CONTINUATION METHOD FOR $\alpha$-SUBLINEAR MAPPINGS

YONG-ZHUO CHEN

(Communicated by Dale Alspach)

Abstract. Let $B$ be a real Banach space partially ordered by a closed convex cone $P$ with nonempty interior. We study the continuation method for the monotone operator $A : \bar{P} \to \bar{P}$ which satisfies

$$A(tx) \geq t^{\alpha(a,b)} A(x),$$

for all $x \in \bar{P}$, $t \in [a, b] \subset (0, 1)$, where $\alpha(a, b) \in (0, 1)$. Thompson’s metric is among the main tools we are using.

1. Introduction

Let $B$ be a real Banach space partially ordered by a closed convex cone $P$ with nonempty interior, which is denoted by $\bar{P}$. Suppose $A : \bar{P} \to \bar{P}$ is monotone, i.e., $Ax \geq Ay$ when $x \geq y$, and satisfies

$$A(tx) \geq \varphi(t) A(x),$$

where $t \in (0, 1)$ and $\varphi$ is a positive function on $(0, 1)$. The fixed points of this type of operator were much discussed under various assumptions on $\varphi$. Among them, M. A. Krasnosel’skii studied $u_0$-concave operator (5), where $\varphi(t) = [1 + \eta(x, t)] t$ with $\eta(x, t) > 0$, D. Guo established the existence of the unique fixed point for $\alpha$-concave operators (3), where $\varphi(t) = t^\alpha$ with $\alpha \in (0, 1)$, and U. Krause proved fixed point theorems for ascending operators (3), where $\varphi : [0, 1] \to [0, 1]$ is continuous and $\lambda < \varphi(\lambda)$ for $\lambda \in (0, 1)$. In [1], we investigated the mixed monotone counterpart of the monotone operator $A$ which satisfies

$$A(tx) \geq t^{\alpha(a,b)} A(x),$$

for all $x \in \bar{P}$, $t \in [a, b] \subset (0, 1)$, where $\alpha(a, b) \in (0, 1)$. This class of operator includes Guo’s $\alpha$-concave operator and U. Krause’s ascending operator (see [1, Corollary 3.2]). We say that a monotone operator is $\alpha$-sublinear if it satisfies (2).

One important method for solving an operator equation $F(x) = 0$ is the continuation method, i.e., to continuously deform $F$ to a simpler operator $G$ such that $G(x) = 0$ is easily solved. In the present paper, we intend to discuss the continuation method for $\alpha$-sublinear mappings. Our work is motivated by a paper of A. Granas (3).

Received by the editors September 15, 1997 and, in revised form, April 5, 1999.

1991 Mathematics Subject Classification. Primary 47H07, 47H09; Secondary 47H10.

Key words and phrases. $\alpha$-sublinear, cone, fixed point, generalized contraction, monotone operator, ordered Banach space, Thompson’s metric.

©2000 American Mathematical Society

203
x, y ∈ P − {0} are called comparable if there exist positive numbers λ and µ such that λx ≤ y ≤ µx. This defines an equivalent relationship, and splits P − {0} into disjoint components of P. P̂ is a component of P if P̂ ̸= ∅.

Unless specified otherwise, throughout this paper, we assume that the norm is monotone, i.e., 0 < x ≤ y implies that ∥x∥ ≤ ∥y∥. Hence all the cones in this paper are normal, since P is normal if B has an equivalent norm which is monotone.

Let C be a component of P and x, y ∈ C. Put

\[ M(x/y) = \inf\{\lambda : x ≤ \lambda y\} \quad \text{and} \quad M(y/x) = \inf\{\mu : y ≤ \mu x\}. \]

Thompson’s metric is defined by

\[ d(x, y) = \ln\{\max[M(x/y), M(y/x)]\}. \]

\( \tilde{d}(x, y) \) is a metric on C and C is complete with respect to \( \tilde{d} \) under our assumption on P ([7, Lemma 3]).

