UNBOUNDED QUASI-INTEGRALS

ALF BIRGER RUSTAD

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Abstract. Let $X$ be a locally compact Hausdorff space. We define a quasi-measure in $X$, a quasi-integral on $C_0(X)$, and a quasi-integral on $C_c(X)$. We show that all quasi-integrals on $C_0(X)$ are bounded, continuity properties of the quasi-integral on $C_c(X)$, representation of quasi-integrals on $C_c(X)$ in terms of quasi-measures, and unique extension of quasi-integrals on $C_c(X)$ to $C_0(X)$.

1. Introduction

The notion of a quasi-measure was introduced in [1] by J. F Aarnes. In [1], physical states on commutative unital C*-algebras were represented by quasi-measures. The quasi-measure in [1] was defined as a regular, finitely additive set function on open and closed subsets of a compact Hausdorff space $X$. The quasi-integral (physical state) with respect to a quasi-measure was constructed on the space of continuous functions on $X$ (denoted $C(X)$). The quasi-integrals were shown to be the maps linear on each uniformly closed, singly generated subalgebra of $C(X)$.

Recent results (cf. [2], [4] and [6]) indicate that the quasi-measures are interesting as a generalization of regular Borel measures. The restriction of a quasi-measure to a compact Hausdorff space is therefore unfortunate. Accordingly, the work presented here aims to extend the theory in [1] to $X$ being a locally compact Hausdorff space.

In the sequel we let $X$ denote a locally compact Hausdorff space. A set is called bounded if its closure is compact. $F$ and $O$ denote respectively the class of closed and the class of open subsets of $X$. Similarly, $C$ and $O^*$ denote respectively the class of compact and the class of open bounded subsets. Furthermore we put $A = F \cup O$ and $A^* = C \cup O^*$.

Definition 1.1. A quasi-measure in $X$ is a function $\mu : A \to [0, \infty]$ satisfying the following conditions:
1. $\mu(A) < \infty$ if $A \in A^*$.
2. For any finite, disjoint collection $\{A_i\}_{i=1}^n \subset C \cup O$ with $\bigcup_{i=1}^n A_i \in C \cup O$, then
   $$\mu(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i).$$
3. $\mu(U) = \sup\{\mu(K) : K \subset U, K \in C\}, U \in O$.
4. $\mu(F) = \inf\{\mu(U) : F \subset U, U \in O\}, F \in F$.

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Our quasi-measure corresponds to the quasi-measure in [3], and the reader will find numerous properties of the quasi-measure there. The notion of a quasi-measure in a locally compact Hausdorff space can also be found in [7]. The definition in [7] is more restrictive than ours and does not produce the quasi-integrals given below.

Let \( C_0(X) \) denote the real-valued continuous functions on \( X \) vanishing at infinity and let \( C_c(X) \) be the functions in \( C_0(X) \) with compact support. The support of a function \( a \in C_0(X) \) will be denoted by \( \text{supp} a \) and the range of \( a \) in \( \mathbb{R} \) by \( \text{sp} a \).

If \( a \in C_0(X) \), let \( A_0(a) \) denote the smallest uniformly closed subalgebra of \( C_0(X) \) containing \( a \). For subsets \( K, O \subset X \) we will let \( K \prec a \) and \( a \prec O \) denote that \( a \in C_c(X), \text{sp} a \subset [0, 1] \) and respectively that \( x \in K \Rightarrow a(x) = 1 \) and \( \text{supp} a \subset O \).

**Definition 1.2.** A real-valued function \( \rho \) on \( C_0(X) \) is called a quasi-integral if the following conditions are satisfied:

1. \( b \geq 0 \Rightarrow \rho(b) \geq 0 \) whenever \( b \in C_0(X) \).
2. \( \rho \) is linear on \( A_0(a) \) for each \( a \in C_0(X) \).

When \( \sup \{\rho(a) : a \prec X\} < \infty \) we say that \( \rho \) is bounded. If in addition \( \sup \{\rho(a) : a \prec X\} = 1 \) we say that \( \rho \) is a quasi-state.

