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SERIAL SUBALGEBRAS OF FINITARY LIE ALGEBRAS

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Abstract. A Lie subalgebra L of $\mathfrak{gl}_{\mathbb{K}}(V)$ is said to be *finitary* if it consists of elements of finite rank. We show that, if L acts irreducibly on V, and if V is infinite-dimensional, then every non-trivial ascendant Lie subalgebra of L acts irreducibly on V too. When $\operatorname{Char}\mathbb{K} \neq 2$, it follows that the locally solvable radical of such L is trivial. In general, locally solvable finitary Lie algebras over fields of characteristic $\neq 2$ are hyperabelian.

1. Introduction

Let V be a vector space over the field $\mathbb{K}$. The endomorphisms of finite rank form an ideal in $\operatorname{End}_{\mathbb{K}}(V)$, which becomes a locally finite Lie algebra with respect to the usual Lie bracket $[a,b] = ab - ba$ (see [14, p. 32]). We shall denote this Lie algebra by $\mathfrak{fgl}_{\mathbb{K}}(V)$. A Lie algebra L is said to be *finitary* if there exist a field $\mathbb{K}$ and a $\mathbb{K}$-vector space V such that L is isomorphic to a Lie subalgebra of $\mathfrak{fgl}_{\mathbb{K}}(V)$.

The study of finitary Lie algebras is, in part, motivated by the wealth of results available about *finitary linear* groups, that is, subgroups of $\mathrm{GL}_{\mathbb{K}}(V)$ consisting of elements g such that the endomorphism $g-1$ has finite rank (see [12]). In fact, there is the following relationship between finitary linear groups and finitary Lie algebras. If G is a finitary linear subgroup of $\mathrm{GL}_{\mathbb{K}}(V)$, and if $\mathbb{K}G$ denotes the associative algebra generated by G inside $\operatorname{End}_{\mathbb{K}}(V)$, then the subalgebra

$$[\mathbb{K}G, \mathbb{K}G] = \sum_{g,h\in G} \mathbb{K}[g,h] \; = \sum_{g,h\in G} \mathbb{K}[g-1, h-1]$$

consists of elements of finite rank, so that it becomes a Lie subalgebra of $\mathfrak{fgl}_{\mathbb{K}}(V)$.

Finitary Lie algebras have recently attracted some interest. In [2], [3], [4] simple finitary Lie algebras are investigated. In [13] it is shown that the locally solvable radical of an irreducible Lie subalgebra L of $\mathfrak{fgl}_{\mathbb{K}}(V)$ is trivial whenever V has infinite $\mathbb{K}$-dimension and L is generated by elements of bounded rank. The purpose of this paper is to study serial Lie subalgebras in finitary Lie algebras without specific restrictions on a set of generators.

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Theorem A. *Let L be an irreducible Lie subalgebra of $\mathfrak{fgl}_{\mathbb{K}}(V)$, where $\dim_{\mathbb{K}} V$ is infinite.*

(a) *Let S be a serial Lie subalgebra of L. Either S is nil, or every S-composition series in V has finite length, bounded by the smallest rank of a non-nilpotent element in S.*
(b) *Every non-trivial ascendant Lie subalgebra of L acts irreducibly on V.*
(c) *Every locally nilpotent serial (resp. ascendant) Lie subalgebra of L is nil (resp. trivial). In particular, the Hirsch-Plotkin radical of L is trivial.*
(d) *If $\operatorname{Char}\mathbb{K} \neq 2$, then every locally solvable serial (resp. ascendant) Lie subalgebra of L is nil (resp. trivial). In particular, the locally solvable radical of L is trivial.*

