REALIZING ALTERNATING GROUPS AS MONODROMY GROUPS OF GENUS ONE COVERS

MIKE FRIED, ERIC KLASSEN, AND YAACOV KOPELIOVICH

(Communicated by Michael Stillman)

Abstract. We prove that if $n \geq 4$, a generic Riemann surface of genus 1 admits a meromorphic function (i.e., an analytic branched cover of $\mathbb{P}^1$) of degree $n$ such that every branch point has multiplicity 3 and the monodromy group is the alternating group $A_n$. To prove this theorem, we construct a Hurwitz space and show that it maps (generically) onto the genus one moduli space.

1. Introduction

Associated to any $n$-sheeted branched cover of $\mathbb{P}^1$ with branch set $B \subset \mathbb{P}^1$ is a homomorphism $\pi_1(\mathbb{P}^1 - B) \to S_n$ (the symmetric group) called the monodromy representation of the branched cover. The image of this homomorphism in $S_n$ is simply called the monodromy group of the cover (this group is well-defined up to conjugacy in $S_n$). If $\Sigma$ is a compact Riemann surface and $\phi$ is a nonconstant meromorphic function on $\Sigma$, then $\phi : \Sigma \to \mathbb{P}^1$ is a branched cover and so we may speak of the monodromy group of $\phi$. In [GN], it is stated that "Thompson (private correspondence) has verified that $A_4$ is the monodromy group of the generic Riemann surface of genus 1 (as far as we are aware, this is the only known example of a cover of a generic genus $g > 0$ surface with monodromy group different from a symmetric group)". Our main result in this paper (Theorem 1, stated formally and proved in Section 4) states that this is true for all $A_n$, where $n \geq 4$. More precisely, Theorem 1 asserts that if $n \geq 4$, then a generic Riemann surface of genus one admits a meromorphic function of degree $n$ whose monodromy group is the alternating group $A_n$ and all of whose branch points have multiplicity 3. By generic, we mean that for a given $n$, all but a finite number of genus 1 Riemann surfaces admit such functions.

It also says, the set of Riemann surfaces of genus $g \geq 1$, with functions admitting branch points of multiplicity only 3, defines an algebraic set of dimension $\geq 1$ in the moduli space of curves of genus $g$ (Section 4, Comment 0). There is only one Riemann surface of genus one which admits a meromorphic function with monodromy $A_3$: it is the Fermat curve $x^3 + y^3 + z^3 = 0$, and the meromorphic function is projection onto any one of the three coordinate axes in $\mathbb{P}^2$. To see that there is only one such curve, note that, first, the location of the three branch points in $\mathbb{P}^1$...
is irrelevant to the moduli and, second, the combinatorics is completely determined by the monodromy requirements (since the only way to select three 3-cycles in $A_3$ whose product is 1 is to select the same 3-cycle three times).

We now give a brief summary of our proof. Given a topological branched cover $\phi : \Sigma \to \mathbb{P}^1$, one may form the corresponding Hurwitz space $\mathcal{H}$, a moduli space whose points represent those branched covers $\Sigma \to \mathbb{P}^1$ which may be obtained from $\phi$ by moving around the images of the branch points in $\mathbb{P}^1$ while holding constant the combinatorial branch structure over these points as they move. Each of these branched covers gives rise to a permutation $\pi \in \text{Sym}(r)$ of the points over which branching occurs, and choose a basepoint $x_0$.

Let $w_1, \ldots, w_r$ denote simple closed curves in $\mathbb{P}^1_0 := \mathbb{P}^1 - \{x_1, \ldots, x_r\}$, all based at $x_0$, which satisfy (see Figure 1):

1. Each $w_i$ bounds a disc $D_i \subset \mathbb{P}^1$ such that $D_i \cap \{x_1, \ldots, x_r\} = \{x_i\}$.
2. If $i \neq j$, then $D_i \cap D_j = \{x_0\}$.
3. Each $w_i$ is oriented counterclockwise as the boundary of $D_i$.
4. $\prod_{i=1}^r w_i = 1$ in $\pi_1(\mathbb{P}^1_0, x_0)$.

Label the points in $\phi^{-1}(x_0)$ by the numbers $\{1, \ldots, n\}$. Then each loop $w_i$ gives rise to a permutation $\rho_i \in S_n$, the symmetric group. Think of $\rho_i$ as acting on $\{1, \ldots, n\}$ from the right. Define a group homomorphism $\rho : \pi_1(\mathbb{P}^1_0, x_0) \to S_n$ by $\rho(w_i) = \rho_i$ (this $\rho$ is the monodromy representation of $\phi$). Define the signature of the branched cover $\phi : \Sigma \to \mathbb{P}^1$ to be the $n$-tuple of permutations $(\rho_1, \ldots, \rho_r)$.

