THE SORGENFREY LINE HAS A LOCALLY PATHWISE CONNECTED CONNECTIFICATION

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Abstract. We answer a question of Alas, Tkacenko, Tkachuk and Wilson by constructing a connected locally pathwise connected Hausdorff space in which the Sorgenfrey line can be densely embedded.

A connectification of a $T_2$-space $X$ is a connected Hausdorff space $Y$ in which $X$ can be densely embedded. In such a case $Y$ is called a connectification of $X$, and $X$ is said to be connectifiable. In 1977 Emeryk and Kulpa showed, in response to a question of van Douwen, that the Sorgenfrey line cannot be densely embedded in a connected $T_3$-space, although it is connectifiable (see [2]). In a recent paper Alas, Tkacenko, Tkachuk and Wilson asked if the Sorgenfrey line has a locally connected connectification ([1]).

The aim of this paper is to give a strongly positive answer to the above question. We refer the reader to [3] for notations and terminology not explicitly given.

Recall that a collection of pairwise disjoint non-empty open subsets of a space $X$ is called a cellular family.

Theorem 1. The Sorgenfrey line has a locally pathwise connected connectification.

Proof. Let $S = (\mathbb{R}, \tau)$ be the Sorgenfrey line and let $D = \{d_n : n \in \mathbb{N}\}$ be a countable dense subset of $S$, and for every $n, i \in \mathbb{N}$ set $B(n, i) = [d_n, d_n + \frac{1}{i+1}]$ and $C(n, i) = [d_n + \frac{1}{i+1}, d_n + \frac{1}{i}]$.

Let $\Omega$ be the subset of $\mathbb{N}^3$ consisting of all $\omega = (n, m, i)$ with $n < m$ and $B(n, i) \cap B(m, i) = \emptyset$. Set $\Lambda = \Omega \times (\mathbb{Z} \setminus \{0\})$.

By induction we can choose, for every $\lambda = (n, m, i, k) \in \Lambda$, a free open filter $\mathcal{F}_\lambda$ on $S$ with a countable base such that:

i) if $k > 0$, then $C(n, i + k - 1) \in \mathcal{F}_\lambda$;

ii) if $k < 0$, then $C(m, i - k - 1) \in \mathcal{F}_\lambda$;

iii) the family $\Phi = \{\mathcal{F}_\lambda : \lambda \in \Lambda\}$ is Hausdorff separated (i.e., if $\lambda \neq \lambda'$, then there are $F \in \mathcal{F}_\lambda$ and $F' \in \mathcal{F}_{\lambda'}$ such that $F \cap F' = \emptyset$).

Note that $B(n, i) \cup B(m, i) \in \mathcal{F}_\lambda$.

Now let $H = [0, 1] \cap \mathbb{Q} \setminus \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{1 - \frac{1}{n} : n \in \mathbb{N}\}$, and set $\Gamma = \Omega \times H$. 

For each \( q \in H \) set
\[
k_q = \begin{cases} 
    n - 1, & \text{if } \frac{1}{n+1} < q < \frac{1}{n}, \\
    1 - n, & \text{if } \frac{1}{n} \leq 1 - \frac{1}{n} < q < 1 - \frac{1}{n+1}.
\end{cases}
\]

For each \( \gamma = (\omega, q) \in \Gamma \) let \( \lambda_\gamma = (\omega, k_q) \in \Lambda \). Clearly, for every \( \lambda \in \Lambda \), there are countably many \( \gamma \in \Gamma \) such that \( \lambda_\gamma = \lambda \).

Now for every \( \gamma \in \Gamma \) choose a countably generated open filter \( \mathcal{G}_\gamma \) on \( S \) finer than \( \mathcal{F}_\gamma \) so that, for every \( \lambda \in \Lambda \), the family \( \{ \mathcal{G}_\gamma : \gamma \in \Gamma, \lambda_\gamma = \lambda \} \) is totally Hausdorff separated, i.e., for every \( \gamma \) such that \( \lambda_\gamma = \lambda \), there is a \( A_\gamma \in \mathcal{G}_\gamma \) such that \( \{ A_\gamma : \gamma \in \Gamma, \lambda_\gamma = \lambda \} \) is a cellular family (an easy proof of the existence of the family \( \{ \mathcal{G}_\gamma : \gamma \in \Gamma, \lambda_\gamma = \lambda \} \) can be found in [4]).

