A MATRIX-VALUED CHOQUET–DENY THEOREM

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Abstract. Let $\sigma$ be a positive matrix-valued measure on a locally compact abelian group $G$ such that $\sigma(G)$ is the identity matrix. We give a necessary and sufficient condition on $\sigma$ for the absence of a bounded non-constant matrix-valued function $f$ on $G$ satisfying the convolution equation $f \ast \sigma = f$. This extends Choquet and Deny’s theorem for real-valued functions on $G$.

1. Introduction

Let $\sigma$ be a probability measure on a locally compact abelian group $G$. A real Borel function $f$ on $G$ is called $\sigma$-harmonic if it satisfies the integral equation

$$f(x) = (f \ast \sigma)(x) = \int_G f(x - y)d\sigma(y) \quad (\text{for all } x \in G).$$

A celebrated theorem of Choquet and Deny \[2\] asserts that every bounded $\sigma$-harmonic function on $G$ is constant if (and only if) $\sigma$ is adapted; that is, the support of $\sigma$ generates a dense subgroup of $G$. Choquet and Deny’s theorem plays an important role in probability theory and has been extended to various non-abelian groups (see, for example, \[3, 4, 5, 7, 8, 10\]). Recently in \[11\], a vector-valued version of the Choquet–Deny theorem has been proved and used to obtain a vector-valued renewal theorem for the study of the $L^p$ dimension of some vector-valued self-similar measures. In a related paper \[3\], the equation $f \ast \sigma = f$ has been studied under the assumption that both $\sigma$ and $f$ are operator-valued, but with commuting ranges. In this paper, we remove the restriction of commuting ranges and prove a Choquet–Deny type theorem for matrix-valued functions defined on $G$. This theorem uses positive definite matrices and differs from that of \[11\] where matrices with non-negative entries are considered instead, and consequently different techniques are used.

Let $M_n$ be the $C^*$-algebra of $n \times n$ complex matrices. The pure states of $M_n$ are exactly the vector states $\rho(\cdot) = (\cdot \xi, \xi)$ where $\xi$ is a unit vector in $\mathbb{C}^n$. Let $M_n^+$ be the positive cone of $M_n$, consisting of all self-adjoint matrices with non-negative eigenvalues. An $M_n^+$-valued measure $\sigma$ on $G$ will be called a positive $M_n$-valued measure and its support is defined to be

$$\text{supp } \sigma = \{ x \in G : \sigma(V) \neq 0 \text{ for all open sets } V \text{ containing } x \}.$$ 

We say that $\sigma$ is adapted if $\rho \circ \sigma$ is adapted on $G$ for every pure state $\rho$ of $M_n$. We note that $\text{supp } (\rho \circ \sigma) \subseteq \text{supp } \sigma$. We can write $\sigma = (\sigma_{ij})$ where each $\sigma_{ij}$ is a

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complex-valued measure on $G$. A function $f : G \rightarrow M_n$ can also be denoted by $f = (f_{ij})$ where each $f_{ij}$ is a complex-valued function on $G$. The convolution $f \ast \sigma$ can be defined naturally in terms of $f_{kl} \ast \sigma_{ij}$ via matrix multiplication. Details are given later. An $M_n$-valued function $f$ on $G$ is called \( \sigma \)-\text{harmonic} if it satisfies the convolution equation $f \ast \sigma = f$. We can now state our main result.

**Theorem 1.** Let $\sigma$ be a positive $M_n$-valued measure on $G$ such that $\sigma(G)$ is the identity matrix. The following conditions are equivalent:

1. every bounded $\sigma$-harmonic $M_n$-valued function on $G$ is constant;
2. $\sigma$ is adapted.

