A CHARACTERIZATION OF ALGEBRAS WITH POLYNOMIAL GROWTH OF THE CODIMENSIONS

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(Communicated by Ken Goodearl)

Abstract. Let $A$ be an associative algebras over a field of characteristic zero. We prove that the codimensions of $A$ are polynomially bounded if and only if any finite dimensional algebra $B$ with $\text{Id}(A) = \text{Id}(B)$ has an explicit decomposition into suitable subalgebras; we also give a decomposition of the $n$-th cocharacter of $A$ into suitable $S_n$-characters.

We give similar characterizations of finite dimensional algebras with involution whose $*$-codimension sequence is polynomially bounded. In this case we exploit the representation theory of the hyperoctahedral group.

§1. Introduction

Let $F$ be a field of characteristic zero and $F(X) = F(x_1, x_2, \ldots)$ the free algebra of countable rank over $F$. If $A$ is a PI-algebra over $F$, that is, an algebra satisfying a polynomial identity, we let $\text{Id}(A)$ be the $T$-ideal of $F(X)$ of identities of $A$. It is well known that $\text{Id}(A)$ is completely determined by the multilinear polynomials it contains; if $V_n = \text{Span}\{x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \sigma \in S_n\}$ is the space of multilinear polynomials in $x_1, \ldots, x_n$, then the sequence $c_n(A) = \dim_F V_n \cap \text{Id}(A)$, $n = 1, 2, \ldots$, is called the sequence of codimensions of $A$ and it is an important numerical invariant of $\text{Id}(A)$.

It was proved by Regev in [R] that for any PI-algebra $A$, $c_n(A)$ is exponentially bounded, i.e., there exist constants $a, \alpha > 0$ such that $c_n(A) \leq a \alpha^n$ for all $n$.

In this paper we study algebras $A$ whose codimension sequence is polynomially bounded i.e., such that for all $n$, $c_n(A) \leq a t^n$ for some constants $a, t$. Kemer in [K1] gave a characterization of such $T$-ideals in the language of the representation theory of $S_n$. It also follows from [K2] that if $c_n(A)$ is polynomially bounded, then $\text{Id}(A) = \text{Id}(B)$ for a suitable finite dimensional algebra $B$.

For any finite dimensional algebra $A$ over an algebraically closed field we shall prove that $A$ has polynomial growth of the codimensions if and only if $A = A_0 \oplus A_1 \oplus \cdots \oplus A_m$ where $A_0, A_1, \ldots, A_m$ are F-algebras such that 1) for $i = 1, \ldots, m$, $A_i = B_i + J_i$, where $B_i \cong F$ and $J_i$ is a nilpotent ideal of $A_i$, 2) $A_0, J_1, \ldots, J_m$ are nilpotent right ideals of $A$ and 3) $A_iA_k = 0$ for all $i, k \in \{1, \ldots, m\}, i \neq k$ and $B_iA_0 = 0$.

Received by the editors December 1, 1998 and, in revised form, March 26, 1999.
1991 Mathematics Subject Classification. Primary 16R10, 16R50; Secondary 16P99.
The first author was partially supported by the CNR and MURST of Italy; the second author was partially supported by RFFI, grants 96-01-00146 and 96-15-96050.

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Another description of such algebras is given, as in Kemer’s paper, in the language of the cocharacters as follows. The symmetric group $S_n$ acts on the left on $V_n$ by $\sigma f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$, $\sigma \in S_n$, $f(x_1, \ldots, x_n) \in V_n$. This action induces a structure of left $S_n$-module on $\frac{V_n}{\text{Id}(A)}$, and we write $\chi_n(A)$ for its $S_n$-character; $\chi_n(A)$ is called the $n$-th cocharacter of $A$. Let $\chi_\lambda$ denote the irreducible $S_n$-character associated to the partition $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$ and write $\chi_n(A) = \sum_{\lambda} m_\lambda \chi_\lambda$ where $m_\lambda \geq 0$ are the corresponding multiplicities. We shall prove that if $\dim F < \infty$, $A$ has polynomial growth of the codimensions if and only if

$$\chi_n(A) = \sum_{\lambda} m_\lambda \chi_\lambda$$

where $J$ is the Jacobson radical of $A$ and $J^2 = 0$.