Note that the situation is quite different for finitary linear groups. The infinite iterated restricted wreath product $C_p \mathrm{wr} C_p \mathrm{wr} C_p \mathrm{wr} ...$ of cyclic groups C_p of prime order p is a locally nilpotent group which can be represented as a totally imprimitive irreducible finitary linear group over any field of characteristic different from p containing a p-th root of unity. In fact, Theorem A shows that infinite-dimensional irreducible finitary Lie algebras behave pretty much like non-linear primitive periodic finitary linear groups. There, every non-trivial ascendant subgroup is irreducible [10, Proposition 7.4]. In analogy with the group situation we therefore conjecture that every infinite-dimensional irreducible finitary Lie algebra embeds into the Lie algebra of derivations of a simple finitary Lie algebra (cf. [7, Theorem B]).[1]

Theorem B. *Let L be a finitary Lie algebra. If $\operatorname{Char}\mathbb{K} = 2$, let S denote the Hirsch-Plotkin radical in L. Otherwise let S be the locally solvable radical in L. Then S is the extension of the largest nil ideal in L by a subdirect sum of finite-dimensional Lie algebras. Moreover S is the union of an ascending chain of ideals in L with abelian factors.*

Note that even over algebraically closed fields of positive characteristic, non-abelian nilpotent Lie algebras admit irreducible representations of degree larger than one [16].

All notions in our paper are standard and can be found in [1], [5], [15].

2. Proof of Theorem A

A Lie subalgebra S of a Lie algebra L is said to be a *subideal* in L if there exists a finite series $S = S_0 \leq S_1 \leq ... \leq S_n = L$ (where the expression *series* comprises that each S_i is an ideal in S_{i+1}). As usual, an L-module V is said to have *finite length* if it has an L-composition series with finitely many factors. Recall also that the so-called theorem of Jordan and Hölder applies to modules of finite length.

The proof of Theorem A depends on the following key observation.

Proposition 1. *Let L be an irreducible Lie subalgebra of $\mathfrak{fgl}_{\mathbb{K}}(V)$, and let S be a subideal in L. If the S-module V has finite length greater than one, then V is finite-dimensional, and the S-composition factors in V are mutually isomorphic.*

Proof. We may assume without loss that S is an ideal in L. Because V is an S-module of finite length, there exists a minimal S-submodule M in V. Then

[1] *Added in proof.* This conjecture has been confirmed in the meantime over fields of characteristic zero [8] and over fields of characteristic $p > 7$ [9].

$Ma \not\subseteq M$ for some $a \in L$. The inclusion $MaS \subseteq MSa + M[S,a] \subseteq Ma + M$ shows that $Ma + M$ is S-invariant. Consider the $\mathbb{K}$-epimorphism $\varphi : M \longrightarrow (Ma + M)/M$ given by $m\varphi = ma + M$ for all $m \in M$. For every $s \in S$ we have $msa - mas = m[s,a] \in M$, whence φ is an S-epimorphism. Since M is an irreducible S-module, φ is bijective, and $(Ma + M)/M \cong_S M$. However, a has finite rank, and so $(Ma + M)/M$ and M are finite-dimensional. It follows that $\dim_{\mathbb{K}} Ma \leq \dim_{\mathbb{K}} M = \dim_{\mathbb{K}} Ma/(Ma \cap M) \leq \dim_{\mathbb{K}} Ma$. Thus, $Ma + M$ is the direct sum of Ma and M.

Let $M_1 = M + ML$. The theorem of Jordan and Hölder implies that M_1/M is the direct sum of finitely many irreducible S-modules $(Ma+M)/M$ (where $a \in L$) isomorphic to M. We now define $M_i = M + ML + ... + ML^i$ and show by induction that M_{i+1} is S-invariant and that M_{i+1}/M_i is a direct sum of finitely many irreducible S-modules $(Ma_1...a_{i+1} + M_i)/M_i$ (where $a_1, ..., a_{i+1} \in L$) isomorphic to M. In the induction step, let $b = a_1...a_i$ and $a = a_{i+1}$. Suppose that $Mba \not\subseteq M_i$. The inclusion $MbaS \subseteq MbSa + Mb[S,a] \subseteq M_ia + M_i$ shows that $Mba + M_i$ is S-invariant. As above, the $\mathbb{K}$-epimorphism $(Mb + M_{i-1})/M_{i-1} \longrightarrow (Mba + M_i)/M_i$, given by $(mb + M_{i-1}) \longmapsto mba + M_i$ for all $m \in M$, is an S-isomorphism, and the result follows.