The following theorem is just the monotone operator version of Theorem 3.1 in [1], which was proved by appealing to Thompson’s metric.

**Theorem 1.1.** Let C be a component of P, and A : C → C be α-sublinear. Then A has exactly one fixed point \( x^* \) in C, and for any point \( x_0 \in C \), we have \( A^n(x_0) \to x^* \) as \( n \to \infty \).

We also need the following two lemmas.

**Lemma 1.2** (Thompson [7]). If the norm is monotone, then

\[ \|x − y\| ≤ 3be^{d(x,y)−1} \]

for all x, y ∈ P with ∥x∥ ≤ b and ∥y∥ ≤ b.

**Lemma 1.3.** Let \( u \in \text{P̂} \) and \( B(u, r) \subset P \), where \( B(u, r) = \{x \in B : \|x−u\| < r\} \). Then

\[ \tilde{d}(x, u) ≤ \ln \{\max\left(\frac{r + \|x−u\|}{r}, \frac{r}{r − \|x−u\|}\right)\} \]

for all \( x \in B(u, r) \).

**Proof.** Without loss of generality, we assume \( x ≠ u \). Then \( x \in B(u, r) \) implies that \( u ± \frac{r(x−u)}{\|x−u\|} \in P \). It follows that

\[ x ≤ \frac{r + \|x−u\|}{r} u \quad \text{and} \quad u ≤ \frac{r}{r − \|x−u\|} x. \]

Hence

\[ \tilde{d}(x, u) ≤ \ln \{\max\left(\frac{r + \|x−u\|}{r}, \frac{r}{r − \|x−u\|}\right)\}. \]

\[ \square \]

Let \( (X, d) \) be a complete metric space and \( D \subset X \) a closed subset. We say that \( T : D \to X \) is a generalized contraction if for each \( (a, b) \subset (0, \infty) \), there exists \( L(a, b) ∈ (0, 1) \) such that

\[ d(Tx, Ty) ≤ L(a, b) d(x, y), \]

where \( x, y ∈ D \) and \( a ≤ d(x, y) ≤ b \). The following theorem is due to M. A. Krasnosel’skii ([5 Theorem 34.5], see also [2 Theorem (1.3.3)]).
Suppose and so. We claim that \( f(x^*) \) is a fixed point of \( T \), where \( x_n = T^n x, n = 1, 2, \ldots \).

This paper is organized as follows. In Section 2, we generalize A. Granas’s main theorem in \([3]\) to generalized contraction mappings. Section 3 discusses the continuation method for \( \alpha \)-sublinear mappings. An example of application is given in Section 4.

### 2. Topological transversality for generalized contraction mappings

In this section, \( U \) stands for a bounded open set of \( X \). Let \( G(U) \) be the set of all generalized contraction mappings \( T : U \to X \), and \( G_0(U) = \{ T \in G(U) : (\text{Fix} \; T) \cap \partial U = \emptyset \} \), where \( \text{Fix} \; T = \{ x \in U : x = Tx \} \). We denote \( \text{diam} \; U = \text{sup} \{ \| x - y \| : x, y \in U \} \) and \( \text{dist}(A_1, A_2) = \inf \{ \| x - y \| : x \in A_1, y \in A_2 \} \), where \( A_1 \) and \( A_2 \) are subsets of \( X \).

We say \( T \in G_0(U) \) is traverse or essential (cf. \([2, \text{pp. 58-60}] \) and \([3]\) if \( T \) has a fixed point, i.e., the graph of \( T \) crosses or traverses the diagonal of \( U \times X \). The following theorem discusses the topological transversality for operators in \( G_0(U) \).

**Theorem 2.1.** Suppose \( \{ H_t \} \subset G_0(U) \), \( t \in [0, 1] \), satisfy:

(\text{H1}) For each \( (a, b) \subset (0, \infty) \), there exists \( L(a, b) \in (0, 1) \) such that
\[
\frac{d(H_t(x_1), H_t(x_2))}{d(x_1, x_2)} \leq L(a, b) d(x_1, x_2)
\]
for all \( t \in [0, 1] \) and \( x_1, x_2 \in U \) with \( a \leq d(x_1, x_2) \leq b \), where \( L(a, b) \) is independent of \( t \).