2. Choquet–Deny type theorem

We need some vector measure preliminaries. Let $B$ be the algebra of Borel sets in $G$. By an $M_n$-valued measure on $G$, we mean a (norm) countably additive function $\sigma : B \rightarrow M_n$. If we use the matrix notation $\sigma = (\sigma_{ij})$, then each $\sigma_{ij}$ is a complex-valued measure on $G$. We adopt the definition of a complex measure in [13] and note that complex-valued measures are not only bounded, but also of bounded total variation [13, Theorem 6.4]. As in [6, p. 2], we define the \text{semivariation} of $\sigma$ to be a non-negative function $k$ whose value on a set $S \in B$ is given by

$$k(S) = \sup \left\{ |\rho \circ \sigma|(S) : \rho \in M_n^*, \|\rho\| \leq 1 \right\}$$

where $|\rho \circ \sigma|$ is the total variation of the complex measure $\rho \circ \sigma$. We will write $\rho \circ \sigma$ for $\rho \circ \sigma$. We say that $\sigma$ is of \text{bounded semivariation} if $\|\sigma\|(G) < \infty$ which is equivalent to the condition that $\sigma(B)$ is norm-bounded in $M_n$ by [6, Proposition 11]. Given any $A = (a_{ij}) \in M_n$, we have

$$\|A\| \leq \left( \sum_{i,j} |a_{ij}|^2 \right)^{1/2} \leq n\|A\|.$$  

The above inequality together with the boundedness of complex measures implies that every $M_n$-valued measure on $G$ is of bounded semivariation.

Given a Borel function $\lambda : G \rightarrow \mathbb{C}$, the vector integral $\int_S \lambda \, d\sigma$ is defined in the usual way (see [6, Definition 12]). If $\lambda$ is bounded, then we have

$$\left\| \int_G \lambda \, d\sigma \right\| \leq \|\lambda\|_\infty \|\sigma\|(G).$$

In general, we note that the bound of the above inequality cannot be sharpened to $\|\lambda\|_\infty \|\sigma(G)\|$, as the following example shows, although it can be if $\sigma$ is positive with commuting range.

**Example 1.** Let $\alpha = -\frac{\pi}{4}$ and $\beta = \frac{3\pi}{4}$. Let $\sigma$ be the following $M_2$-valued measure on $\mathbb{R}$,

$$\sigma = A\delta_\alpha + B\delta_\beta$$

where $\delta_x$ is the point mass at $x \in \mathbb{R}$,

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$
and $\sigma(\mathbb{R})$ is the identity matrix. Let

$$D = \begin{pmatrix} 1 & i \\ 1 & i \end{pmatrix}, \quad |D| = \sqrt{D^*D} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

and define $\rho: M_2 \rightarrow \mathbb{C}$ by

$$\rho(X) = \frac{1}{2} \text{trace}(DX)$$

for $X \in M_2$. Then $\|\rho\| = \frac{1}{2} \text{trace}(|D|) = 1$. Therefore

$$\|\sigma\| (\mathbb{R}) = |\rho\sigma| (\mathbb{R}) \geq |\rho(A)| + |\rho(B)| = \frac{\sqrt{8}}{2} + \frac{\sqrt{2}}{2} > 1 = \|\sigma(\mathbb{R})\|.$$

Given $\lambda(x) = e^{ix}$, we have

$$\left\| \int_{\mathbb{R}} \lambda(x) d\sigma(x) \right\| \geq |e^{ix} \rho(A) + e^{i\beta} \rho(B)| = \frac{\sqrt{8}}{2} + \frac{\sqrt{2}}{2} > 1 = \|\lambda\| \|\sigma(\mathbb{R})\|.$$

As in \cite{1,3}, using the natural bilinear map

$$(A, B) \in M_n \times M_n \mapsto AB \in M_n,$$

we can define the $\sigma$-integrable functions $f: G \rightarrow M_n$ and the bilinear vector integrals $\int_S f d\sigma$ for $S \in \mathcal{B}$. For our purpose, we can simplify the construction in the following way. Given $\sigma = (\sigma_{ij})$ and $f = (f_{ij})$, we say that $f$ is $\sigma$-integrable on $S$ if the integral $\int_S f_{ik} d\sigma_{kj}$ exists for all $i,j,k$ in which case we define

$$\int_S f d\sigma = \left( \sum_{k=1}^n \int_S f_{ik} d\sigma_{kj} \right) \in M_n.$$

