \in S$, $\beta/\gamma \in V$ with $(\beta, \gamma)V = \mathbf{n}^k$. Let $S_1 = S[\mathbf{p}/a]_{\mathbf{p}_1}$, where $a \in \mathbf{p}$ is such that $\mathbf{p}V = aV$ and $\mathbf{p}_1 := \mathbf{n} \cap S[\mathbf{p}/a]$; that is, S_1 is the first local quadratic transform of S along V. Then $\beta_1 := \beta/a$ and $\gamma_1 := \gamma/a$ are in S_1. Thus $a \in \mathbf{n}$ implies $(\beta_1, \gamma_1)V = \mathbf{n}^j$ where $0 \leq j < k$, so by induction

$$\beta/\gamma = \beta_1/\gamma_1 \in (S_1)_j = S_{j+1} \subseteq S_k,$$

which completes the proof.

In this article we prove the following theorem:

1.2 Main theorem. *Let k be a field of characteristic zero and let $(V, \mathbf{n})$ be a DVR containing k with $V/\mathbf{n} = k$. Assume that the fraction field L of V has finite transcendence degree s over k. Then for every integer d with $1 \leq d \leq s$, there*

[1] It may happen that $R_n = V$ for some n, in which case $R_{n+i} = R_n$ for all $i \geq 0$.

exists a nested family $\{C_n^{(\alpha)} : n \in \mathbb{N}, \alpha \in \Gamma\}$ *of* d*-dimensional regular local* k*-subalgebras of* V *such that* V *is the directed union of the* $C_n^{(\alpha)}$ *and* V *dominates each* $C_n^{(\alpha)}$.

If the field L *is finitely generated over* k*, then* V *is a countable union* $\bigcup_{n=1}^{\infty} C_n$*, where, for each* $n \in \mathbb{N}$*,*

(1) C_n *is a* d*-dimensional regular local* k*-subalgebra of* V*,*
(2) C_n *has fraction field* L*,*
(3) C_{n+1} *dominates* C_n *and*
(4) V *dominates* C_n.

We have the following corollary to the main theorem.

1.3 Corollary. *Let* k *be a field of characteristic zero and let* $(R, \mathbf{m})$ *be a local domain essentially of finite type over* k *with coefficient field* $k = R/\mathbf{m}$ *and field of fractions* L*. Let* $(V, \mathbf{n})$ *be a DVR birationally dominating* R *with* $V/\mathbf{n} = k$*. For every integer* d *with* $1 \le d \le s$*,* $s = trdeg_k(L)$*, there exists a sequence of* d*-dimensional regular local* k*-subalgebras* C_n *of* V *such that* $V = \bigcup_{n=1}^{\infty} C_n$*, and for each* $n \in \mathbb{N}$*,* C_{n+1} *dominates* C_n *and* V *dominates* C_n*. Moreover* C_n *dominates* R *for all sufficiently large* n*.*

Proof. By Theorem 1.2, $V = \bigcup_{n=1}^{\infty} C_n$, where the C_n are regular local rings satisfying all the conditions of Corollary 1.3 except possibly that of dominating R. Since R is essentially of finite type over k and is dominated by V, it follows that R is dominated by C_n for all sufficiently large n. □

1.4 Discussion. (1) If L/k is finitely generated of transcendence degree s, then the fact that V is a directed union of s-dimensional regular local rings follows from classical theorems of Zariski. The local uniformization theorem of Zariski [Z1] implies the existence of a regular local domain $(R, \mathbf{m})$ containing the field k such that V birationally dominates R. Since k is a coefficient field for V, we have

(a) $k \hookrightarrow V \twoheadrightarrow V/\mathbf{n} \cong k$; thus k is relatively algebraically closed in L.
(b) $R/\mathbf{m} = k$ (because V dominates R).
(c) Every iterated local quadratic transform of R along V has dimension s.

Now by 1.1(4), V is a directed union of s-dimensional RLRs.

(2) If $d = 1$, the main theorem is trivially true by taking each $C_n = V$. Thus if L/k is finitely generated of transcendence degree $s = 2$, then the theorem is saying nothing new.

(3) If $s > 2$, then the classical local uniformization theorem says nothing about expressing V as a directed union of d-dimensional RLRs, where $2 \le d \le s - 1$. If $(S, \mathbf{p})$ is a local Noetherian domain containing k and birationally dominated by V with $\dim(S) = d < s$, then S does not satisfy the dimension formula. It follows that S is not essentially finitely generated over k [M1, page 119].