In the second part of the paper we address ourself to similar questions in the setting of algebras with involution. Let $F(X, \ast) = F(x_1, x_1^*, x_2, x_2^*, \ldots)$ be the free algebra with involution $\ast$ and $V_n(\ast)$ the space of multilinear $\ast$-polynomials in $x_1, x_1^*, \ldots, x_n, x_n^*$. For an algebra with involution $A$ we let $\text{Id}(A, \ast)$ be the ideal of $\ast$-identities of $A$; then the sequence of $\ast$-codimensions is

$$c_n(A, \ast) = \dim F \frac{V_n(\ast)}{V_n(\ast) \cap \text{Id}(A, \ast)}, \quad n = 1, 2, \ldots.$$ 

In the last section we characterize finite dimensional algebras $A$ such that $c_n(A, \ast)$ is polynomially bounded. Moreover by using the representation theory of the hyperoctahedral group $Z_2 \sim S_n$, we shall obtain a characterization of $A$ in terms of the $\ast$-cocharacter sequence of $A$.

§2. ALGEBRAS WITH POLYNOMIAL GROWTH OF THE CODIMENSIONS

In this section we shall characterize algebras with polynomial growth of the codimensions. The following reduction is due to Kemer.

**Theorem 1.** Let $A$ be a PI-algebra. If for all $n$, $c_n(A) \leq a_n t^n$ for some constants $a, t$, then there exists a finite dimensional algebra $B$ such that $\text{Id}(A) = \text{Id}(B)$.

**Proof.** Let $G$ be the Grassmann algebra of countable dimension over $F$. By $c_n(G) = 2^{n-1}$, hence, for $n$ large, $c_n(A) < c_n(G)$. This implies that $\text{Id}(A) \not\subseteq \text{Id}(G)$ and, by a theorem of Kemer [K2] Theorem 2.3] there exists a finite dimensional algebra $B$ such that $\text{Id}(A) = \text{Id}(B).$ \hfill \Box

We remark that $c_n(A)$ does not change upon extension of the ground field $F$. In fact, if $K$ is an extension field of $F$, then $\text{Id}(A) \otimes_F K = \text{Id}(A \otimes_F K).$ Therefore in studying properties of $c_n(A)$ we may as well assume that $F$ is an algebraically closed field.

**Theorem 2.** Let $A$ be a finite dimensional algebra over an algebraically closed field $F$. Then the sequence of codimensions $\{c_n(A)\}_{n \geq 1}$ is polynomially bounded if and only if

1. $A = A_0 \oplus A_1 \oplus \cdots \oplus A_m$ a vector space direct sum of $F$-algebras where for $i = 1, \ldots, m$, $A_i = B_i + J_i$, $B_i \cong F$, $J_i$ a nilpotent ideal of $A_i$ and $A_0, J_1, \ldots, J_m$ are nilpotent right ideals of $A$;
2. for all $i, k \in \{1, \ldots, m\}, i \neq k$, $A_i A_k = 0$ and $B_i A_0 = 0$. 


Proof. Let $A = B + J$ be the Wedderburn-Malcev decomposition of $A$ ([CR Theorem 72.19]) where $B$ is a semisimple subalgebra of $A$ and $J = J(A)$ its Jacobson radical. Write $B = B_1 \oplus \cdots \oplus B_m$ with $B_1, \ldots, B_m$ simple $F$-algebras. Since $c_n(A)$ is polynomially bounded, by [GZ], $B_iJB_k = 0$ for all $i \neq k$ and $\dim_F B_i = 1$, $i, k = 1, \ldots, m$.

Let $e = e_1 + \cdots + e_m$ be the decomposition of the unit element of $B$ into orthogonal central (in $B$) idempotents; thus $e_i B = B_i \cong F$. Define for all $i = 1, \ldots, m$, $J_i = e_i J$ and $J_0 = \{ x \in J \mid Bx = 0 \}$. It is easy to show that $A = B + J = (B_1 + J_1) \oplus \cdots \oplus (B_m + J_m) \oplus J_0$, let $A_i = B_i + J_i$ and $A_0 = J_0$.

For $i \neq k \in \{ 1, \ldots, m \}$, $A_i A_k = (B_i + J_i)(B_k + J_k) = 0$ since $e_i e_k = 0$ and $B_iJB_k = 0$. Also, for $i \neq 0$, $B_i A_0 = 0$.