In the C*-algebra setting this corresponds to the commutative, nonunital case, where \( \rho \) is characterized by linearity on closed subalgebras generated by self-adjoint elements.

If \( a \in C_c(X) \), then we have \( A_0(a) \subset C_c(X) \). Hence we may define a quasi-integral on \( C_c(X) \) similarly as above:

**Definition 1.3.** A real-valued function \( \rho \) on \( C_c(X) \) is called a quasi-integral if:

1. \( f \geq 0 \Rightarrow \rho(f) \geq 0 \) whenever \( f \in C_c(X) \).
2. \( \rho \) is linear on \( A_0(f) \) for each \( f \in C_c(X) \).

If in addition \( \sup \{\rho(f) : f \prec X\} < \infty \), then \( \rho \) is bounded and we put \( ||\rho|| = \sup \{\rho(f) : f \prec X\} \).

The only difference between the definition above and Definition 1.2 is that \( \rho \) is now restricted to \( C_c(X) \). However we will show that if \( \rho \) is bounded, these two definitions coincide (Corollary 3.10). The key results in this article are boundedness of quasi-integrals on \( C_0(X) \) and a representation theorem between the quasi-measures in \( X \) and the quasi-integrals on \( C_c(X) \). The representation is a generalization of the Riesz Representation Theorem in [5].

The section below presents some preparatory results on the quasi-measures and quasi-integrals on \( C_0(X) \). The section ends with the boundedness theorem for quasi-integrals on \( C_0(X) \). The next and last section presents construction of the quasi-integral on \( C_c(X) \) with respect to a quasi-measure. Monotonicity and continuity properties of the quasi-integral are given. The section highlights with the representation theorem for quasi-measures and quasi-integrals on \( C_c(X) \). Finally, unique extension to \( C_0(X) \) of quasi-integrals on \( C_c(X) \) is given.

2. **QUASI-INTEGRALS ON \( C_0(X) \)**

Throughout this article we will assume that \( X \) is a locally compact Hausdorff space. The results in the following proposition were given in [3]. We will only give a brief outline of the proofs here.
Proposition 2.1. Let $\mu$ be a quasi-measure in $X$.
1. $\mu(\emptyset) = 0$.
2. $A \subset B \Rightarrow \mu(A) \leq \mu(B)$, $A, B \in A$.
3. If $K \in C, F \in F$ are disjoint, then $\mu(F \cup K) = \mu(F) + \mu(K)$.
4. $\mu$ is countably additive on open sets.
5. Let $\mu$ be a quasi-measure in $X$. For any increasing family of open sets $\{V_\lambda\}$, if $V_\lambda \not\subset V$ (i.e. $\bigcup V_\lambda = V$), then $\mu(V_\lambda) \not\subset \mu(V)$.

Proof. With $A_1 = A_2 = \emptyset$ in item 2 of Definition 1.1 we get $\mu(\emptyset) = 0$. The monotonicity follows from regularity (items 3 and 4 of Definition 1.1). The third statement follows from regularity and a Urysohn’s lemma argument. The fifth statement follows from regularity (item 3 of Definition 1.1). The fifth statement and finite additivity (item 2 of Definition 1.1) imply the fourth statement.

Proposition 2.2. A set function $\mu : A \to [0, \infty]$ satisfying items 1, 3, and 4 of Definition 1.1 is a quasi-measure if and only if the following are satisfied:
1. If $O_1, O_2 \in O$ are disjoint, then $\mu(O_1 \cup O_2) = \mu(O_1) + \mu(O_2)$.
2. If $K \subset O \in O$ with $K$ compact, then $\mu(O) = \mu(O \setminus K) + \mu(K)$.