Since the S-module V has finite length, $V = M_n$ for some n. □

We now aim to reduce our considerations to Lie algebras over algebraically closed fields. The next result can be proved in precisely the same way as its counterpart for finitary linear groups [6].

Proposition 2. *Let L be an irreducible Lie subalgebra of $\mathfrak{fgl}_{\mathbb{K}}(V)$, and let $\tilde{\mathbb{K}}$ be a maximal subfield of the division ring $\operatorname{End}_L(V)$.*

(a) *Then $\tilde{L} = \tilde{\mathbb{K}}L$ is an irreducible Lie subalgebra of $\mathfrak{fgl}_{\tilde{\mathbb{K}}}(V)$ satisfying $\operatorname{End}_{\tilde{L}}(V) = \tilde{\mathbb{K}} \cdot \mathrm{id}_V$.*
(b) *Va is $\tilde{\mathbb{K}}$-invariant for every $a \in L$. In particular, the degree of the field extension $\tilde{\mathbb{K}}/\mathbb{K}$ is finite and divides the rank of every $a \in L$.*
(c) *Let $\overline{\mathbb{K}}$ denote the algebraic closure of the field $\tilde{\mathbb{K}}$, and let $\overline{V} = \overline{\mathbb{K}} \otimes_{\tilde{\mathbb{K}}} V$. Then $\overline{L} = \overline{\mathbb{K}} \otimes_{\tilde{\mathbb{K}}} \tilde{\mathbb{K}}L$ is an irreducible Lie subalgebra of $\mathfrak{fgl}_{\overline{\mathbb{K}}}(\overline{V})$.*

In the sequel we shall use the notation introduced in Proposition 2 without further comment. Suppose now that L is an irreducible Lie subalgebra of $\mathfrak{fgl}_{\mathbb{K}}(V)$ and that $\dim_{\mathbb{K}} V$ is infinite. Since every element in L has finite rank, we can choose for every finitely generated Lie subalgebra F of L a finite-dimensional $\tilde{\mathbb{K}}$-subspace V_F of V such that $VF = V_FF \subseteq V_F$. Because $V = VL$, every finite-dimensional subspace of V is contained in one of the so-chosen V_F.

Proposition 3. *For every finitely generated Lie subalgebra F of L there exist a finitely generated Lie subalgebra F^* in L containing F, and a $\tilde{\mathbb{K}}F^*$-composition factor M_F/N_F in $V_F^* = V_F + V_{F^*}$, such that $V_F \cap N_F = 0$ and $V_F \subseteq M_F$.*

Proof. Let $\{v_1, ..., v_n\}$ be a $\tilde{\mathbb{K}}$-basis of V_F. Since Proposition 2 implies that $\tilde{L}$ acts densely on $\tilde{V}$, there exist elements a_{jk} $(j, k \in \{1, ..., n\})$ in the associative subalgebra $\tilde{\mathbb{K}} \cdot \mathrm{id}_V + \sum_{i \geq 1} \tilde{L}^i$ of $\operatorname{End}_{\tilde{\mathbb{K}}}(V)$ generated by $\tilde{L} + \tilde{\mathbb{K}} \cdot \mathrm{id}_V$, such that $v_ia_{jk} = \delta_{ij}v_k$ for all i, j, k. Choose $F^* \subseteq L$ such that the elements a_{jk} are contained in the associative subalgebra of $\mathrm{End}_{\tilde{\mathbb{K}}}(V)$ generated by $\tilde{\mathbb{K}}F^* + \tilde{\mathbb{K}} \cdot \mathrm{id}_V$. Then

$$V_F \subseteq w(\tilde{\mathbb{K}} \cdot \mathrm{id}_V + \sum_{i \geq 1} (\tilde{\mathbb{K}}F^*)^i) \qquad \text{for all } w \in V_F \setminus 0.$$