Viceversa, let $A$ satisfy 1) and 2). From the relations $A_i A_k = 0$ and $B_i A_0 = 0$ it follows that $J = A_0 + J_1 + \cdots + J_m$ is a nilpotent two-sided ideal of $A$ and $A = B_1 \oplus \cdots \oplus B_m \oplus J$ where $B_i \cong F$ for all $i$. Since from the defining relations $A_i A_k = 0$ and $B_i A_0 = 0$ it follows that $B_i JB_k = 0$ for all $i \neq k$, then $c_n(A)$ is polynomially bounded by [GZ].

As an immediate consequence of the above result we get

**Corollary 1.** Let $A$ be the algebra described in the previous theorem. Let $J$ be the Jacobson radical of $A$ and, for $i = 1, \ldots, m$, let $C_i = A_i \oplus A_0$. Then

$$Id(A) = Id(C_1) \cap \ldots \cap Id(C_m) \cap Id(J).$$

Proof. Let $f \in Id(C_1) \cap \ldots \cap Id(C_m) \cap Id(J)$ and suppose that $f \not\in Id(A)$. We may clearly assume that $f$ is multilinear and let $r_1, \ldots, r_s \in A$ be such that $f(r_1, \ldots, r_s) \neq 0$.

If $r_1, \ldots, r_s \in J$, then $f \not\in Id(J)$, a contradiction. Hence there exists $r_i \not\in J_i$ by linearity we may assume that $r_i \in B_k$ for some $k$. Recall that, for all $l$, $B_l A_0 = 0$, $J_l$ is a right ideal of $A$ and, in case $l \neq k$, $A_k A_l = A_l A_k = 0$. From an easy calculation it follows that $r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_s \in A_k \cup A_0$. But then $f \not\in Id(C_k)$, a contradiction. \[\square\]

What can be said if $F$ is not algebraically closed?

Let $A$ be a finite dimensional algebra over a field $F$ and write $A = B + J$, $B = B_1 \oplus \cdots \oplus B_m$ with the $B_i$’s simple algebras. If $\overline{F}$ is the algebraic closure of $F$, we write $\overline{A} = A \otimes_F \overline{F}$; moreover, since $J(\overline{A}) = J(A) \otimes_F \overline{F}$ (see [Ro] Theorem 2.5.36), we get that

$$\overline{A} \cong \overline{B}_1 \oplus \cdots \oplus \overline{B}_m + J(\overline{A})$$

where $\overline{B}_i = B_i \otimes_F \overline{F}$ are semisimple algebras.

Let $Z(B_i)$ be the center of $B_i$ and $t_i = \dim_F Z(B_i)$. Then $\overline{B}_i \cong C_{i1} \oplus \cdots \oplus C_{it_i}$ where $C_{i1} \cong \cdots \cong C_{i k}$ are central simple algebras over $\overline{F}$.

In case $c_n(\overline{A}) = c_n(\overline{A})$ is polynomially bounded, by [GZ], $C_{ik} \cong \overline{F}$ for all $i, k$ and $C_{ik} J(\overline{A}) C_{iv} = 0$ if $(i, k) \neq (u, v)$. It follows that for all $i = 1, \ldots, m$, $B_i$ is a field extension of $F$ of degree $t_i$. Since $\text{char} F = 0$, we write $B_i = F(a_i)$, a simple algebraic extension of $F$ of degree $t_i$. We have shown that if $F$ is any field and $A$ is an $F$-algebra with polynomial growth of the codimensions, then

$$A \cong F(a_1) \oplus \cdots \oplus F(a_m) + J(A)$$

and for all $i \neq k$, $F(a_i) J(A) F(a_k) = 0$. 

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We next give a characterization of polynomial growth in terms of the cocharacter sequence of the algebra.

In the sequel for \( \lambda \vdash n \) we also write \( |\lambda| = n \). We write \( \chi_\lambda(1) = d_\lambda \) for the degree of the irreducible \( S_n \)-character \( \chi_\lambda \) and, if \( T_\lambda \) is a tableau of shape \( \lambda \), we let \( e_{T_\lambda} \) be the corresponding essential idempotent of \( FS_n \). Notice that if \( \lambda = (\lambda_1, \lambda_2, \ldots) \vdash n \), then \( |\lambda| - \lambda_1 \) denotes the number of boxes below the first row of the diagram of \( \lambda \).