(\text{H2}) For any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( d(H_t(x), H_{t_2}(x)) \leq \varepsilon \) for all \( x \in U \) and \( t_1, t_2 \in [0, 1] \) with \( |t_1 - t_2| < \delta \), where \( \delta \) is independent of \( x \).

If \( H_0 \) has a fixed point in \( U \), then so does \( H_t \) for each \( t \in [0, 1] \).

**Proof.** Let \( \Lambda = \{ \lambda \in [0, 1] : x = H_\lambda(x) \text{ for some } x \in U \} \). \( \Lambda \neq \emptyset \) since \( 0 \in \Lambda \).

(i) \( \Lambda \) is closed in \([0, 1]\).

Let \( \lambda_n \to \lambda_0 \) with \( \lambda_n \in \Lambda \) and \( x_n \in U \) such that \( x_n = H_{\lambda_n} x_n \). Then
\[
d(x_n, x_m) \leq d(H_{\lambda_n}(x_n), H_{\lambda_m}(x_n)) + d(H_{\lambda_m}(x_n), H_{\lambda_m}(x_m)).
\]
We claim that \( \{ x_n \} \) is a Cauchy sequence. Otherwise, for any \( k > 0 \), there exist \( n_k, m_k > k \) such that \( d(x_{n_k}, x_{m_k}) \geq \delta \), where \( \delta \) is a positive constant. Let \( M = \text{diam} \; U \). (5) leads to
\[
d(x_n, x_m) \leq d(H_{\lambda_n}(x_{n_k}), H_{\lambda_n}(x_{n_k})) + L(\delta, M) d(x_{n_k}, x_{m_k}),
\]
and so
\[
\delta \leq d(x_{n_k}, x_{m_k}) \leq \frac{d(H_{\lambda_n}(x_{n_k}), H_{\lambda_n}(x_{n_k}))}{1 - L(\delta, M)}.
\]
By (H2), \( d(H_{\lambda_{n_k}}(x_{n_k}), H_{\lambda_{n_k}}(x_{n_k})) \to 0 \) as \( k \to \infty \). We reach a contradiction from (6). Hence there exists \( x_0 \in U \) such that \( x_n \to x_0 \).

On the other hand,
\[
d(x_n, H_{\lambda_0}(x_0)) \leq d(H_{\lambda_n}(x_n), H_{\lambda_0}(x_n)) + d(H_{\lambda_0}(x_n), H_{\lambda_0}(x_0)) \leq d(H_{\lambda_n}(x_n), H_{\lambda_0}(x_0)) + d(x_n, x_0) \to 0 \text{ as } n \to \infty.
\]
Thus \( x_0 = H_{\lambda_0}(x_0) \). Since \( (\text{Fix} \; H_{\lambda_0}) \cap \partial U = \emptyset \), \( x_0 \in U \) and \( \lambda_0 \in \Lambda \).
(ii) $\Lambda$ is open in $[0, 1]$. Let $\lambda_0 \in \Lambda$ and $x_0 = H_{\lambda_0}(x_0)$, where $x_0 \in U$. Choose $r > 0$ such that $r < \text{dist} (x_0, \partial U)$. There exists $\varepsilon_1 > 0$ so that $d(H_{\lambda}(x_0), H_{\lambda_0}(x_0)) < \frac{r}{2}$ when $\|\lambda - \lambda_0\| < \varepsilon_1$. Hence for $\lambda \in [0, 1] \cap (\lambda_0 - \varepsilon_1, \lambda_0 + \varepsilon_1)$ and $x \in B(x_0, \frac{r}{2})$,

$$d(H_{\lambda}(x), x_0) \leq d(H_{\lambda}(x), H_{\lambda_0}(x_0)) + d(H_{\lambda_0}(x_0), H_{\lambda_0}(x_0)) \leq d(x, x_0) + \frac{r}{2} \leq r.$$ 