We use the following notation in the proof of the main theorem.

1.5 Endpieces and related localized polynomial rings. Let $(R, \mathbf{m})$ be a local Noetherian domain with fraction field F. Let $y \in \mathbf{m}$ be a nonzero element, let $R^* = \widehat{(R, (y))}$ be the (y)-adic completion of R. Suppose $\tau_1, \ldots, \tau_s \in yR^*$ are regular elements[2] of R^* that are algebraically independent over K. We consider

[2] We say an element of a ring is a *regular element* if it is not a zero divisor.

the localized polynomial ring

$$B_0 := R[\tau_1, \dots, \tau_s]_{(\mathbf{m}, \tau_1, \dots, \tau_s)}.$$

It is clear that B_0 is a local Noetherian domain with $\dim(B_0) = \dim(R) + s$ such that R^* dominates B_0 and B_0 dominates R.

For every $\gamma \in R^*$ and every $n > 0$, we define the nth-endpiece γ_n with respect to y of γ to be

$$(1.5.1) \qquad \gamma_n := \sum_{j=n+1}^{\infty} c_j y^{j-n}, \text{ where } \gamma := \sum_{j=1}^{\infty} c_j y^j \text{ and each } c_j \in R.$$

In particular, we represent each of the τ_i by a power series expansion in y; we use these representations to obtain for each positive integer n the nth-endpieces τ_{in} and corresponding nth-localized polynomial ring B_n. For $1 \le i \le s$, and $\tau_i := \sum_{j=1}^{\infty} a_{ij} y^j$, where the $a_{ij} \in R$, we have

$$\tau_{in} := \sum_{j=n+1}^{\infty} a_{ij} y^{j-n}, \qquad B_n := R[\tau_{1n}, \dots, \tau_{sn}]_{(\mathbf{m}, \tau_{1n}, \dots, \tau_{sn})}, \text{ for each } n \in \mathbb{N}.$$

The definition of B_n is independent of the representation of the τ_i as power series in y [HRW1, (2.3)]. The localized polynomial ring B_{n+1} birationally dominates B_n. We define

$$(1.5.2) \qquad B := \bigcup_{n=0}^{\infty} B_n = \varinjlim B_n \qquad \text{and} \qquad A := F(\tau_1, \dots, \tau_s) \cap R^*.$$

It is readily seen that A birationally dominates B.

In the proof of our main theorem, we make use of the following result from previous work that we summarize as Theorem 1.6.

1.6 Theorem ([HRW1] or [HRW2]). *Suppose $(R, \mathbf{m})$ is a local Noetherian domain with fraction field F, $y \in \mathbf{m}$ is a nonzero element, and $\tau_1, \dots, \tau_s$ are elements of the (y)-adic completion R^* of R that are algebraically independent over R. Then the following conditions are equivalent:*

1. *The ring $R^*[1/y]$ is flat over $B_0 = R[\tau_1, \dots, \tau_s]_{(\mathbf{m}, \tau_1, \dots, \tau_s)}$.*
2. *The ring $R^*[1/y]$ is flat over $R[\tau_1, \dots, \tau_s]$.*
3. *The directed union $B := \bigcup_{n=0}^{\infty} B_n$ is Noetherian.*

Moreover, if these equivalent conditions hold, then the domain $A := F(\tau_1, \dots, \tau_s) \cap R^*$ is equal to B, and thus both are Noetherian.

2. Proof of the main theorem

Let $y \in \mathbf{n}$ be such that $yV = \mathbf{n}$. Then the $\mathbf{n}$-adic completion $\widehat{V}$ of V is $k[[y]]$. Hence we have

$$k[y]_{(y)} \subseteq V \subseteq k[[y]].$$

Then $V = L \cap k[[y]]$, for example, since $V \hookrightarrow k[[y]]$ is flat. Since $\operatorname{trdeg}_{k(y)} L = s-1$, there are $s-1$ elements $\sigma_1, \dots, \sigma_{s-d}, \tau_1, \dots, \tau_{d-1} \in yV$ such that L is algebraic over $F := k(y, \sigma_1, \dots, \sigma_{s-d}, \tau_1, \dots, \tau_{d-1})$.