Conversely, suppose we are just given the points $\{x_1, \ldots, x_r\} \subset \mathbb{P}^1$, the loops $w_1, \ldots, w_r$ (as above), and the permutations $\rho_1, \ldots, \rho_r \in S_n$ satisfying $\prod_{i=1}^r w_i = 1$. Reconstruct the surface $\Sigma$ and the branched covering $\phi : \Sigma \to \mathbb{P}^1$ as follows. First construct the (unbranched) cover $\phi_0 : \Sigma_0 \to \mathbb{P}^1_0$ corresponding to $\rho$ using covering space theory. Then fill in one point for each end of $\Sigma_0$ to obtain $\Sigma$, and extend $\phi_0$ continuously to $\phi$ on $\Sigma$ in the only possible way.

2. A TOPOLOGICAL CONSTRUCTION OF THE BRANCHED COVERING

We begin by reminding the reader how any given $n$-sheeted branched covering $\phi : \Sigma \to \mathbb{P}^1$ may be described combinatorially. Let $\{x_1, \ldots, x_r\} \subset \mathbb{P}^1$ denote the points over which branching occurs, and choose a basepoint $x_0 \in \mathbb{P}^1$ disjoint from the other $x_i$'s. Let $w_1, \ldots, w_r$ denote simple closed curves in $\mathbb{P}^1_0 := \mathbb{P}^1 - \{x_1, \ldots, x_r\}$, all based at $x_0$, which satisfy (see Figure 1):

1. Each $w_i$ bounds a disc $D_i \subset \mathbb{P}^1$ such that $D_i \cap \{x_1, \ldots, x_r\} = \{x_i\}$.
2. If $i \neq j$, then $D_i \cap D_j = \{x_0\}$.
3. Each $w_i$ is oriented counterclockwise as the boundary of $D_i$.
4. $\prod_{i=1}^r w_i = 1$ in $\pi_1(\mathbb{P}^1_0, x_0)$.

Label the points in $\phi^{-1}(x_0)$ by the numbers $\{1, \ldots, n\}$. Then each loop $w_i$ gives rise to a permutation $\rho_i \in S_n$, the symmetric group. Think of $\rho_i$ as acting on $\{1, \ldots, n\}$ from the right. Define a group homomorphism $\rho : \pi_1(\mathbb{P}^1_0, x_0) \to S_n$ by $\rho(w_i) = \rho_i$ (this $\rho$ is the monodromy representation of $\phi$). Define the signature of the branched cover $\phi : \Sigma \to \mathbb{P}^1$ to be the $n$-tuple of permutations $(\rho_1, \ldots, \rho_r)$. Conversely, suppose we are just given the points $\{x_1, \ldots, x_r\} \subset \mathbb{P}^1$, the loops $w_1, \ldots, w_r$ (as above), and the permutations $\rho_1, \ldots, \rho_r \in S_n$ satisfying $\prod_{i=1}^r w_i = 1$. Reconstruct the surface $\Sigma$ and the branched covering $\phi : \Sigma \to \mathbb{P}^1$ as follows. First construct the (unbranched) cover $\phi_0 : \Sigma_0 \to \mathbb{P}^1_0$ corresponding to $\rho$ using covering space theory. Then fill in one point for each end of $\Sigma_0$ to obtain $\Sigma$, and extend $\phi_0$ continuously to $\phi$ on $\Sigma$ in the only possible way.
Thus, to create a branched covering with certain properties, one needs to produce permutations with corresponding properties. Hence the following lemma:

**Lemma 1.** Let \( n \geq 4 \), and consider \( \rho_1 = (123) \) and \( \rho_2 = (132) \) in \( S_n \). Then it is possible to choose \( \rho_3, \ldots, \rho_n \in S_n \) such that:

1. \( \rho_i \) is a 3-cycle for each \( i \).
2. \( \prod_{i=1}^{n} \rho_i = 1 \).
3. The number “1” doesn’t occur in any of the 3-cycles \( \rho_1, \ldots, \rho_n \): all fix 1.
4. The subgroup of \( S_n \) generated by \( \{ \rho_3, \ldots, \rho_n \} \) acts transitively on \( \{2, \ldots, n\} \).
5. \( \{ \rho_1, \ldots, \rho_n \} \) generate \( A_n \).

**Proof.** We will denote by \( \tilde{\rho}_n \) the \( n \)-tuple \((\rho_1, \ldots, \rho_n)\). Let

\[ \tilde{\rho}_4 = ((123), (132), (234), (243)) \]

and

\[ \tilde{\rho}_5 = ((123), (132), (234), (245), (253)). \]

It is easily verified that these signatures satisfy the five conditions specified in the lemma. Inductively, if \( n > 5 \) define \( \tilde{\rho}_n \) by adjoining the permutations \( \rho_{n-1} = (2, n-1, n) \) and \( \rho_n = (2, n-2, 1) \) to the \((n-2)\)-tuple \( \tilde{\rho}_{n-2} \). It is an elementary exercise (which we omit) to show that \( \tilde{\rho}_n \) satisfies the conditions of the theorem for all \( n \). This completes the proof of Lemma 1.