**Theorem 3.** Let \( A \) be a finite dimensional algebra over a field \( F \). Then \( \{c_n(A)\}_{n \geq 1} \) is polynomially bounded if and only if

\[
\chi_n(A) = \sum_{|\lambda| - \lambda_1 < q} m_\lambda \chi_\lambda
\]

where \( J(A)^q = 0 \).

**Proof.** Note that the decomposition of \( \chi_n(A) \) into irreducible components does not change when extending the base field. Therefore, since \( J(A \otimes F \overline{F})^q = 0 \), we may assume, without loss of generality, that \( F \) is algebraically closed.

Suppose first that the codimensions of \( A \) are polynomially bounded. Let \( \lambda \) be a partition of \( n \) such that \( |\lambda| - \lambda_1 \geq q \) and suppose by contradiction that \( m_\lambda \neq 0 \). Then there exists a tableau \( T_\lambda \) such that \( e_{T_\lambda}(x_1, \ldots, x_n) \notin \text{Id}(A) \). Let \( \lambda' = (\lambda'_1, \ldots, \lambda'_t) \) be the conjugate partition of \( \lambda \). Then \( e_{T_\lambda}(x_1, \ldots, x_n) \) is a linear combination of polynomials each alternating on \( t \) disjoint sets of \( \lambda'_1, \ldots, \lambda'_t \) variables, respectively. We shall reach a contradiction by proving that each such polynomial \( f \) vanishes in \( A \).

Fix a basis of \( A \) which is the union of bases of \( B_1, \ldots, B_m \) and \( J \) respectively. Since \( B_iB_k = B_iJB_k = 0 \) for all \( i \neq k \), in order to get a non-zero value of \( f \) we must replace all the variables with elements of \( J \) and of one simple component, say, \( B_i \). Also, since \( \dim B_i = 1 \), we can substitute at most one element of \( B_i \) in each alternating set. Hence we can substitute in all at most \( t = \lambda_1 \) elements from \( B_i \). It follows that in order to get a non-zero value, we must substitute at least \( |\lambda| - \lambda_1 \geq q \) elements from \( J \). Since \( J^q = 0 \), we get that \( f \equiv 0 \) and with this contradiction the proof of the first part of the theorem is complete.

Suppose now that \( \chi_n(A) = \sum_{|\lambda| - \lambda_1 < q} m_\lambda \chi_\lambda \) and \( m_\lambda = 0 \) whenever \( |\lambda| - \lambda_1 \geq q \). By [BR] the multiplicities \( m_\lambda \) are polynomially bounded; hence

\[
c_n(A) = \sum_{|\lambda| - \lambda_1 < q} m_\lambda d_\lambda \leq Cn^t \sum_{|\lambda| - \lambda_1 < q} d_\lambda
\]

and the proof now follows from the hook formula for the degrees \( d_\lambda \).

The previous theorem says that \( A \) has polynomial growth of the codimensions if and only if all the irreducible characters appearing with non-zero multiplicity in \( \chi_n(A) \) have associated diagram with at most \( q - 1 \) boxes below the first row where \( J^q = 0 \).

### §3. Finite dimensional algebras with involution

In this section we shall prove that if \( A \) is a finite dimensional algebra with involution \( * \), then in the decomposition \( A = B + J \) we can choose \( B \) to be invariant under \( * \). Beside its own interest, this result will be used in the next section. Throughout we shall assume that \( \text{char} F \neq 2 \).
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**Theorem 4.** Let $A$ be a finite dimensional algebra with involution $*$ over $F$ and $J$ its Jacobson radical. Then $J^* = J$ and there exists a maximal semisimple subalgebra $B$ such that $B = B^*$ and $A = B + J$.

**Proof.** It is obvious that $J^* = J$. Let $A = B + J$ with $B$ a semisimple subalgebra of $A$ and suppose first that $J^2 = 0$.