Proof. The proof of the third statement in Proposition 2.1 holds for $\mu$. Hence by induction $\mu$ is finitely additive on $C$. Similarly, $\mu$ is finitely additive on $O$ by assumption. Let $\{A_i\}_{i=1}^n \subset C \cup O$ with disjoint union $A = \bigcup_{i=1}^n A_i \in C \cup O$. We may split the union to a disjoint union of a compact and an open set by $A = (\bigcup_{A_i \in C} A_i) \cup (\bigcup_{A_i \notin C} A_i)$. If $A$ is open, then $\mu(A) = \mu(\bigcup_{A_i \in C} A_i) + \mu(\bigcup_{A_i \notin C} A_i)$ by assumption. With $\mu$ finitely additive on $C$ and on $O$ we obtain $\mu(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$. If $A$ is compact we may use a similar argument. Hence it suffices to show that if $O$ is open and $O \subset K \in C$, then $\mu(K) = \mu(K \setminus O) + \mu(O)$. Let $K' \subset O$ be compact. Then since $\mu$ is monotone $\mu(K) \geq \mu(K \setminus O) + \mu(K')$. Taking supremum of all $K' \subset O$, regularity yields $\mu(K) \geq \mu(K \setminus O) + \mu(O)$. Conversely, given $\epsilon > 0$ pick an open set $U \supset K \setminus O$ with $\mu(U) < \mu(K \setminus O) + \epsilon$. Observing that $K \setminus U \subset O$ yields

$$
\mu(K) \leq \mu(U \cup O) = \mu(U) + \mu(K \setminus U) < \mu(K \setminus O) + \mu(O) + \epsilon.
$$

Equality follows. We have shown finite additivity for $\mu$ on $C \cup O$ which completes the proof.

Lemma 2.3. A quasi-integral $\rho$ on $C_0(X)$ is bounded on $A_0(a)$ for each $a \in C_0(X)$.

Proof. Suppose $\sup\{ \rho(f) : 0 \leq f \leq 1, f \in A_0(a) \} = \infty$ for some $a \in C_0(X)$. Choose $\phi_i$ with $\phi_i \circ a \in A_0(a)$, $\rho(\phi_i(a)) > 2^i$ and $0 \leq \phi_i(a) \leq 1$ for $i = 1, 2, \ldots$. Then with $\phi = \sum_{i=1}^{\infty} 2^{-i} \phi_i$ we have $\phi \circ a \in A_0(a)$, $0 \leq \phi(a) \leq 1$ and $\rho(\phi(a)) = \infty$, which is a contradiction. Hence we must have $\rho$ bounded on $A_0(a)$.

Remark 1. Note that $\rho$ is a linear functional on $A_0(a)$ and thus boundedness implies that $\rho$ is continuous on $A_0(a)$ for each $a \in C_0(X)$. Hence

$$
\sup\{ \rho(a) : a \prec X \} = \sup\{ \rho(a) : 0 \leq a \leq 1, a \in C_0(X) \}
$$

for all quasi-integrals $\rho$ on $C_0(X)$. Moreover, the complexification of $A_0(a)$ is a $C^*$-algebra so Lemma 2.3 is not a new result. We included it for completeness and the reader’s convenience.
Lemma 2.4. Suppose $a \in C_c(X)$ with $0 \leq a \leq 1$. Then there is a function $f \in C_c(X)$ with $\text{supp } a < f$. Moreover, $\text{supp } a < f < X$ implies that $a, f \in A_0(a+f)$ and $\rho(a) \leq \rho(f)$.

Proof. If $a \in C_c(X)$, then $\text{supp } a = K$ is compact. There is an open bounded set $V$ containing $K$. By Urysohn’s lemma there is a function $f$ with $K < f < V$ which implies that $f \in C_c(X)$. Define $\phi_1$ and $\phi_2$ by

$$
\phi_1(x) = \begin{cases} 
1, & x \geq 1, \\
2^{i}(x-2^{-i}), & 2^{-i} \leq x < 2^{-i+1}, \\
0, & x < 1.
\end{cases}
$$

Then $\phi_1(a+f) = f$ and $\phi_2(a+f) = a$; thus $a, f \in A_0(a+f)$ and we get $\rho(a) \leq \rho(f)$.

Theorem 2.5 (Boundedness of quasi-integrals). All quasi-integrals on $C_0(X)$ are bounded.