Set $K := k(y, \sigma_1, \dots, \sigma_{s-d})$ and $R := V \cap K$. Then R is a DVR and the (y)-adic completion of R is $R^* = k[[y]]$. In the notation of 1.5, $B_0 := R[\tau_1, \dots, \tau_{d-1}]_{(y, \tau_1, \dots, \tau_{d-1})}$ is a d-dimensional regular local ring and $V_0 := V \cap F$ is a DVR that birationally dominates B_0 and has y-adic completion $\widehat{V_0} = k[[y]]$. The following diagram displays these domains:

$$\begin{array}{ccccccccc}
k & \xrightarrow{\subseteq} & K & \xrightarrow{\subseteq} & F & = & F & \xrightarrow{\subseteq} & L := \mathcal{Q}(V) \\
\| \uparrow & & \cup\!\!\mid \uparrow & & \cup\!\!\mid \uparrow & & \cup\!\!\mid \uparrow & & \cup\!\!\mid \uparrow \\
k & \xrightarrow{\subseteq} & R := V \cap K & \xrightarrow{\subseteq} & B_0 & \xrightarrow{\subseteq} & V_0 := V \cap F & \xrightarrow{\subseteq} & V
\end{array}$$

Let τ_{in} be the nth-endpiece of τ_i for each i; since $V_0 = F \cap k[[y]]$, the $\tau_{in} \in V_0$ and $B_n = R[\tau_{1n}, \ldots, \tau_{d-1,n}]_{(y,\tau_{1n},\ldots,\tau_{d-1,n})}$ is, for each $n \in \mathbb{N}$, the first quadratic transform of B_{n-1} along V_0.

Now R^*_y is a field and thus is flat as an $R[\tau_1, \ldots, \tau_{d-1}]$-module. By Theorem 1.6, $V_0 = \bigcup_{n=1}^{\infty} B_n$. An alternate way to justify this description of V_0 is to use the statement 1.1(4) where the R of 1.1(4) is B_0, the V of 1.1(4) is V_0, and each $R_n = B_n$. We have

$$V_0 \underset{\text{alg}}{\longrightarrow} V \longrightarrow k[[y]],$$

where V_0 and V are DVRs of characteristic zero having completion $k[[y]]$. Since V_0 and V are excellent, their Henselizations V_0^h and V^h are the set of elements of $k[[y]]$ algebraic over V_0 or V [N2, (44.3)]. Thus $V_0^h = V^h$ and V is a directed union of étale extensions of V_0 [R, (1.3)].

2.1 Proposition. *Let B_n^h denote the Henselization of B_n. Then $\bigcup_{n=1}^{\infty} B_n^h = V_0^h = V^h$.*

Proof. The ring $C := \bigcup B_n^h$ is Henselian and contains V_0, so $V_0^h = V^h \subseteq C$. Moreover, the inclusion map $V \to C = \bigcup B_n^h$ extends to a map $V^h \xrightarrow{\sigma} C = \bigcup B_n^h$.

On the other hand, the maps $B_n \to V$ extend to maps $B_n^h \to V^h$ yielding a map $\rho : C \to V^h$ with $\sigma\rho = 1_C$, and $\rho\sigma = 1_{V^h}$. Thus $\bigcup_{n=1}^{\infty} B_n^h = V^h$. □

Continuation of the proof of the main theorem. First assume L is finitely generated over k. It follows that L is finite algebraic over F. Since $\bigcup_{n=1}^{\infty} B_n^h = V^h$, we have $\bigcup_{n=1}^{\infty} \mathcal{Q}(B_n^h) = \mathcal{Q}(V^h)$ and $L \subseteq \mathcal{Q}(V^h)$. Since L/F is finite algebraic, $L \subseteq \mathcal{Q}(B_n^h)$ for all sufficiently large n. By relabeling, we may assume $L \subseteq \mathcal{Q}(B_n^h)$ for all n. Let $C_n := B_n^h \cap L$. Since B_n is a regular local ring, C_n is a regular local ring with $C_n^h = B_n^h$ [R, (1.3)].

Claim. For every n, C_{n+1} dominates C_n and V dominates C_n. Also $\bigcup_{n=1}^{\infty} C_n = V$.

Proof of the claim. Since B_{n+1} dominates B_n, we see that B_{n+1}^h dominates B_n^h and hence $C_{n+1} = B_{n+1}^h \cap L$ dominates $C_n = B_n^h \cap L$. Since $C_n = B_n^h \cap L \subseteq V^h \cap L = V$, it follows that V dominates C_n and $V_0 \subseteq \bigcup_{n=1}^{\infty} C_n \subseteq V$. Since V birationally dominates $\bigcup_{n=1}^{\infty} C_n$, it suffices to show that $\bigcup_{n=1}^{\infty} C_n$ is a DVR.