Since $B^*$ is also a maximal semisimple subalgebra of $A$, by the Wedderburn-Malcev theorem there exists $y \in J$ such that $B^* = (1 - y)B(1 + y)$. For $b \in B$ let $\overline{b} \in B$ be such that $b^* = (1 - y)\overline{b}(1 + y)$. Then $b = b^{**} = (1 + y^*)\overline{b}^*(1 - y^*) = (1 + y^*)(1 - y)b(1 + y)(1 - y^*)$, for a suitable $\overline{b} \in B$. It follows that we can write $b = \overline{b} + j$ for a suitable $j \in J$; hence $b - \overline{b} \in B \cap J = 0$ and $b = \overline{b}$ follows. But then, from the above, $b = (1 + y)y^*\overline{b}^*(1 - y^*) = (1 + y^*)(1 - y)b(1 + y)(1 - y^*) = (1 - y + y^*)b(1 + y - y^*)$ since $J^2 = 0$. This says that $y - y^*$ commutes with $b$. Therefore by writing $y = \frac{y + y^*}{2} + \frac{y - y^*}{2}$, we get $b^* = (1 - y)\overline{b}(1 + y) = (1 - \frac{y + y^*}{2})\overline{b}(1 + \frac{y + y^*}{2})$.

We have proved that $B^* = (1 - s)B(1 + s)$ for a suitable symmetric element $s = s^* \in J$. At this stage it is easy to check that $B^* = (1 - s)B(1 + s)$ is the desired invariant subalgebra of $A$.

Suppose now that $J^n = 0, J^{n-1} \neq 0, n > 2$. Set $J^{n-1} = I$. Since $I^* = I$, $A/I$ has an induced involution; also $J(A/I) = J/I$ and $J(A/I)^{n-1} = 0$. Therefore, by induction on $n$, $A/I = B/I + J/I$ for a suitable semisimple subalgebra $B/I = (B/I)^*$. It follows that we can write $B = C + I$ where $C$ is a semisimple subalgebra of $B$ and, since $B^* = B$, by the first part we may assume that $C^* = C$. By counting dimensions we get that $C$ is a maximal semisimple subalgebra of $A$ and $A = C + J$ is the desired decomposition.

Recall that an algebra with involution $A$ is $*$-simple if $A$ has no proper $*$-invariant ideals (i.e., ideals $I$ such that $I^* = I$). It is well known and easy to prove that if $A$ is $*$-simple, then either $A$ is simple or $A \cong A_1 \oplus A_1^{op}$ where $A_1$ is a simple homomorphic image of $A$ and $*$ on $A_1 \oplus A_1^{op}$ is the exchange involution $(a, b)^* = (b, a)$ (see [Ro Proposition 2.13.24]).

**Remark 1.** If $B$ is a semisimple algebra with involution and $\dim_F B < \infty$, then $B = B_1 \oplus \cdots \oplus B_t$ where $B_1, \ldots, B_t$ are $*$-simple algebras.

**Proof.** Let $B = C_1 \oplus \cdots \oplus C_m$ be the decomposition of $B$ into simple components. Let $e_1, \ldots, e_m$ be the corresponding orthogonal central idempotents. Let $i \in \{1, \ldots, m\};$ if $e_i^* = e_i$, then $C_i = C_i^* = e_i B$ is $*$-simple. If $e_i^* \neq e_i$, then $e_i^* B$ is still a minimal ideal of $B$ which implies that $e_i^* = e_j$ for some $j \in \{1, \ldots, m\}$. Hence $C_i \oplus C_j$ is $*$-simple.

§4. $*$-CODIMENSIONS WITH POLYNOMIAL GROWTH

Throughout this section $F$ will be a field of characteristic zero, and $A$ an $F$-algebra with involution $*$. Let $A^+ = \{a \in A \mid a = a^*\}$ and $A^- = \{a \in A \mid a = -a^*\}$ be the sets of symmetric and skew elements of $A$ respectively.