Proof. Let $\rho$ be a quasi-integral on $C_0(X)$ and suppose $\sup \{\rho(a) : a < X\} = \infty$. By Lemma 2.4 recursively construct a sequence $\{a_i\}_{i=1}^{\infty}$ where $\rho(a_i) \geq 2^{i+1}$ and $\text{supp } a_i < a_{i+1} < X$ for each $i$. Let $f = \sum_{i=1}^{\infty} 2^{-i} a_i$. Then $f \in C_0(X)$ since $C_0(X)$ is complete. Define $\phi_i$ for $i = 1, 2, \ldots$ by

$$
\phi_i(x) = \begin{cases} 
1, & x \geq 2^{-i+1}, \\
2^{i}(x-2^{-i}), & 2^{-i} \leq x < 2^{-i+1}, \\
0, & x < 2^{-i},
\end{cases}
$$

we have $\phi_i \in C(\text{sup } f)$, $\phi_i(0) = 0$ and $\phi_i(f) = a_i$ for each $i$ and thus $\{a_i\}_{i=1}^{\infty} \subset A_0(f)$. Finally $f \geq 2^{-i} a_i$ implies $\rho(f) \geq 2^{-i} \rho(a_i) \geq 2^i$ for $i = 1, 2, \ldots$ which in turn implies that $\rho(f) = \infty$. This is a contradiction since $\rho$ is supposed to be a quasi-integral on $C_0(X)$; we may conclude that $\sup \{\rho(a) : a < X\} < \infty$ so $\rho$ is bounded.

Remark 2. Theorem 2.5 shows that the local linearity of the quasi-integrals impose strong restrictions on their global behavior. This suggests that unbounded quasi-integrals on $C_c(X)$ may exhibit nice properties. Indeed, this is what we will devote the next and last section to.

3. QUASI-INTEGRALS ON $C_c(X)$

Proposition 3.1. Suppose that $\mu$ is a quasi-measure in $X$ and $f \in C_c(X)$. Then there is a unique bounded regular Borel measure $\mu_f$ on $R \setminus \{0\}$ with $\mu_f(O) = \mu(f^{-1}(O))$ for all open sets $O \subset R \setminus \{0\}$.

Proof. Let $\tilde{f}(x) = \mu(f^{-1}(-\infty, x) \setminus \{0\})$ which implies that $\tilde{f}$ is increasing. Since $f \in C_c(X)$ we have that $f(\text{supp } f)$ is compact. Hence $\tilde{f}$ is constant outside an interval $[a, b]$ for some $a, b \in \mathbb{R}$. By Proposition 2.1 $f(x^-) = \tilde{f}(x)$ for each $x \in \mathbb{R}$, so $\tilde{f}$ is continuous from the left. Thus $\mu_f(x, y) = \tilde{f}(y) - \tilde{f}(x)$ uniquely determines a regular Borel measure in $\mathbb{R}$ and by regularity $\mu_f(x, y) = \tilde{f}(y) - \tilde{f}(x^+)$. If $x_\lambda \in (x, y)$, then $\tilde{f}(y) \geq \mu(f^{-1}([x_\lambda, y) \setminus \{0\})) + \tilde{f}(x_\lambda)$ since $\mu$ is monotone; hence $\tilde{f}(y) - \tilde{f}(x^+) \geq \mu(f^{-1}([x, y) \setminus \{0\}])$. Conversely finite additivity and monotonicity of $\mu$ yields

$$
\tilde{f}(y) = \mu(f^{-1}([x, y) \setminus \{0\}]) + \mu_f^{-1}((x) \setminus \{0\}) + \tilde{f}(x) \\
\leq \mu(f^{-1}([x, y) \setminus \{0\}]) + \tilde{f}(x_\lambda),
$$

so $\tilde{f}(y) - \tilde{f}(x^+) \leq \mu(f^{-1}([x, y) \setminus \{0\}])$. We have $\mu(f^{-1}([x, y) \setminus \{0\}]) = \tilde{f}(y) - \tilde{f}(x^+) = $ and since both $\mu$ and $\mu_f$ are countably additive on open sets, the proof is complete.
Remark 3. We will call $\mu_f$ the measure corresponding to $\mu$ and $f$. Notice that Proposition 3.1 is stated only for open sets not containing $\{0\}$, whereas the proof produces a measure on $\mathbb{R}$ with $\mu_f(\{0\}) = 0$. This is convenient when the quasi-measure is an extended real-valued function. In fact, Lemma 3.3 is not valid unless zero is omitted.