Since V_F is finite-dimensional, there exists a $\tilde{\mathbb{K}}F^*$-composition factor M_F/N_F in V_F^* such that $M_F \cap V_F \neq 0 = N_F \cap V_F$. From the above, $V_F \subseteq M_F$. □

Note that, in the situation of Proposition 3, the F-module V_F embeds into the F^*-module $W_F = M_F/N_F$ via $V_F \ni v \longmapsto v + N_F \in W_F$.

Using the notation introduced above, we proceed to the proof of Theorem A.

Proof of part (a) *of Theorem* A. Suppose that S is not nil. Let $s \in S$ be a non-nilpotent element of minimal rank d. Assume that $0 = V_0 < V_1 < ... < V_{d+1} = V$ is a S-series in V. Choose $F = \mathbb{K} \cdot s$. We may assume without loss that V_F is large enough so that $V_{i-1} \cap V_F < V_i \cap V_F$ for $1 \leq i \leq d+1$. Then the $(V_i \cap M_F) + N_F/N_F$ $(0 \leq i \leq d+1)$ form a properly ascending $(S \cap F^*)$-series in W_F. In particular, W_F has an $(S \cap F^*)$-composition series of length $\geq d+1$.

Now $S \cap F^*$ is a subideal in F^*. From multiplying each term of a series between $S \cap F^*$ and F^* with $\tilde{\mathbb{K}}$, we see that $\tilde{\mathbb{K}}(S \cap F^*)$ is a subideal in $\tilde{\mathbb{K}}F^*$. Therefore Proposition 1 yields that all $\tilde{\mathbb{K}}(S \cap F^*)$-composition factors in W_F are mutually isomorphic. As a $\tilde{\mathbb{K}}$-linear transformation of V, the element $s \in S$ is non-nilpotent of rank $\tilde{d} = d/[\tilde{\mathbb{K}} : \mathbb{K}]$. But $(V_F + N_F)/N_F \leq W_F$, and so any $\tilde{\mathbb{K}}(S \cap F^*)$-composition series in W_F has length at most $\tilde{d}$. In order to derive a contradiction it suffices to show that every $\tilde{\mathbb{K}}(S \cap F^*)$-composition factor U in W_F has length at most $n = d/\tilde{d} = [\tilde{\mathbb{K}} : \mathbb{K}]$ as an $(S \cap F^*)$-module. But if U_0 is an irreducible $(S \cap F^*)$-submodule of U, and if $\{x_1, ..., x_n\}$ is a $\mathbb{K}$-basis of $\tilde{\mathbb{K}}$, then $U = x_1 U_0 + ... + x_n U_0$ is a direct sum of at most n copies of U_0. □

Proof of part (b) *of Theorem* A. Let $\{S_\alpha\}_{\alpha \leq \kappa}$ be an ascending series of non-trivial Lie subalgebras in L with $S_\kappa = L$, where $S_\alpha = \bigcup_{\beta < \alpha} S_\beta$ for each limit ordinal $\alpha \leq \kappa$. Choose the smallest α such that V is an irreducible S_α-module. Assume that $\alpha > 0$.