We consider $F(X, *) = F(x_1, x_1^*, x_2, x_2^*, \ldots)$ the free algebra with involution $*$ of countable rank. Recall that $f(x_1, x_1^*, \ldots, x_n, x_n^*) \in F(X, *)$ is a $*$-polynomial identity for $A$ if \[ f(a_1, a_1^*, \ldots, a_n, a_n^*) = 0 \] for all $a_1, \ldots, a_n \in A$. The set $Id(A, *)$ of all $*$-polynomial identities of $A$ is a T-ideal of $F(X, *)$, i.e., an ideal invariant
under all endomorphisms of $F(X, *)$ commuting with the involution. Let
\[ V_n(*) = \text{Span}\{x_{\sigma(1)}^{a_1} \cdots x_{\sigma(n)}^{a_n} \mid \sigma \in S_n, \ a_i \in \{1, \ast\}\} \]
be the space of multilinear $*$-polynomials in $x_1, x_1', \ldots, x_n, x_n'$.

If we set $s_i = x_i + x_i'$ and $k_i = x_i - x_i'$, $i = 1, 2, \ldots$, then, since $\text{char} F \neq 2$, we can also write
\[ V_n(*) = \text{Span}\{w_{\sigma(1)} \cdots w_{\sigma(n)} \mid \sigma \in S_n, \ w_i = s_i \text{ or } w_i = k_i, \ i = 1, \ldots, n\}. \]
Let $H_n$ be the hyperoctahedral group. Recall that $H_n = \mathbb{Z}_2 \ast S_n$ is the wreath product of $\mathbb{Z}_2 = \{1, \ast\}$, the multiplicative group of order 2, and $S_n$. We write the elements of $H_n$ as $(a_1, \ldots, a_n; \sigma)$ where $a_i \in \mathbb{Z}_2, \ \sigma \in S_n$. The action of $H_n$ on $V_n(*)$ defined in [GR] can be rewritten (see [DG]) as follows: for $h = (a_1, \ldots, a_n; \sigma) \in H_n$ define $hs_i = s_{\sigma(i)}$, $hk_i = k_{\sigma(i)} = \pm k_{\sigma(i)}$ and then extend this action diagonally to $V_n(*)$. Hence $V_n(*)$ becomes a left $H_n$-module and, since $V_n(*) \cap \text{Id}(A, *)$ is a subspace invariant under this action, we can view $V_n(*)/(V_n(*) \cap \text{Id}(A, *))$ as an $H_n$-module. Let $\chi_n(A, *)$ be its character.

The sequence $c_n(A, *) = \chi_n(A, *)(1) = \dim_F V_n(*)/\text{Id}(A, *), \ n = 1, 2, \ldots$ is called the sequence of $*$-codimensions of $A$.

Recall that there is a one-to-one correspondence between irreducible $H_n$-characters and pairs of partitions $(\lambda, \mu)$, where $\lambda \vdash r, \ \mu \vdash n - r$, for all $r = 0, 1, \ldots, n$. If $\chi_{\lambda, \mu}$ denotes the irreducible $H_n$-character corresponding to $(\lambda, \mu)$, then we can write
\[ \chi_n(A, *) = \sum_{r=0}^{n} \sum_{\lambda \vdash r, \mu \vdash n-r} m_{\lambda, \mu} \chi_{\lambda, \mu} \]
where $m_{\lambda, \mu} \geq 0$ are the corresponding multiplicities.

Now, for $r = 0, \ldots, n$, we let $V_{r,n-r} = \text{Span}\{w_{\sigma(1)} \cdots w_{\sigma(n)} \mid \sigma \in S_n, \ w_i = s_i \text{ for } i = 1, \ldots, r \text{ and } w_i = k_i \text{ for } i = r+1, \ldots, n\}$. Thus $V_{r,n-r}$ is the space of multilinear polynomials in $s_1, \ldots, s_r, k_{r+1}, \ldots, k_n$. It is clear that in order to study $V_n(*) \cap \text{Id}(A, *)$ it is enough to study $V_{r,n-r} \cap \text{Id}(A, *)$ for all $r$.

If we let $S_r$ act on the symmetric variables $s_1, \ldots, s_r$ and $S_{n-r}$ on the skew variables $k_{r+1}, \ldots, k_n$, we obtain an action of $S_r \times S_{n-r}$ on $V_{r,n-r}$ and
\[ V_{r,n-r}(A, *) = \frac{V_{r,n-r}}{V_{r,n-r} \cap \text{Id}(A, *)} \]
becomes a left $S_r \times S_{n-r}$-module. Let $\psi_{r,n-r}(A, *)$ be its character and
\[ c_{r,n-r}(A, *) = \psi_{r,n-r}(A, *)(1) = \dim_F V_{r,n-r}(A, *). \]

We write $\psi_{\lambda, \mu}$ for the irreducible $S_r \times S_{n-r}$-character associated to the pair $(\lambda, \mu)$ with $\lambda \vdash r, \mu \vdash n - r$. The following result holds.