For $(1 - L(\frac{r}{2}, r)) r$, there exists $\varepsilon_2 > 0$ such that $d(H_{\lambda}(x_0), H_{\lambda_0}(x_0)) < (1 - L(\frac{r}{2}, r)) r$ when $\|\lambda - \lambda_0\| < \varepsilon_2$. Then for $\lambda \in [0, 1] \cap (\lambda_0 - \varepsilon_2, \lambda_0 + \varepsilon_2)$ and $x \in U$ with $\frac{r}{2} \leq d(x, x_0) \leq r$,

$$d(H_{\lambda}(x), x_0) \leq d(H_{\lambda}(x), H_{\lambda_0}(x_0)) + d(H_{\lambda_0}(x_0), H_{\lambda_0}(x_0)) \leq L(\frac{r}{2}, r) d(x, x_0) + (1 - L(\frac{r}{2}, r)) r \leq r.$$ 

Put $\varepsilon = \max \{\varepsilon_1, \varepsilon_2\}$. For all $\lambda \in [0, 1] \cap (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$, we have $H_{\lambda} : \overline{B}(x_0, r) \rightarrow \overline{B}(x_0, r)$. By the Generalized Contraction Principle, there exists $x \in \overline{B}(x_0, r) \cap U$ such that $H_{\lambda}(x) = x$. We conclude that $\lambda \in \Lambda$, and $[0, 1] \cap (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \subset \Lambda$.

Therefore $\Lambda \neq \emptyset$ is both open and closed, and consequently $\Lambda = [0, 1]$. \hfill $\square$

The following example illustrates that Theorem 2.1 is indeed more general than Theorem 3.1 in $\mathbb{R}^n$.

Example 2.2. Let $X = [0, \infty)$, $d(x, y) = \|x - y\|$ and $G_t(x) = \frac{x - t}{1 + x} + t$, where $x, y \in X$ and $t \in [0, 1]$. Consider $U = [0, 2)$. Then $\partial U = \{2\}$ For each $t \in [0, 1]$, we have

$$d(G_t(x), G_t(y)) = \frac{\|x - y\|}{(1 + x)(1 + y)} \leq \frac{\|x - y\|}{1 + \|x - y\|} = \frac{1}{1 + d(x, y)} d(x, y).$$

For $0 < a \leq d(x, y) \leq b < \infty$, we can put $L(a, b) = \frac{1}{1 + a}$. Hence $G_t$ is a generalized contraction, however it is not a contraction in the usual sense. Since $2$ is not a fixed point for any $G_t$ and $G_0$ has a fixed point $0 \in U$, we apply Theorem 2.1 to conclude that $G_t$ has a fixed point in $U = [0, 2)$ for each $t \in [0, 1]$.

3. Continuation method for $\alpha$-sublinear mappings

In this section, we will use Thompson’s metric and Theorem 2.1 as tools to study $\alpha$-sublinear mappings.

Theorem 3.1. Let $S_t : \hat{P} \rightarrow \hat{P}$ be monotone for each $t \in [0, 1]$, and satisfy:

(H1) For each $[a, b] \subset (0, 1)$, there exists $\alpha(a, b) \in (0, 1)$ such that

$$S_t(cx) \geq c^{\alpha(a, b)} S_t(x)$$

for all $x \in \hat{P}$ and $c \in [a, b]$, where $\alpha(a, b)$ is independent of $t \in [0, 1]$.

(H2) There exists a bounded open set $U$ with $\overline{U} \subset \hat{P}$ and $\text{dist}(\overline{U}, \partial \hat{P}) = r > 0$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|S_{t_1}(x) - S_{t_2}(x)\| < \varepsilon$$

for all $x \in \overline{U}$ and $t_1, t_2 \in [0, 1]$ with $|t_1 - t_2| < \delta$, where $\delta$ is independent of $x$. 