Definition 3.2. A map $f \mapsto \mu_f$ from $C_c(X)$ into the regular Borel measures in $\mathbb{R}\setminus\{0\}$ is consistent if $\mu_{\phi f} = \mu_f \circ \phi^{-1}$ for each $f \in C_c(X)$ and $\phi \in C(\text{sp} f), \phi(0) = 0$.

Lemma 3.3. Let $\mu$ be a quasi-measure in $X$. Let $\mu_f$ denote the measure corresponding to $\mu$ and $f \in C_c(X)$. Then the map $f \mapsto \mu_f$ is consistent.

Proof. Let $f \in C_c(X), \phi f \in A_0(f)$ and $K \subset \mathbb{R}\setminus\{0\}$ be compact. Now $0 \notin \phi^{-1}(K)$ implies:

$$
\mu_{\phi f}(K) = \mu((\phi \circ f)^{-1}(K)) = \mu(f^{-1}(\psi^{-1}(K))) = \mu_f(\phi^{-1}(K)).
$$

Note that since $K$ is compact in $\mathbb{R}\setminus\{0\}$, then $K$ is compact in $\mathbb{R}$ by the identity map. So $f^{-1}(\phi^{-1}(K))$ is a closed subset of $\text{sp} f$, and thus compact. The result now follows from the regularity of $\mu_f$.

In the sequel we will assume that the measure corresponding to a quasi-measure $\mu$ and a function $f \in C_c(X)$ is extended to $\mathbb{R}$ by $\mu_f(\{0\}) = 0$.

Proposition 3.4. Let $\rho$ be a quasi-integral on $C_c(X)$. If $f, g \in C_c(X)$ and $f \leq g$, then $\rho(f) \leq \rho(g)$.

Proof. Given $\delta > 0$, suppose $f \geq 0$ and $g(x) \geq \delta + f(x)$ when $x \in \text{sp} f$. Pick a natural number $n$ such that $n\delta > \max g$ and define $\phi_i \in C(\text{sp} f), 1 \leq i \leq n$ by:

$$
\phi_i(x) = \begin{cases} 
0, & x \leq (i-1)\delta, \\
(x-(i-1)\delta), & (i-1)\delta < x < i\delta, \\
\delta, & x \geq i\delta.
\end{cases}
$$

Then $x \in \text{sp} \phi_i(f) \Rightarrow \phi_i(g(x)) = \delta$; thus $\phi_i(f), \phi_i(g) \in A_0(\phi_i(f) + \phi_i(g))$ which imply $\rho(\phi_i(f)) \leq \rho(\phi_i(g))$. Now $\sum \phi_i(f) = f, \sum \phi_i(g) = g$ implies $\rho(f) \leq \rho(g)$. Given $\epsilon > 0$, suppose now that $0 \leq f \leq g$, choose $h$ and $\delta > 0$ with $\text{supp} g < h \subset C_c(X)$ and $\rho(\delta h) < \epsilon$. We have $\rho(f) \leq \rho(g + \delta h) < \rho(g) + \epsilon$. Let $f \leq g \in C_c(X)$ be arbitrary. Then $f^+, f^- \in A(f)$ and $f^+ \leq g^+, f^- \geq g^-$. We have $\rho(f) = \rho(f^+) - \rho(f^-) \leq \rho(g^+) - \rho(g^-) = \rho(g)$ by the previous argument. The proof is complete.