Set \( Y = \Omega \times [0, 1] \), and for every \( y = (\omega, r) \in Y \) and any \( \epsilon > 0 \), let us denote by \( [y - \epsilon, \omega + \epsilon] \) the set \( \{ (\omega, s) : |s - r| < \epsilon, \ s \in [0, 1] \} \).

Moreover \( [y - \epsilon, \omega + \epsilon] \) will denote the set \( \{ (\omega, s) : |s - r| \leq \epsilon, \ s \in [0, 1] \} \).

Put \( T = \mathbb{R} \cup Y \) and let \( \tau^* \) be the topology on \( T \) generated by the base \( \{ A^* : A \in \tau \} \), where \( A^* \) is the subset of \( T \) characterized by the following properties:
1) \( A^* \cap \mathbb{R} = A \);
2) for every \( y \in Y, \ y \in A^* \iff (\exists x > 0 : \gamma \in [y - \epsilon, y + \epsilon] \cap \Gamma \Rightarrow A \in \mathcal{G}_\gamma) \).

Observe that for every \( y \in A^* \cap Y \) there is some \( \epsilon > 0 \) such that \( [y - \epsilon, y + \epsilon] \subset A^* \). We claim that, endowed with the topology \( \tau^* \), is a connected locally pathwise connected (and hence pathwise connected) \( T_2 \)-space in which \( S \) is densely embedded.

Clearly \( S \) is a dense subspace of \( T \).

We first show that for every \( \omega = (n, m, i) \in \Omega \) the function \( f_\omega : [0, 1] \to T \) defined by
1) \( f_\omega(0) = d_n, \ f_\omega(1) = d_m; \)
2) \( f_\omega(t) = (\omega, t) \) for every \( t \in [0, 1] \)

is a path.

Let \( t \in [0, 1], \ y = f_\omega(t) = (\omega, t) \) and take a \( G \in \tau \) such that \( G^* \) is a neighbourhood of \( y \) in \( T \). Then there is a positive \( \epsilon \) such that \( G \in \mathcal{G}_\gamma \) for every \( \gamma \in [y - \epsilon, y + \epsilon] \cap \Gamma \).

Hence \( f_\omega([t - \epsilon, t + \epsilon] \cap [0, 1]) \) is connected and locally pathwise connected it is enough to show that \( A^* \) is pathwise connected for every \( A \in \tau \).

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Now let us prove that \( f_\omega \) is continuous at 0. Let \( G \in \tau \) such that \( d_n \in G^* \). Since \( d_n \in G \), there is a \( k > 0 \) such that \( B(n, i + k - 1) \subset G \). Observe that \( G \in F_{(\omega, h)} \) for every \( h \geq k \) (note that \( F_{(\omega, h)} \supset C(n, i + h - 1) \subset B(n, i + h - 1) \subset B(n, i + k - 1) \) for every \( h \geq k \)). Therefore \( G \in G_{(\omega, i)} \) for every \( t \in [0, \frac{1}{n+k}] \) (in fact let \( h \) be such that \( t \in [\frac{1}{2n+k}, \frac{1}{1+n+k}] \cap H \); then \( G_{(\omega, i)} \) finer than \( F_{(\omega, h)} \)).

Now let \( t \in [0, \frac{1}{1+k}] \), \( y = f_\omega(t) = (\omega, t) \) and take \( \epsilon > 0 \) such that \( [t - \epsilon, t + \epsilon] \subset [0, \frac{1}{1+k}] \) and \( G \in \mathcal{G}_\gamma \) for every \( \gamma \in [y - \epsilon, y + \epsilon] \cap \Gamma \). Then \( y \in G^* \). Hence \( f_\omega([0, \frac{1}{1+k}] \subset G^* \), and \( f_\omega \) is continuous at 0. Similarly it can be shown that \( f_\omega \) is continuous at 1. Therefore \( f_\omega \) is a path.