For example, if $f_{ij} = \delta_{ij} \lambda$ where $\delta_{ij}$ is the Kronecker delta, then we have the $(i,j)$-th entry $(\int_S f d\sigma)_{ij} = \int_S \lambda d\sigma_{ij}$ and $\int_S f d\sigma = \int_S \lambda d\sigma$. We can also define the following convolution of $f$ and $\sigma$ if it exists:

$$f * \sigma(x) = \int_G f(x-y) d\sigma(y).$$

Given two matrix-valued measures $\sigma = (\sigma_{ij})$ and $\gamma = (\gamma_{ij})$, their convolution can be defined as

$$\sigma * \gamma = \left( \sum_k \sigma_{ik} * \gamma_{kj} \right).$$

Given a complex-valued Borel measure $\mu$ on $G$, the integral $\int_S f d\mu \in M_n$ denotes the Bochner integral of $f = (f_{ij})$ whenever it is well defined (see \cite{5} p. 44).

In the sequel, we will always assume that all $M_n$-valued measures $\sigma$ are regular which means that each $\rho\sigma$ is a regular Borel measure on $G$, for every $\rho \in M_n^*$. Let $\sigma = (\sigma_{ij})$ be an $M_n$-valued measure on $G$. We define its Fourier transform $\hat{\sigma}$ on the dual group $\hat{G}$ by

$$\hat{\sigma}(\lambda) = \int_G \lambda(-x) d\sigma(x) \in M_n$$

for $\lambda \in \hat{G}$. We also define the determinant $\det \sigma$ by convolution

$$\det \sigma = \prod_{\pi} \text{sgn}(\pi) \sigma_{1\pi(1)} * \cdots * \sigma_{n\pi(n)}$$
where \( \pi \) is a permutation. But for an \( M_n \)-valued function \( f = (f_{ij}) \) on \( G \), we define the determinant \( \det f \) by pointwise multiplication

\[
\det f = \sum_{\pi} \text{sgn}(\pi) f_{1 \pi(1)} \cdots f_{n \pi(n)}.
\]

Using the above notation, we have

\[
\det \sigma(\lambda) = \hat{\det} \sigma(\lambda).
\]

We also have

\[
\rho(\hat{\sigma}(\lambda)) = \int_{\hat{G}} \lambda(-x) d\rho(x) = \hat{\rho}(\lambda)
\]

for \( \rho \in M_n^* \).

The proof of Theorem 1 is achieved by writing the equation \( f = f \ast \sigma \) in the form \( f \ast \mu = 0 \) and convolving it with a judiciously chosen \( M_n \)-valued measure which reduces the equation to simultaneous scalar convolution equations, but convoluted by the same scalar measure, as in the proof of Lemma 3 ii) \( \Rightarrow \) i), to which one can apply the \( L^1(\hat{G}) \)-Tauberian theorem \([9, 39.27]\) and get the result readily. The setting where the Tauberian theorem applies is the following lemma which can be proved as in \([12, \text{Theorem 2}]\).

**Lemma 2.** Let \( \mu \) be a complex-valued measure on a locally compact abelian group \( G \). The following conditions are equivalent:

i) for every \( f \in L^\infty(G) \), \( f \ast \mu = 0 \) implies that \( f \) is constant;

ii) \( \hat{\mu}(\lambda) \neq 0 \) for \( \lambda \in \hat{G} \setminus \{i\} \) where \( i \) is the identity in \( \hat{G} \).

We need to extend the above lemma to the following matrix-valued setting.

**Lemma 3.** Let \( \mu \) be an \( M_n \)-valued measure on a locally compact abelian group \( G \). The following conditions are equivalent:

i) for every bounded \( M_n \)-valued function \( f \) on \( G \), \( f \ast \mu = 0 \) implies that \( f \) is constant;

ii) \( \det \hat{\mu}(\lambda) \neq 0 \) for \( \lambda \in \hat{G} \setminus \{i\} \).