Assume further that α is a limit ordinal. Since S_α cannot be nil [13, Lemma 2], there exists $\beta < \alpha$ such that S_β is not nil. Due to part (a), the S_γ-module V has finite length d_γ for every γ between β and α. Let $d = \min\{d_\gamma \mid \beta \leq \gamma < \alpha\}$. We may assume that $d_\gamma = d$ for $\beta \leq \gamma < \alpha$. It follows that, whenever $\beta \leq \gamma_1 \leq \gamma_2 < \alpha$, every S_{γ_2}-composition series in V is a S_{γ_1}-composition series. In particular, if U_γ denotes the sum of all finite-dimensional irreducible S_γ-submodules in V, then $U_{\gamma_2} \subseteq U_{\gamma_1}$. Since each U_γ is finite-dimensional, we may assume that $U_{\gamma_1} = U_{\gamma_2}$ whenever $\beta \leq \gamma_1 \leq \gamma_2 < \alpha$. But then U_β is a finite-dimensional S_α-submodule in V, whence $U_\beta = 0$. It follows that V contains an infinite-dimensional irreducible S_β-submodule M. Since S_γ is an ideal in $S_{\gamma+1}$, a recursive application of the first argument in the proof of Proposition 1 yields that M is S_γ-invariant for all $\gamma < \alpha$. In particular, M is S_α-invariant, and $V = M$ is an irreducible S_β-module.

This contradiction to the choice of α shows that $\alpha = \beta + 1$ for some β. But then S_β is an ideal in the irreducible Lie subalgebra S_α of $\mathfrak{fgl}_{\mathbb{K}}(V)$. Hence S_β cannot be nil [13, Lemma 2]. Now part (a) and Proposition 1 imply that V is finite-dimensional, in contradiction to the hypothesis of Theorem A. □

Proof of part (c) *of Theorem* A. Because of parts (a) and (b), it suffices to show that a locally nilpotent Lie subalgebra L of $\mathfrak{fgl}_{\mathbb{K}}(V)$ can only act irreducibly on V when V is finite-dimensional. Now $\dim_{\mathbb{K}} V = [\tilde{\mathbb{K}} : \mathbb{K}] \dim_{\tilde{\mathbb{K}}} V = [\tilde{\mathbb{K}} : \mathbb{K}] \dim_{\overline{\mathbb{K}}} \overline{V}$, and $[\tilde{\mathbb{K}} : \mathbb{K}]$ is finite. We may therefore assume without loss that the field $\mathbb{K}$ is algebraically closed.

Assume that V is infinite-dimensional, and that L acts irreducibly on V. Consider $a \in L$, let $F = \mathbb{K} \cdot a$, and choose again V_F large enough so that $\dim_{\mathbb{K}} V_F$ exceeds the rank of a. From [5, p. 41, Corollary of Zassenhaus], every element in F^* acts like the sum of a scalar and a nilpotent element on W_F. By the choice of $\dim_{\mathbb{K}} V_F$, the element a induces a singular transformation on W_F. Thus a acts nilpotently on W_F, hence also on V_F and on V. This shows that L is nil, in contradiction to [13, Lemma 2]. □

Proof of part (d) *of Theorem* A. As in the proof of part (c), it suffices to show that a locally solvable Lie subalgebra L of $\mathfrak{fgl}_{\mathbb{K}}(V)$ can only act irreducibly on V when V is finite-dimensional. Again we may assume without loss that the field $\mathbb{K}$ is algebraically closed. In the case when $\operatorname{Char} \mathbb{K} = 0$, such a Lie algebra L has locally nilpotent derived subalgebra $[L, L]$ (see [5, p. 51, Corollary 1]), and hence the assertion follows from part (c). It remains to consider the case when $\operatorname{Char} \mathbb{K} = p > 2$.

Assume that V is infinite-dimensional, and that L acts irreducibly on V. Since L is not nil [13, Lemma 2], we can choose a non-nilpotent element $x \in L$. Let $F = \mathbb{K} \cdot x$, and choose V_F large enough so that $\dim_{\mathbb{K}} V_F > 3 \cdot \dim_{\mathbb{K}} Vx$. Let bars denote images in F^* modulo the kernel of the representation on W_F. Replace $\overline{F}$ and $\overline{F^*}$ by the smallest p-subalgebras of $\mathfrak{gl}_{\mathbb{K}}(W_F)$ containing $\overline{F}$ resp. $\overline{F^*}$ (see [15, p. 65]). In this way, $\overline{F}$ and $\overline{F^*}$ become restricted Lie algebras which are still solvable [15, Proposition 2.1.3], and $\dim_{\mathbb{K}} W_F > 3 \cdot \dim_{\mathbb{K}} W_F \overline{x}$. Moreover, the representation of $\overline{F^*}$ on W_F has character 0 in the sense of [15, p. 210].