Corollary 3.5. Let $\rho$ be a quasi-integral on $C_c(X)$ and let $K$ be an arbitrary compact subset of $X$. Then there is a $k \in \mathbb{R}$ such that whenever $\text{supp} f_i \subset K, f_i \in C_c(X)$ for $i = 1, 2$, we have:

$$
|\rho(f_1) - \rho(f_2)| \leq k\|f_1 - f_2\|.
$$

Proof. Pick a $g > K$ and let $\rho(g) = k$. Then $f_1 \leq f_2 + g\|f_1 - f_2\|$ which implies that $|\rho(f_1) - \rho(f_2) \leq \rho(g)\|f_1 - f_2\|$ and conversely $\rho(f_2) - \rho(f_1) \leq \rho(g)\|f_1 - f_2\|$. But then we must have $|\rho(f_1) - \rho(f_2)| \leq k\|f_1 - f_2\|$.
Remark 4. In general $\rho$ is not uniformly continuous (since it is a generalization of regular Borel measures). However, $\rho$ is continuous with respect to the topology of uniform convergence on compacta. Hence this is a sharp result; we cannot expect stronger continuity properties.

Corollary 3.6. Let $\rho$ be a bounded quasi-integral on $C_c(X)$. Then for each pair $f_1, f_2 \in C_c(X)$ we have

$$|\rho(f_1) - \rho(f_2)| \leq \|\rho\| \|f_1 - f_2\|.$$  

Proof. Pick a function $g > \text{supp } f_1 \cup \text{supp } f_2$. Then $\rho(g) \leq \|\rho\|$ and the result follows from Corollary 3.5.

Proposition 3.7. Let $\mu$ be a quasi-measure in $X$. Define

$$\rho(f) = \int i \, d\mu_f$$  

for each $f \in C_c(X)$, where $\mu_f$ is the measure corresponding to $\mu$ and $i$ is the identity map on $R$. Then $\rho$ is a quasi-integral on $C_c(X)$.

Proof. By the transformation theorem for integrals and Lemma 3.8, the result follows.

Lemma 3.8. Let $\mu$ be a quasi-measure in $X$ and let $\rho$ be the corresponding quasi-integral. Then for each open set $O \subset X$ we have:

$$\mu(O) = \sup \{ \rho(f) : f \prec O \}.$$  

Moreover, if $\mu(X) < \infty$, then $\rho$ is bounded and $\|\rho\| = \mu(X)$.

Proof. First suppose $\mu(O) < \infty$. Choose a compact set $K \subset O$ with $\mu(K) > \mu(O) - \epsilon$, and a function $f$ with $K \prec f \prec O$. We have:

$$\rho(f) = \int_{\text{sp } f} i \, d\mu_f = \int_{\{1\}} d\mu_f + \int_{\{0,1\}} i \, d\mu_f$$

$$\geq \int_{\{1\}} d\mu_f = \mu_f(\{1\}) = \mu(f^{-1}\{1\})$$

$$\geq \mu(K) \text{ since } K \subset f^{-1}\{1\}.$$  

On the other hand we have:

$$\rho(f) \leq \int_{\text{sp } f} d\mu_f$$

$$= \mu_f(\text{sp } f \setminus \{0\})$$

$$= \mu(f^{-1}(0,\infty)) (f^{-1}(0,\infty) = f^{-1}(\text{sp } f \setminus \{0\}))$$

$$\leq \mu(O) \text{ since } f \prec O \Rightarrow f^{-1}(0,\infty) \subset O.$$  

These together imply $\mu(O) = \sup \{ \rho(f) : f \prec O \}$. If $\mu(O) = \infty$, then there is, for every natural number $n$, a compact set $K \subset O$ with $\mu(K) > n$. By the previous argument there is then a function $f$ with $K \prec f \prec O$ and $\rho(f) > n$. Hence $\mu(O) = \sup \{ \rho(f) : f \prec O \} = \infty$. If $\mu(X) < \infty$, put $O = X$ in the previous argument. Then $\mu(X) = \sup \{ \rho(f) : f \prec X \} = \|\rho\| < \infty$. The proof is complete.

Theorem 3.9 (The representation theorem). Let $X$ be a locally compact Hausdorff space.