**Theorem 5** ([DG] Theorem 1.3). Let $A$ be an algebra with involution; then, for all $n$,
\[ \chi_n(A, *) = \sum_{r=0}^{n} \sum_{\lambda \vdash r, \mu \vdash n-r} m_{\lambda, \mu} \chi_{\lambda, \mu}, \quad \text{and} \quad \psi_{r,n-r}(A, *) = \sum_{\lambda \vdash r, \mu \vdash n-r} m_{\lambda, \mu} \psi_{\lambda, \mu}. \]
Moreover
\[ c_n(A, \ast) = \sum_{r=0}^{n} \binom{n}{r} c_{r, n-r}(A, \ast). \]

We next characterize finite dimensional algebras A with polynomial growth of the \( \ast \)-codimensions.

**Theorem 6.** Let A be a finite dimensional algebra with involution over an algebraically closed field F. Then the sequence of \( \ast \)-codimensions \( \{c_n(A, \ast)\}_{n \geq 1} \) is polynomially bounded if and only if

1) the sequence of codimensions \( \{c_n(A)\}_{n \geq 1} \) is polynomially bounded;
2) \( A = B + J \), where B is a maximal semisimple subalgebra of A and \( b = b^* \) for all \( b \in B \).

**Proof.** By [GR, Lemma 4.4], for all \( n \), \( c_n(A) \leq c_n(A, \ast) \leq an^t \) for some constants \( a, t \), and the sequence of codimensions is polynomially bounded. From [GZ] it follows that \( A = B + J \), where \( B = B_1 \oplus \cdots \oplus B_m \) and \( B_i \cong F \), \( b_i \in B_i \) for all \( i \neq k \). By Theorem 4 we may also assume that \( B^* = B \).

Suppose by contradiction that \( \ast \) is not the identity map on B; then, by Remark 1, there exist \( B_i, B_k \) such that \( C = B_i \oplus B_k \cong F \oplus F \) is \( \ast \)-simple with involution \( (a, b)^* = (b, a) \). Notice that, for all \( \sigma \in S_n \) and \( a_1, \ldots, a_n \in \{1, \ast\} \),
\[ x_{\sigma(1)}^{a_1} \cdots x_{\sigma(n)}^{a_n} \equiv x_1^{a_1} \cdots x_n^{a_n} \pmod{Id(C, \ast)}. \]
Moreover the set \( \{x_1^{a_1} \cdots x_n^{a_n} \mid a_i \in \{1, \ast\}\} \) is linearly independent modulo \( Id(C, \ast) \).
It follows that \( c_n(C, \ast) = 2^n \). Since \( c_n(C, \ast) \leq c_n(A, \ast) \) we get a contradiction.

Suppose now that \( A = B + J \), \( c_n(A) \) is polynomially bounded and \( \ast \) is the identity on B. In this case, if \( a \in A \), write \( a = b + j, \ b \in B, j \in J \). Then \( a - a^* = j - j^* \in J \) and \( A^- \subseteq J \).

Notice that if \( f(x_1, \ldots, x_n) \in V_n \cap Id(A) \), then, for every \( r = 0, \ldots, n \),
\[ f(s_1, \ldots, s_r, k_{r+1}, \ldots, k_n) \in V_{r, n-r} \cap Id(A, \ast). \]
Hence \( c_{r, n-r}(A, \ast) \leq c_n(A) \leq an^t \), for some \( a, t \), for all \( r \). Let \( J^q = 0 \). Since \( A^- \subseteq J \), then, for all \( r \leq n - q \), \( V_{r, n-r} \cap Id(A, \ast) = V_{r, n-r} \) and \( c_{r, n-r}(A, \ast) = 0 \) follows. By Theorem 5 for all \( n \) we obtain
\[ c_n(A, \ast) = \sum_{r=0}^{n} \binom{n}{r} c_{r, n-r}(A, \ast) \leq an^t \sum_{r=n-q+1}^{n} \binom{n}{r} \]
\[ = an^t \sum_{r=0}^{q-1} \binom{n}{r} \leq an^{t+q} \]
and \( c_n(A, \ast) \) is polynomially bounded. \( \square \)