Under the hypothesis that $\operatorname{Char} \mathbb{K} > 2$, it now follows from [15, Theorem 5.8.4] that there is a p-subalgebra Q in $\overline{F^*}$ and a one-dimensional Q-submodule $\mathbb{K}u$ in W_F such that

$$W_F \cong \operatorname{Ind}_Q^{\overline{F^*}}(\mathbb{K}u, 0) = \mathbb{K}u \otimes_{\mathrm{u}(Q)} \mathrm{u}(\overline{F^*}),$$

where $\mathrm{u}(\overline{F^*})$, resp. $\mathrm{u}(Q)$, denotes the restricted universal enveloping algebra of $\overline{F^*}$ resp. Q (see [15, p. 226, and Sections 5.3 and 2.5]). Let $\{e_n, ..., e_1\}$ be a $\mathbb{K}$-basis in $\overline{F^*}$ such that $\{e_n, ..., e_{m+1}\}$ is a $\mathbb{K}$-basis of Q for some $m \geq 1$. From [15, Theorem 2.5.1], the restricted universal enveloping algebras $\mathrm{u}(\overline{F^*})$ and $\mathrm{u}(Q)$ have $\mathbb{K}$-bases

$$\{e_n^{\alpha_n} ... e_1^{\alpha_1} \mid 0 \leq \alpha_i \leq p-1\} \qquad \text{resp.} \qquad \{e_n^{\alpha_n} ... e_{m+1}^{\alpha_{m+1}} \mid 0 \leq \alpha_i \leq p-1\}.$$

Hence $\mathbb{K}u \otimes_{\mathrm{u}(Q)} \mathrm{u}(\overline{F^*})$ has $\mathbb{K}$-basis

$$\{u \otimes e_m^{\alpha_m} ... e_1^{\alpha_1} \mid 0 \leq \alpha_i \leq p-1\}.$$

Assume now that $\overline{x} \notin Q$. We may choose $e_1 = \overline{x}$. For $0 \leq \alpha_1 \leq p-2$, the element $\overline{x}$ acts via

$$(u \otimes e_m^{\alpha_m} ... e_1^{\alpha_1}) \cdot \overline{x} = u \otimes e_m^{\alpha_m} ... e_1^{\alpha_1+1}.$$

Therefore, as a $\mathbb{K}$-linear transformation of W_F, the element $\overline{x}$ has rank at least $p^{m-1}(p-1)$, although $\dim_{\mathbb{K}} W_F = p^m$. This contradicts the choice of V_F.

Assume next that $[\overline{a},\overline{x}] \notin Q$ for some $a \in F^*$. In this case we may choose $e_1 = [\overline{a},\overline{x}]$, and the above argument shows that

$$\dim_{\mathbb{K}} W_F \leq \tfrac{3}{2} \cdot \dim_{\mathbb{K}} W_F[\overline{a},\overline{x}] \leq 3 \cdot \dim_{\mathbb{K}} W_F\overline{x}.$$

Again we have a contradiction to the choice of V_F. Thus, both $\overline{x}$ and $[\overline{F^*},\overline{x}]$ are contained in Q.