Suppose \((\text{Fix} \, S_t) \cap \partial U = \emptyset\) for all \(t \in [0, 1]\). If \(S_0\) has a fixed point in \(U\), then so does \(S_t\) for each \(t \in [0, 1]\), and the sequence \(\{S^n_t(x)\}\) converges to the unique fixed point of \(S_t\) for any \(x \in \hat{P}\).

Proof. Let \(x, y \in \hat{P}\) with \(d(x, y) \in [-\ln b, -\ln a]\), where \([a, b] \subset (0, 1)\). Without loss of generality, assume \(M(x/y) \geq M(y/x)\). Then \(d(x, y) = \ln M(x/y)\) and \(\frac{1}{a} \leq M(x/y) \leq \frac{1}{b}\). Now

\[
S_t(x) \geq S_t(M(y/x)^{-1}y) \\
\geq S_t(M(x/y)^{-1}y) \\
\geq M(x/y)^{-\alpha(a, b)}S_t(y).
\]

Thus \(M(S_t(y)/S_t(x)) \leq M(x/y)^{\alpha(a, b)}\). On the other hand,

\[
S_t(y) \geq S_t(M(x/y)^{-1}x) \\
\geq M(x/y)^{-\alpha(a, b)}S_t(x)
\]

implies that \(M(S_t(x)/S_t(y)) \leq M(x/y)^{\alpha(a, b)}\). Hence

\[
\bar{d}(S_t(x), S_t(y)) \leq \ln[M(x/y)^{\alpha(a, b)}] \\
= \alpha(a, b) \ln M(x/y) \\
= L(-\ln b, -\ln a) \bar{d}(x, y),
\]

where \(L(-\ln b, -\ln a) = \alpha(a, b)\).

Lemma 1.2 and Lemma 1.3 imply that \(U\) is also open in Thompson’s metric, and its closure \(\overline{U}\) and boundary \(\partial U\) are identical in both the norm topology and Thompson’s metric topology.

Let \(\varepsilon > 0\) be given. There exists \(\varepsilon_1 \in (0, r)\) such that

\[
\ln \{\max \left(\frac{r + \beta}{r}, \frac{r}{r - \beta}\right)\} < \varepsilon
\]

for all \(\beta \in [0, \varepsilon_1]\). By (H2), there exists \(\delta > 0\) such that \(\|S_{t_1}(x) - S_{t_2}(x)\| < \varepsilon_1\) for all \(x \in \overline{U}\) and \(t_1, t_2 \in [0, 1]\) with \(|t_1 - t_2| < \delta\). Using Lemma 1.3, we have

\[
\bar{d}(S_t(x), S_{t_{1/2}}(x)) \leq \ln \{\max \left(\frac{r + \beta}{r}, \frac{r}{r - \beta}\right)\} < \varepsilon
\]

for all \(x \in \overline{U}\) and \(t_1, t_2 \in [0, 1]\) with \(|t_1 - t_2| < \delta\). If \((\text{Fix} \, S_t) \cap \partial U = \emptyset\) for all \(t \in [0, 1]\) and \(S_0\) has a fixed point in \(U\), then we can apply Theorem 2.1 to conclude that \(S_t\) has a fixed point in \(U\) for each \(t \in [0, 1]\).

(H1) implies that the sequence \(\{S^n_t(x)\}\) converges to the unique fixed point of \(S_t\) for any \(x \in \hat{P}\) by Theorem 1.1. \(\Box\)

The following is a nonlinear alternative theorem for \(\alpha\)-sublinear mappings.

Theorem 3.2. Let \(A : \hat{P} \rightarrow \hat{P}\) be an \(\alpha\)-sublinear mapping and \(U\) be a nonempty open bounded subset with \(\overline{U} \subset \hat{P}\) and \(\text{dist}(\overline{U}, \partial \hat{P}) > 0\). If \(A(\overline{U})\) is bounded, then \(A\) has at least one of the following properties:

(i) \(A\) has a unique fixed point in \(\overline{U}\), and the sequence \(A^n(x)\) converges to that fixed point for any \(x \in \hat{P}\).