Next we want to get an analogue of Theorem 3 above by using the representation theory of the hyperoctahedral group \( H_n \). We write \( \chi_{\lambda, \mu}(1) = d_{\lambda, \mu} \) for the degree of the irreducible \( H_n \)-character \( \chi_{\lambda, \mu} \). Recall that if \( \lambda \vdash n, \mu \vdash n - r \), then \( d_{\lambda, \mu} = \binom{n}{r} d_{\lambda} d_{\mu} \) (see [DG]).
Theorem 7. Let $A$ be a finite dimensional algebra with involution over a field $F$. Then the sequence of $*$-codimensions $\{c_n(A,*)\}_{n \geq 1}$ is polynomially bounded if and only if

$$\chi_n(A,*) = \sum_{|\lambda|+|\mu|=n \atop n-\lambda_1 < q} m_{\lambda,\mu} \chi_{\lambda,\mu}$$

where $J(A)^q = 0$.

Proof. Since the decomposition of $V_n(*)$ into irreducible $H_n$-modules does not change by extending the scalars, as in the proof of Theorem 3 we may assume that $F$ is algebraically closed.

Suppose that the $*$-codimensions of $A$ are polynomially bounded and let $\lambda = (\lambda_1, \ldots, \lambda_t) + r, \mu = n - r$ be such that $n - \lambda_1 \geq q$. Suppose by contradiction that $m_{\lambda,\mu} \neq 0$; then there exist tableaux $T_\lambda, T_\mu$ such that $e_{T_\lambda} e_{T_\mu}$ has a non-trivial action on $V_{r,n-r}(A,*)$. This says that there exists a non-trivial polynomial $f \in \mathcal{P}_{r,n-r}(A,*)$ such that $f = f(s_1, \ldots, s_r, k_{r+1}, \ldots, k_n)$ is not a $*$-identity of $A$.

We have that $f$ is a linear combination of polynomials each alternating on $t$ disjoint sets of $\lambda_1^t, \ldots, \lambda_t^t$ symmetric variables respectively. Let $g$ be one such polynomial; it is clear that, in order to finish the proof, it is enough to show that $g \equiv 0$ in $A$.

Since $B_i B_k = B_i JB_k = 0$ for all $i \neq k$, we get $g \equiv 0$ on $A$ unless we substitute for the symmetric variables elements from one simple component, say $B_i$, and from $J$. Also, since $\dim B_i = 1$, only one element of $B_i$ can be replaced for a variable in each alternating set. Thus, since $A^- \subseteq J$, in all we substitute at least $|\lambda| - \lambda_1$ elements from $J$. Since $J^q = 0$ we get that $g \equiv 0$ also in this case and the proof of the first part is complete.

Suppose now that $\chi_n(A,*) = \sum_{|\lambda|+|\mu|=n-\lambda_1 < q} m_{\lambda,\mu} \chi_{\lambda,\mu}$. By a result of Berele ([B] Theorem 15) the multiplicities $m_{\lambda,\mu}$ are polynomially bounded. By recalling that if $|\lambda| - \lambda_1$ is bounded by a constant, then $d_{\lambda}$ is polynomially bounded, we get

$$c_n(A,*) = \sum_{|\lambda|+|\mu|=n \atop n-\lambda_1 < q} m_{\lambda,\mu} d_{\lambda,\mu} \leq \alpha n^t \sum_{r=n-q}^{n} \sum_{\lambda'} \binom{n}{r} d_{\lambda'} d_{\mu} \leq \alpha_1 n^{t_1} \sum_{r=0}^{q} \binom{n}{r} \leq \alpha_1 n^{t_1} n^{q+1}.$$ 

The previous theorem says that a finite dimensional algebra with involution $A$ has polynomial growth of the $*$-codimensions if and only if all the irreducible $H_n$-characters appearing with non-zero multiplicity in $\chi_n(A,*)$ are such that the diagram of $\lambda$, without the first row, and the diagram of $\mu$ contain in all at most $q$ boxes.

References


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