1. To each quasi-measure $\mu$ in $X$ there is a unique quasi-integral $\rho$ on $C_c(X)$ such that for any $f \in C_c(X)$ we have

$$\rho(\phi(f)) = \int \phi(i) \, d\mu_f.$$
for all \( \phi \in \{ \phi \in C(\text{sp} f) : \phi(0) = 0 \} \). Here \( \mu_f \) is the regular Borel measure in \( \mathbb{R} \) corresponding to \( \mu \) and \( f \).

2. Conversely, for any quasi-integral \( \rho \) on \( C_c(X) \) there is a unique quasi-measure \( \mu \) in \( X \) such that \( \rho \) is the quasi-integral corresponding to \( \mu \). Specifically we have, for any open set \( O \subset X \):

\[
\mu(O) = \sup\{ \rho(f) : f < O \}.
\]

**Proof.** The first part of the theorem follows from Proposition 3.4. Suppose \( \rho \) is a quasi-integral on \( C_c(X) \). Define a set function \( \mu : O \to [0, \infty] \) by (3.1). Extend \( \mu \) to the closed subsets \( F \) of \( X \) by \( \mu(F) = \inf\{ \mu(O) : F \subset O, O \text{ is open} \} \). Notice that this implies \( \mu(K) = \inf\{ \rho(f) : f > K \} \) when \( K \) is compact by Urysohn’s lemma and the monotonicity of \( \rho \). We will show that \( \mu \) is a quasi-measure in \( X \). Note that \( \mu(A) < \infty \) when \( A \in \mathcal{A}^* \) by Urysohn’s lemma and Corollary 3.3. Suppose that \( O_1 \) and \( O_2 \) are open disjoint subsets of \( X \). Pick \( f_i \) with \( f_i < O_i \) and \( \rho(f_i) > \mu(O_i) - \epsilon \) for \( i = 1, 2 \). We have \( f_1 f_2 = 0 \) which implies \( f_1, f_2 \in \mathcal{A}(f_1 - f_2) \) and thus

\[
\mu(O_1 \cup O_2) \geq \rho(f_1 + f_2) = \rho(f_1) + \rho(f_2) \geq \mu(O_1) + \mu(O_2) + 2\epsilon.
\]

Conversely if \( f < O_1 \cup O_2 \), the opposite equality follows from observing that \( f = f_1 + f_2 \) where \( f_i(x) = f(x) \) if \( x \in O_i \) and elsewhere zero. Let \( K \subset O \subset X \) where \( K \) is compact and \( O \) is open. By Urysohn’s lemma there is an open bounded set \( U \) and functions \( f_K, f_U \) such that \( K \subset U \subset O \), \( K < f_k < O \) and \( U < f_U < O \) with \( \rho(f_U) > \mu(O) - \epsilon \). Then \( f_K, f_U \in \mathcal{A}(f_K + f_U) \) and \( f_U - f_K < O \setminus K \); thus

\[
\mu(O \setminus K) \geq \rho(f_U - f_K) = \rho(f_U) - \rho(f_K) > \mu(O) - \mu(K) - \epsilon,
\]

which yields \( \mu(O) \leq \mu(O \setminus K) + \mu(K) \) when \( \mu(O) < \infty \) and equality when \( \mu(O) = \infty \). Conversely, if \( f < O \setminus K \) with \( \rho(f) > \mu(O \setminus K) - \epsilon \), then \( K' = \text{supp} f \subset O \setminus K \), so \( (X \setminus K') \cap U \) is an open set containing \( K \). Pick \( f_K \) such that \( K < f_K < (X \setminus K') \cap U \); then \( \int f f_K = 0 \). We have

\[
\mu(O) \geq \rho(f_K + f) = \rho(f_K) + \rho(f) > \mu(K) + \mu(O \setminus K) - \epsilon.
\]