(ii) \(A(\partial U)\) contains a point of some exterior ray, i.e., there exists \(x_0 \in U\) such that \(A_{y_0} = x_0 + \tau(y_0 - x_0)\) for some \(\tau > 1\) and \(y_0 \in \partial U\).
Proof. Let \( x_0 \in U \) and consider \( S_t(x) = tAx + (1 - t)x_0 \). By the definition of \( \alpha \)-sublinear mapping, for each \([a, b] \subset (0, 1)\), there exists \( \alpha(a, b) \in (0, 1) \) such that for all \( x \in \overset{\circ}{P} \) and \( c \in [a, b] \),

\[
S_t(cx) = tA(cx) + (1 - t)x_0 \\
\geq t\alpha(a, b)Ax + (1 - t)x_0 \\
\geq \alpha(a, b)(tAx + (1 - t)x_0) \\
= \alpha(a, b)S_t(x).
\]

Let \( M = \sup\{\|y\| : y \in A(U)\} \). For any \( \varepsilon > 0 \), choose \( \delta = \frac{\varepsilon}{2\max\{M, \|x_0\|\}} \). Then for \( x \in \bar{U} \) and \( t_1, t_2 \in [0, 1] \) with \( |t_1 - t_2| < \delta \),

\[
\|S_{t_1}(x) - S_{t_2}(x)\| = \|(t_1 - t_2)Ax - (t_1 - t_2)x_0\| \\
\leq |t_1 - t_2|\|Ax\| + |t_1 - t_2|\|x_0\| \\
< \varepsilon.
\]

Note that \( S_0 \) has a fixed point \( x_0 \in U \). Assume that \( A \) does not have a fixed point in \( \bar{U} \), then by Theorem 3.1, there exists \( y_0 \in \partial U \) and \( t \in (0, 1) \) such that \( S_t(y_0) = y_0 \), i.e., \( tAy_0 + (1 - t)x_0 = y_0 \). It follows that \( Ay_0 = x_0 + \tau(y_0 - x_0) \), where \( \tau = \frac{1}{t} > 1 \).

Remark. The distinction between cases (i) and (ii) in Theorem 3.2 cannot be sharpened to a proper alternative. Let’s consider the so-called square root version of Fibonacci’s rabbit population model:

\[
\overset{\circ}{P} = R^2_+, \quad A(a, b) = (\sqrt{a} + \sqrt{b}, \sqrt{a}), \quad (a, b) \in R^2_+.
\]

Suppose \( U = (3, 4) \times (1, 2) \subset R^2_+ \). It is easy to check that \( A \) has a fixed point \((a^*, b^*) \approx (3.08, 1.75) \in U \) and \( A(U) = \left[1 + \sqrt{3}, 2 + \sqrt{2}\right] \times \left[\sqrt{3}, 2\right] \). Take \( y_0 = (3.144) \in \partial U \); then \( Ay_0 = (1.2 + \sqrt{3}, \sqrt{3}) \). Now there exists \( x_0 = (4.8 - \sqrt{3}, 2.88 - \sqrt{3}) \in U \) such that \( A(y_0) = x_0 + \tau(y_0 - x_0) \) with \( \tau = 2 \). Hence cases (i) and (ii) in Theorem 3.2 are not mutually exclusive.

4. Example

The following example illustrates the application of Theorem 3.2 to the Dirichlet problem for a uniformly elliptic differential operator.