But by the same argument as before, $\bigcup_{n=1}^{\infty} C_n^h = (\bigcup_{n=1}^{\infty} C_n)^h = V^h$. This shows that $\bigcup_{n=1}^{\infty} C_n$ is a DVR, and therefore $\bigcup_{n=1}^{\infty} C_n = V$. Thus in the case where L/k is finitely generated we have completed the proof of the main theorem including the "Moreover" statement.

2.2 *Remark.* An alternate approach to the definition of C_n is as follows. Since V is a directed union of étale extensions of V_0 and $\mathbb{Q}(V) = L$ is finite algebraic over $\mathbb{Q}(V_0) = F$, V is étale over V_0 and therefore $V = V_0[\theta] = V_0[X]/(f(X))$, where $f(X)$ is a monic polynomial such that $f(\theta) = 0$ and $f'(\theta)$ is a unit of V. Let B_n' denote the integral closure of B_n in L and let $C_n = (B'_n)_{(\mathbf{n} \cap B'_n)}$. Since $\bigcup_{n=1}^{\infty} B_n = V_0$, it follows that $\bigcup_{n=1}^{\infty} C_n = V$. Moreover, for all sufficiently large n, $f(X) \in B_n[X]$ and $f'(\theta)$ is a unit of C_n. Therefore C_n is a regular local ring for all sufficiently large n [N2, (38.6)]. As we note in Remark 2.3 below, this allows us to deduce a version of the main theorem also in the case where k has characteristic $p > 0$, provided the field F can be chosen so that L/F is separable.

Completion of the proof of the main theorem. If L is not finitely generated over k, we choose a nested family of fields L_α, with $\alpha \in \Gamma$, such that

(1) $F \subseteq L_\alpha$, for all α.
(2) L_α is finite algebraic over F.
(3) $\bigcup_{\alpha \in \Gamma} L_\alpha = L$.

The rings $V_\alpha = L_\alpha \cap V$ are DVRs with $\bigcup_{\alpha \in \Gamma} V_\alpha = V$ and $V_\alpha^h = V^h$, since $V_0 \subseteq V_\alpha$, for each $\alpha \in \Gamma$.

As above, $\bigcup_{n=1}^{\infty} B_n^h = V^h$, $\bigcup_{n=1}^{\infty} \mathcal{Q}(B_n^h) = \mathcal{Q}(V^h)$ and $L \subseteq \mathcal{Q}(V^h)$. Thus we see that for each $\alpha \in \Gamma$, there is an $n_\alpha \in \mathbb{N}$ such that $L_\alpha \subseteq \mathcal{Q}(B_n^h)$ for all $n \geq n_\alpha$.

Put $C_n^{(\alpha)} = L_\alpha \cap B_n^h$ for each $n \geq n_\alpha$. Then $V_\alpha = \bigcup_{n=n_\alpha}^{\infty} C_n^{(\alpha)}$ and V_α birationally dominates $C_n^{(\alpha)}$. Hence

$$V = \bigcup_{\alpha \in \Gamma, n \geq n_\alpha} C_n^{(\alpha)}.$$

This completes the proof of the main theorem. □

2.3 *Remark.* If the characteristic of k is $p > 0$, then the Henselization V_0^h of $V_0 = F \cap k[[y]]$ may not equal the Henselization V^h of $V = L \cap k[[y]]$, because the algebraic field extension L/F may not be separable. But in the case where L/F is separable algebraic, the fact that the DVRs V and V_0 have the same completion implies that V is a directed union of étale extensions of V_0 (see, for example, [AH, Theorem 2.7]). Therefore in the case where L/F is separable algebraic, V is a directed union of regular local rings of dimension d.

Thus for a local domain $(R, \mathbf{m})$ essentially of finite type over a field k of characteristic $p > 0$, a result analogous to Corollary 1.3 is true provided there exists a subfield F of L such that F is purely transcendental over k, L/F is separable algebraic, and F contains a generator for the maximal ideal of V.

In characteristic $p > 0$, with V excellent and the extension separable, the ring V_0 need not be excellent (see for example [HRS, (3.3) and (3.4)]).

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