We have shown that \( \mu \) is a quasi-measure in \( X \). The uniqueness of \( \mu \) follows from Lemma 3.8. Let \( \rho_\mu \) denote the quasi-integral corresponding to \( \mu \); it remains to prove that \( \rho_\mu \) is equal to \( \rho \). Let \( f \in C_c(X) \) be arbitrary. Then \( \rho_f : \phi \to \rho(\phi(f)) \) is a functional on \( \{ \phi : \phi \in C(\text{sp} f), \phi(0) = 0 \} \). Extend \( \rho_f \) to a functional \( F \) on \( C(\text{sp} f) \) by \( F(\phi) = \rho_f(\phi - \phi(0)) + \phi(0), \phi \in C(\text{sp} f) \). Now \( \text{sp} f \) is compact and thus \( F \) determines a unique regular Borel measure \( \nu_f \) on \( \text{sp} f \) such that

\[
F(\phi) = \rho(\phi(f)) = \int \phi(x) \, d\nu_f(x) \text{ when } \phi \in C(\text{sp} f) \text{ and } \phi(0) = 0.
\]

So by regularity it suffices to show that \( \mu_f = \nu_f \) on the open subsets of \( E = \text{sp} f \setminus \{ 0 \} \). Suppose \( \epsilon > 0 \) and \( U \subset E \) open are arbitrary. Pick a compact set \( K \subset U \) such that \( \nu_f(K) > \nu_f(U) - \epsilon \). Choose an Urysohn function \( K < \phi < U \); then \( \nu_f(U) - \epsilon < \nu_f(K) \leq \rho(\phi(f)) \leq \mu(f^{-1}(U)) = \mu_f(U) \) since \( \phi \circ f < f^{-1}(U) \). Conversely, pick a compact set \( K \subset f^{-1}(U) \) such that \( \mu(K) > \mu(f^{-1}(U)) - \epsilon \) and a function \( \phi \) with \( f(K) < \phi < U \). Then since \( \phi \circ f < f^{-1}(K) \) we have

\[
\mu_f(U) - \epsilon = \mu(f^{-1}(U)) - \epsilon < \mu(K) \leq \rho(\phi \circ f) = \rho_f(\phi) \leq \nu_f(U).
\]

The proof is complete.
Corollary 3.10. Let \( \rho \) be a quasi-integral on \( C_c(X) \). If \( \rho \) is bounded, then \( \rho \) has a unique extension to a quasi-integral on \( C_0(X) \).

Proof. By Corollary 3.6 \( \rho \) is uniformly continuous. Extend \( \rho \) by continuity to a function \( \rho_0 : C_0(X) \to \mathbb{R} \). For example by the functions \( \phi_\epsilon \) defined by

\[
\phi_\epsilon(x) = \begin{cases} 
0, & x < \epsilon, \\
2x - 2\epsilon, & \epsilon \leq x \leq 2\epsilon, \\
x, & x > 2\epsilon.
\end{cases}
\]

Obviously \( \rho_0(\alpha f) = \alpha \rho_0(f) \) for all \( \alpha \in \mathbb{R}, f \in C_0(X) \). Suppose \( f \in C_0(X) \) and \( \phi_1(f), \phi_2(f) \in A_0(f) \). Then \( \phi_i(\phi_\epsilon(f)) \in A_0(\phi_\epsilon(f)) \) for all \( \epsilon > 0 \) and \( i = 1, 2 \). Note that \( \phi_i(\phi_\epsilon(f)) \) converges uniformly to \( \phi_i(f) \) when \( \epsilon \) tends to zero. Hence by continuity

\[
\rho(\phi_1(f) + \phi_2(f)) = \lim_{\epsilon \to 0} \rho(\phi_1(\phi_\epsilon(f)) + \phi_2(\phi_\epsilon(f)))
\]

\[
= \lim_{\epsilon \to 0} [\rho(\phi_1(\phi_\epsilon(f))) + \rho(\phi_2(\phi_\epsilon(f)))]
\]

\[
= \rho(\phi_1(f)) + \rho(\phi_2(f)).
\]

We have shown that \( \rho_0 \) is a quasi-integral on \( C_0(X) \). The uniqueness of the extension is immediate from the continuity of \( \rho_0 \). The proof is complete.

References