Let \( \Omega \) be a bounded convex domain in \( R^n \) \((n \geq 2)\), whose boundary \( \partial \Omega \) belongs to \( C^{2+\mu} \) \((0 < \mu < 1)\) and consider the Dirichlet problem

\[
\begin{cases}
Lu = f(x, u), \\
u|_{\partial \Omega} = 0,
\end{cases}
\]

where

\[
Lu = -\sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial u}{\partial x_i} + c(x)u
\]

is a uniformly elliptic differential operator, i.e., there exists \( \nu > 0 \) such that

\[
\sum_{i,j=1}^n a_{ij}(x)\xi_i \xi_j \geq \nu |\xi|^2, \quad x \in \bar{\Omega}, \quad \xi = (\xi_1, \cdots, \xi_n) \in R^n,
\]
and \( a_{ij}(x) = a_{ji}(x), \ c(x) \geq 0, \) all coefficients \( a_{ij}, b_i, c \in C^0(\Omega). \)

Suppose \( f(x, u) > 0 \) is continuous for all \( x \in \bar{\Omega} \) and \( u \geq 0. \) The solution of (8) is equivalent to the fixed point of the integral operator

\[
Au(x) = \int_{\Omega} G(x, y) f(y, u(y)) \, dy,
\]

where \( G(x, y) \) is the corresponding Green function which satisfies

\[
0 < G(x, y) < \begin{cases} 
  k_0 |x - y|^{2-n}, & n > 2, \\
  k_0 \ln |x - y|, & n = 2
\end{cases}
\]

where \( x, y \in \Omega \) and \( x \neq y. \)

It is well known that \( A \) is monotone and completely continuous from \( P \) into \( P \) (see [4] pp. 60-62), where \( P = \{ u \in C(\Omega) \mid u(x) \geq 0, \forall x \in \Omega \}. \) The sup norm of \( C(\Omega) \) is monotone in the partial order introduced by cone \( P. \) Note that \( \hat{P} = \{ u \in C(\Omega) \mid u(x) > 0, \forall x \in \Omega \}, \) and it is easy to see \( A : \hat{P} \to \hat{P}. \) Let \( U = \{ u \in C(\bar{\Omega}) \mid m < u(x) < M, \forall x \in \bar{\Omega} \}, \) where \( m \) and \( M \) are positive constants. Then \( \bar{U} \subset \hat{P} \) and \( A(\bar{U}) \) is bounded due to the complete continuity of \( A. \)

If there exists a lower semicontinuous function \( \phi : (0, 1) \to (0, 1) \) such that \( \phi(r) > r \) and

\[
f(x, tu) \geq \phi(t) f(x, u),
\]

then \( A(tu) \geq \phi(t) A(u). \) This implies that \( A \) is \( \alpha \)-linear by observing \( \phi(t) = \frac{t^\alpha}{\log \phi(t)} \)

and \( \log \phi(t) \) attains its maximum \( \alpha(a, b) \) on each \( [a, b] \subset (0, 1) \) due to the lower semicontinuity of \( \phi. \) Applying Theorem 3.2, we have at least one of the following:

(i) \( A \) has a unique fixed point \( u_0 \in \bar{U} \) and the sequence

\[
u_{n+1}(x) = \int_{\Omega} G(x, y) f(y, u_n(y)) \, dy, \quad n = 1, 2, \ldots,
\]

converges to \( u_0(x) \) in sup norm for any initial function \( u_1 \in C(\Omega) \) with \( u_1(x) > 0 \) for all \( x \in \bar{\Omega}. \)

(ii) \( A(\partial U) \) contains a point of some exterior ray, i.e., there exists \( u_0 \in C(\Omega) \) with \( m < u(x) < M, x \in \bar{\Omega}, \) such that

\[
\int_{\Omega} G(x, y) f(y, v_0(y)) \, dy = u_0 + \tau (v_0(x) - u_0(x))
\]

for some \( \tau > 1 \) and \( v_0 \in \partial U. \)

**Acknowledgement**

The author is very grateful to the referee for many valuable comments and suggestions.

**References**


Division of Natural Sciences, University of Pittsburgh at Bradford, Bradford, Pennsylvania 16701

E-mail address: yong@imap.pitt.edu