**Proof.** i) \( \Rightarrow \) ii). Suppose that \( \det \hat{\mu}(\lambda) = 0 \) for some \( \lambda \in \hat{G} \setminus \{i\} \). Then there exists \( \xi \in \mathbb{C}^n \setminus \{0\} \) such that \( \hat{\mu}(\lambda)^T \xi = 0 \), where \( T \) denotes transpose. Define \( f : G \to M_n \) by \( f(x) = \lambda(x)(\zeta_{ij}) \) where

\[
(\zeta_{i1}, \zeta_{i2}, \cdots, \zeta_{in}) = (\xi_1, \xi_2, \cdots, \xi_n)
\]

for all \( i \). Then \( f \ast \mu = 0 \), but \( f \) is not constant.

ii) \( \Rightarrow \) i). Let \( f = (f_{ij}) \) be a bounded \( M_n \)-valued function such that \( f \ast \mu = 0 \). Let \( \gamma = (\gamma_{ij}) \) be the \( M_n \)-valued measure defined as the adjoint matrix of \( \mu = (\mu_{ij}) \), using convolution, so that

\[
\mu \ast \gamma = \begin{pmatrix}
\det \mu & 0 \\
0 & \ddots & 0 \\
& \ddots & \ddots & \ddots \\
\end{pmatrix}.
\]

Then we have

\[
f \ast \begin{pmatrix}
\det \mu & 0 \\
0 & \ddots & 0 \\
& \ddots & \ddots & \ddots \\
\end{pmatrix} = f \ast \mu \ast \gamma = 0
\]
which gives
\[ f_{ij} \ast \det \mu = 0 \]
for all \( i, j \). Since \( \det \mu(\lambda) = \det \bar{\mu}(\lambda) \neq 0 \) for \( \lambda \in \bar{G} \setminus \{ \iota \} \), Lemma 2 implies that \( f_{ij} \) is constant. \( \square \)

**Lemma 4.** Let \( A \in M_n^+ \) be such that \( \langle A\xi, \xi \rangle = 0 \) for some \( \xi \in \mathbb{C}^n \). Then \( A\xi = 0 \).

**Proof.** We have \( A = B^2 \) for some \( B \in M_n^+ \). Hence \( \langle B^2 \xi, \xi \rangle = 0 \) which gives \( B\xi = 0 \) and \( A\xi = 0 \). \( \square \)

We also need the following well-known result for Theorem 1.

**Lemma 5.** Let \( \nu \) be a probability measure on a locally compact abelian group \( G \). Then \( \nu \) is adapted if, and only if, \( \bar{\nu}(\lambda) \neq 1 \) for every \( \lambda \in \bar{G} \setminus \{ \iota \} \).

Let \( f \) be an \( M_n \)-valued function on \( G \) and let \( \Delta_e \) be the diagonal matrix in which each diagonal entry is the point mass \( \delta_e \) at the identity \( e \) of \( G \). Then we have \( f \ast \sigma = f \) if, and only if, \( f \ast (\sigma - \Delta_e) = 0 \). Given \( \lambda \in \bar{G} \), we have
\[
\left( \det (\sigma - \Delta_e) \right)(\lambda) = \det (\sigma - \bar{\Delta}_e(\lambda)) = \det (\bar{\sigma}(\lambda) - I_n).
\]

Now we are ready to prove Theorem 1.