Now let us show that for every \( A \in \tau \), \( x, y \in A^* \) there is a path in \( A^* \) between \( x \) and \( y \).

i) \( x = d_n \) and \( y = d_m \). Let \( i \) be such that \( B(n, i) \cup B(m, i) \subset A \); then \( A \in \mathcal{G}_\gamma \) for every \( \gamma = (\omega, q) \in \Gamma \), where \( \omega = (n, m, i) \) and \( n < m \). Now set \( z = (\omega, t) \) with
$t \in ]0,1[ \}$ and let us show that $z \in A^*$. Take a positive $\epsilon$; then $A \in \mathcal{G}_\gamma$ for every $\gamma \in [y-\epsilon, y+\epsilon] \cap \Gamma$. Hence $z \in A^*$. Therefore $f_\omega$ is a path between $x$ and $y$ in $A^*$.

ii) $x = y \in A \cap D$ and $y \in A$. Let $\{B_n : n \in \mathbb{N}\}$ be a decreasing base for $S$ in $y$ such that $B_1 = A$, $C_n = B_n \setminus cl_X(B_{n+1}) \neq \emptyset$ and $d \in C_1$. Put $d_{n_1} = d$ and take $d_{n_k} \in C_k \cap D$ with $n_k < n_{k+1}$, for every $k \geq 2$. Now choose, for every $k \in \mathbb{N}$, $i_k$ so that $B(n_k, i_k) \cup B(n_k, i_k-1) \subset C_k$. Set $\omega_k = (n_k, n_{k+1}, i_k)$ and observe that $f_{\omega_k}$ is a path between $d_{n_k}$ and $d_{n_{k+1}}$, contained in $(B(n_k, i_k) \cup B(n_{k+1}, i_k))^r$ (and hence in $B_k^r$). Since $\{B_n^* : n \in \mathbb{N}\}$ is a base for $T$ at $y$ and, for every $n \in \mathbb{N}$, all but finitely many $f_{\omega_k}$ are contained in $B_n$, it is possible to find a path in $A^*$ between $x$ and $y$.

iii) $x = d \in A$ and $y = (\omega, t) \in A^* \setminus A$. Take $\epsilon > 0$ such that $[y-\epsilon, y+\epsilon] \subset A^*$. Since $[y-\epsilon, y+\epsilon]$ is a path, we may assume that $y \in \Gamma$. If $B^*$ is a neighbourhood of $y$, then there is $\epsilon > 0$ such that $B \in \mathcal{G}_\gamma$, for each $\gamma \in [y-\epsilon, y+\epsilon] \cap \Gamma$, in particular $B \in \mathcal{G}_\gamma$.

Now if $B$ is a countable base of $\mathcal{G}_y$, then $\{B^* : B \in \mathcal{B}\}$ is a local $\pi$-base of $T$ at $y$. Arguing as in the previous case we can find a path in $A^*$ between $x$ and $y$.

Now it remains to show that $T$ is a Hausdorff space. So let us consider two distinct points $x$ and $y$ in $T$.

1) $x, y \in \mathbb{R}$. Choose $A, B \in \pi$ such that $A \cap B = \emptyset$, $x \in A$ and $y \in B$. Clearly $A^*$ and $B^*$ are open in $T$, $x \in A^*$ and $y \in B^*$. We claim that $A^* \cap B^* = \emptyset$. Suppose not, and take $p \in A^* \cap B^*$. Since $p \in Y$, there is some $\epsilon > 0$ such that $A, B \in \mathcal{G}_\gamma$ for every $\gamma \in [y-\epsilon, y+\epsilon] \cap \Gamma$, a contradiction.

2) $x \in \mathbb{R}$ and $y = (\omega, r) \in Y$. It is easy to see that there are a sufficiently small $\epsilon > 0$ and $F_1, F_2 \in \Phi$ such that, for every $\gamma \in [y-\epsilon, y+\epsilon] \cap \Gamma$, $F_\omega \cap F_\gamma = F_2$. Since $F_1$ and $F_2$ are free filters, there are $A_1 \in F_1$, $A_2 \in F_2$ and a neighbourhood $B$ of $x$ such that $A_1 \cap B = \emptyset$, $i \in \{1, 2\}$. Set $A = A_1 \cup A_2$ and observe that $A^* \cap B^* = \emptyset$. Clearly $x \in B^*$; moreover $y \in A^*$ (since $A \in F_1$ and $A \in F_2$, it follows that $A \in F_{\omega \cap} \subset \mathcal{G}_y$ for every $\gamma \in [y-\epsilon, y+\epsilon] \cap \Gamma$).