For each $\nu \geq 0$, let U_ν be the $\mathbb{K}$-subspace of $\mathbb{K}u \otimes_{\mathrm{u}(Q)} \mathrm{u}(\overline{F^*})$ generated by $\{u \otimes e_m^{\alpha_m}...e_1^{\alpha_1} \mid \alpha_m + ... + \alpha_1 \leq \nu\}$. Since the restricted universal enveloping algebra $\mathrm{u}(\overline{F^*})$ is a quotient of the universal enveloping algebra of $\overline{F^*}$, it follows from [15, Proposition 1.9.1] that $u \otimes \overline{a}_1...\overline{a}_\nu \in U_\nu$ for all $a_i \in F^*$. Therefore [15, Lemma 5.7.1] yields that, for fixed $0 \leq \alpha_i \leq p-1$, the element $(u \otimes e_m^{\alpha_m}...e_1^{\alpha_1}) \cdot \overline{x}$ is congruent to

$$u \otimes \overline{x}e_m^{\alpha_m}...e_1^{\alpha_1} + \sum_{\{i|\alpha_i\geq 1\}} \alpha_i \cdot u \otimes ([e_i,\overline{x}] \cdot e_m^{\alpha_m}...e_{i+1}^{\alpha_{i+1}} e_i^{\alpha_i - 1} e_{i-1}^{\alpha_{i-1}}...e_1^{\alpha_1})$$

modulo $U_{|\alpha|-1}$, where $|\alpha| = \alpha_m + ... + \alpha_1$. Since $\overline{x}$ and $[e_i,\overline{x}]$ are in Q, we obtain

$$(u \otimes e_m^{\alpha_m}...e_1^{\alpha_1}) \cdot \overline{x} \equiv u\overline{x} \otimes e_m^{\alpha_m}...e_1^{\alpha_1} \qquad \mathrm{mod} \quad U_{|\alpha|-1}$$

for $0 \leq \alpha_i \leq p-1$. In particular, the sections $(\mathbb{K}u \otimes e_m^{\alpha_m}...e_1^{\alpha_1}) + U_{|\alpha|-1}/U_{|\alpha|-1}$ are mutually isomorphic $\mathbb{K}\overline{x}$-modules. Since $\overline{x}$ acts non-nilpotently on W_F, we conclude that $\overline{x}$ must act regularly on W_F, and this is the final contradiction to the choice of V_F. ∎

The proof of part (d) does in fact reveal the following bound for the ranks of elements in irreducible representations of locally solvable finitary Lie algebras.

Proposition 4. *Let L be an irreducible locally solvable Lie subalgebra of $\mathfrak{fgl}_{\mathbb{K}}(V)$, and let d denote the rank of a non-nilpotent element in L.*

(a) *If $\operatorname{Char}\mathbb{K} = 0$, then $d = \dim_{\mathbb{K}} V$.*
(b) *If $\operatorname{Char}\mathbb{K} = p > 2$, then $d \geq \frac{p-1}{2p} \cdot \dim_{\mathbb{K}} V$.*

3. Proof of Theorem B

Let $L \leq \mathfrak{fgl}_{\mathbb{K}}(V)$, and consider an L-composition series $\mathfrak{S}$ in V with factors U_λ $(\lambda \in \Lambda)$. Let $\rho_\lambda : L \longrightarrow \mathfrak{fgl}_{\mathbb{K}}(U_\lambda)$ denote the canonical projection. Then $K = \bigcap_{\lambda\in\Lambda} \ker \rho_\lambda$ is the largest nil ideal in L [13, Lemma 2], and Theorem A implies that $\dim_{\mathbb{K}} U_\lambda$ is finite whenever $S\rho_\lambda$ is non-trivial. Therefore S is the extension of K by a subdirect sum of the finite-dimensional Lie algebras $S\rho_\lambda$ $(\lambda \in \Lambda)$.

It follows as in the proof of [13, Proposition 7] that every finite subset of K is contained in an ideal in L which annihilates the factors of a finite series in V whose terms occur in $\mathfrak{S}$ (see also [11, Proof of Theorem B(vi)]). Hence K is the union of an ascending chain of ideals in L with abelian factors. Further S/K is contained in the direct sum of the solvable radicals of those $L\rho_\lambda$, for which U_λ is finite-dimensional, and the intersections of L/K with larger and larger sums of these radicals gives the desired chain in L/K.

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