**Proof of Theorem 1.** \( i) \implies ii) \). Let \( \rho(\cdot) = \langle \cdot, \xi \rangle \) be a pure state of \( M_n \). We need to show that \( \rho\sigma \) is adapted. Suppose otherwise. By Lemma 5, there exists \( \lambda \in \bar{G} \setminus \{ \iota \} \) such that \( \bar{\rho}\sigma(\lambda) = 1 \); that is \( \langle \bar{\sigma}(\lambda)\xi, \xi \rangle = 1 \). As we do not know if \( ||\bar{\sigma}(\lambda)|| \) is at most 1 (see Example 1), we cannot conclude immediately that \( \bar{\sigma}(\lambda)\xi = \xi \) although it is true but requires the following arguments. Since \( \rho\sigma \) is a probability measure, we have
\[
\rho\sigma \{ x \in G : \lambda(-x) = 1 \} = 1.
\]
Write \( V = \{ x \in G : \lambda(-x) \neq 1 \} \); then
\[
\langle \sigma(V)\xi, \xi \rangle = \rho\sigma(V) = 0.
\]
It follows from Lemma 5 that \( \sigma(V)\xi = 0 \) and hence also
\[
\sigma(G \setminus V)\xi = \xi.
\]
Thus
\[
\bar{\sigma}(\lambda)\xi = \left( \int_G \lambda(-x) \, d\sigma(x) \right) \xi
= \left( \int_{G \setminus V} 1 \, d\sigma(x) \right) \xi + \left( \int_V \lambda(x) \, d\sigma(x) \right) \xi
= \sigma(G \setminus V)\xi = \xi.
\]
Therefore \( \det (\bar{\sigma}(\lambda) - I_n) = 0 \). By Lemma 5, there is a non-constant bounded \( M_n \)-valued function \( f \) such that \( f \ast (\sigma - \Delta_e) = 0 \) which contradicts condition \( i) \).

\( ii) \implies i) \). By Lemma 5, it suffices to show that \( \det (\sigma - \bar{\Delta}_e)(\lambda) \neq 0 \) for all \( \lambda \in \bar{G} \setminus \{ \iota \} \). Suppose otherwise, so that \( \det (\bar{\sigma}(\lambda) - I_n) = 0 \) for some \( \lambda \neq \iota \). Then there is a unit vector \( \xi \in \mathbb{C}^n \) such that \( (\bar{\sigma}(\lambda) - I_n)\xi = 0 \); that is, \( \bar{\sigma}(\lambda)\xi = \xi \). Let
\( \rho(\cdot) = \langle \cdot, \xi, \xi \rangle \). Then we have
\[
\hat{\rho \sigma} (\lambda) = \rho(\hat{\sigma}(\lambda)) = \langle \hat{\sigma}(\lambda) \xi, \xi \rangle = 1.
\]
Therefore, by Lemma 5, \( \rho \sigma \) is not adapted, contradicting condition ii).

We end with an example which shows that condition ii) in Theorem 1 cannot be replaced by the condition that \( \text{supp} \sigma \) generates a dense subgroup of \( G \).

**Example 2.** Let \( \nu \) be any adapted probability measure on \( \mathbb{R} \) with \( \nu\{0\} \geq \frac{1}{2} \) and let
\[
\sigma = \begin{pmatrix}
\nu & \delta_0 - \nu \\
\delta_0 - \nu & \nu
\end{pmatrix}.
\]
Then \( \sigma \) is a positive \( M_2 \)-valued measure on \( \mathbb{R} \) such that \( \sigma(\mathbb{R}) \) is the identity matrix and \( \text{supp} \sigma = \text{supp} \nu \) generates a dense subgroup of \( \mathbb{R} \). A direct calculation reveals that every \( M_2 \)-valued function \( f = (f_{ij}) \), with \( f_{11} = f_{12} \) and \( f_{21} = f_{22} \), is \( \sigma \)-harmonic and need not be constant.

In fact under the change of coordinates \( u = x + y \) and \( v = x - y \), the measure \( \sigma \) is transformed to
\[
\begin{pmatrix}
\delta_0 & 0 \\
0 & 2\nu - \delta_0
\end{pmatrix}.
\]
The bounded solutions of the convolution equation with this measure are then of the form
\[
\begin{pmatrix}
g \\
h
\end{pmatrix},
\]
with \( \alpha \) and \( \beta \) constants, \( g \) and \( h \) any bounded functions. Undoing the change of coordinates we see that all bounded solutions of \( f = f * \sigma \) are the bounded functions \( f = (f_{ij}) \), with \( f_{11} - f_{12} \) and \( f_{21} - f_{22} \) both constant.

**References**


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