3) $x = (\omega, r), y = (\omega', r') \in Y$, with $\omega \neq \omega'$. Take $\epsilon > 0$ and $F_i \in \Phi$ for every $i \in \{1, 2, 3, 4\}$ so that:

i) $F_i \neq F_j$ whenever $i \neq j$;

ii) $F_1 \subset \mathcal{G}_y$, or $F_3 \subset \mathcal{G}_y$, for every $\gamma = (\omega, t) \in [x-\epsilon, x+\epsilon] \cap \Gamma$;

iii) $F_3 \subset \mathcal{G}_y$, or $F_4 \subset \mathcal{G}_y$, for every $\gamma = (\omega', t) \in [y-\epsilon, y+\epsilon] \cap \Gamma$.

Since $\{F_1, F_2, F_3, F_4\}$ is Hausdorff separated, we can choose $A_i \in F_i$ for every $i \in \{1, 2, 3, 4\}$ in such a way that the family $\{A_1, A_2, A_3, A_4\}$ is cellular.

Put $A = A_1 \cup A_2$, $B = A_3 \cup A_4$. It is clear that $A^*$ and $B^*$ are open sets of $T$ such that $x \in A^*$ and $y \in B^*$. Since $A \cap B = \emptyset$, it follows that $A^*$ and $B^*$ are disjoint.

4) $y_1 = (\omega, r_1), y_2 = (\omega, r_2) \in Y$, with $r_1 < r_2$.

If there is some $n \geq 2$ such that $\frac{1}{n} \in [r_1, r_2]$ or $1 - \frac{1}{n} \in [r_1, r_2]$, then there are $\epsilon > 0$ and $F_1, F_2, F_3, F_4$ as in the previous case. So let us suppose that there is $n \geq 1$ such that $\frac{1}{n+1} \leq r_1 < r_2 \leq \frac{1}{n}$ (the case $1 - \frac{1}{n} \leq r_1 < r_2 \leq 1 - \frac{1}{n+1}$ is similar). Then there are $\epsilon > 0$ and $F, F_1, F_2 \in \Phi$ such that:

i) $[y_1 - \epsilon, y_1 + \epsilon] \cap [y_2 - \epsilon, y_2 + \epsilon] = \emptyset$;

ii) $F \subset \mathcal{G}_y$, or $F_1 \subset \mathcal{G}_y$, for every $\gamma \in [y_1 - \epsilon, y_1 + \epsilon] \cap \Gamma$;

iii) $F \subset \mathcal{G}_y$, or $F_2 \subset \mathcal{G}_y$, for every $\gamma \in [y_2 - \epsilon, y_2 + \epsilon] \cap \Gamma$.

Choose $F \in F, F_1 \in F_1$ and $F_2 \in F_2$ so that the family $\{F, F_1, F_2\}$ is cellular.

Since the family $\{\mathcal{G}_\gamma : \gamma \in \Gamma, F \subset \mathcal{G}_\gamma\}$ is totally Hausdorff separated, take $A_3 \subset \mathcal{G}_\gamma$ such that $A_3 \subset F$ and the family $\{A_3 : \gamma \in \Gamma, F \subset \mathcal{G}_\gamma\}$ is cellular.

Set $C_1 = F_1 \cup \bigcup\{A_3 : \gamma \in [y_1 - \epsilon, y_1 + \epsilon] \cap \Gamma, F \subset \mathcal{G}_\gamma\}$ and note that $C_1 \cap C_2 = \emptyset$. 

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Now it is enough to observe that $C_1^*$ and $C_2^*$ are disjoint open subsets of $T$ such that $y_i \in C_i^*$, $i \in \{1, 2\}$.

Remark 1. A space $X$ is called feebly compact if every countable open cover of $X$ has a finite subfamily whose union is dense. Observe that if $G$ is an open subset of a space $X$ whose closure is not feebly compact, then $G$ is an element of a free open filter on $X$ with a countable base (see Lemma 3.7 in [4]). It is easy to see that in the proof of Theorem 1 we use only the fact that the Sorgenfrey line is a first countable separable Hausdorff space in which every non-empty open subset has non feebly compact closure.

Therefore we have the following more general result.

Theorem 2. Every first countable separable Hausdorff space with no non-empty open subsets with feebly compact closure has a locally pathwise connected connectification.

A space is called pathwise connectifiable if it can be densely embedded in a pathwise connected Hausdorff space ([4]). Since every non-empty countable Hausdorff space without isolated points is not feebly compact ([5], Theorem 5.2), we have the following

Corollary 1 ([4]). Every countable first countable Hausdorff space without isolated points is pathwise connectifiable.

References

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