Fix an \( n \geq 4 \), choose \( n \) distinct points \( x_1, \ldots, x_n \in \mathbb{P}^1 \), a basepoint \( x_0 \in \mathbb{P}^1 \), and \( n \) based loops \( w_i \) related to the \( x_i \)'s as described above. Use the signature \( \tilde{\rho}_n \) produced in Lemma 1 to construct a branched cover \( \phi : \Sigma \to \mathbb{P}^1 \), branched over the \( x_i \)'s. By construction, this \( n \)-sheeted cover will be connected and have monodromy group \( A_n \). By the Riemann-Hurwitz formula, genus(\( \Sigma \)) = 1.

### 3. Hurwitz spaces

In this section we give a construction of the Hurwitz space corresponding to a branched cover. (Note: The definition of a Hurwitz space given in this paper corresponds to a single connected component of a Hurwitz space as defined in [F2].) We will start with a general finite-sheeted branched cover, and then specialize to the ones constructed in the last section. So, begin by letting \( \phi : \Sigma \to \mathbb{P}^1 \) be any \( n \)-sheeted branched cover, branched over \( \{x_1, \ldots, x_r\} \). Let \( \text{Homeo}(\mathbb{P}^1) \) denote the...
topological group of orientation preserving self-homeomorphisms of $\mathbb{P}^1$. Define the 
Hilbert space $\mathcal{H}$ corresponding to the branched cover $\phi$ by 
$$
\mathcal{H} = \{ g \circ \phi : \Sigma \rightarrow \mathbb{P}^1 \text{ such that } g \in \text{Homeo}(\mathbb{P}^1) \} / \sim
$$
where $g_1 \circ \phi \sim g_2 \circ \phi$ if and only if there exists a homeomorphism $h : \Sigma \rightarrow \Sigma$ such that the diagram 
$$\begin{array}{ccc}
\Sigma & \xrightarrow{h} & \Sigma \\
g_2 \circ \phi & \downarrow & g_1 \circ \phi \\
\mathbb{P}^1 & \xrightarrow{\phi} & \mathbb{P}^1
\end{array}$$
commutes. Note that $g_1 \circ \phi \sim g_2 \circ \phi$ if and only if there exists an $h \in \text{Homeo}(\Sigma)$ such that $(g_1^{-1}g_2)\phi = \phi h$. Thus we may write $\mathcal{H} \cong \text{Homeo}(\mathbb{P}^1)/G$, where $G \subset \text{Homeo}(\mathbb{P}^1)$ is the subgroup consisting of those homeomorphisms $g$ of $\mathbb{P}^1$ which lift to a homeomorphism $h_g$ of $\Sigma$ making the diagram 
$$\begin{array}{ccc}
\Sigma & \xrightarrow{h_g} & \Sigma \\
\phi & \downarrow & \phi \\
\mathbb{P}^1 & \xrightarrow{\phi} & \mathbb{P}^1
\end{array}$$
commute. Let $S_r$ act on $(\mathbb{P}^1)^r$ by permuting the coordinates, and define $\Delta \subset (\mathbb{P}^1)^r$ by $\Delta = \{(y_1, \ldots, y_r) \in (\mathbb{P}^1)^r : y_i = y_j \text{ for some } i \neq j\}$. Define 
$$\Pi = ((\mathbb{P}^1)^r - \Delta)/S_r.$$ 
Define a map $P : \text{Homeo}(\mathbb{P}^1) \rightarrow \Pi$ by $P(f) = [f(x_1), \ldots, f(x_r)]$. Define the following two subgroups of $\text{Homeo}(\mathbb{P}^1)$:
$$G = P^{-1}[x_1, \ldots, x_r],$$
$$G_0 = \text{ the identity component of } G.$$ 
We now observe that 
$$G_0 \subseteq G \subseteq G.$$ 
The second of these inclusions is completely elementary; since $\phi \circ h$ and $g \circ \phi$ are two ways of writing the same branched cover, they must have the same branch locus in $\mathbb{P}^1$. Hence, $g[x_1, \ldots, x_r] = [x_1, \ldots, x_r]$.

To prove the first inclusion, $G_0 \subseteq G$, we quote two lemmas from [KIKO] Lemmas 2 and 3:

**Lemma 2.** If $r \geq 3$, then $\pi_i(G_0) = 0$ for all $i$.

We omit the proof of Lemma 2; the reader is referred to [KIKO].

**Lemma 3.** Given $g \in G_0$, there is a homeomorphism $h_g : \Sigma \rightarrow \Sigma$ such that $g \circ \phi = \phi \circ h_g$. If $r \geq 3$, then $h_g$ is uniquely determined by $g$ and, in fact, $g \mapsto h_g$ defines a continuous group homomorphism $G_0 \rightarrow \text{Homeo}(\Sigma)$ such that $\phi$ is equivariant with respect to the resulting action of $G_0$ on $\Sigma$.

**Proof.** Let $g \in G_0$. Choose a path $g_t$ in $G_0$ from the identity to $g$. Let $y \in \phi^{-1}(\mathbb{P}^1)$. Let $\alpha : I \rightarrow \phi^{-1}(\mathbb{P}^1)$ be the lift of the path $g_t(\phi(y))$ which starts at $y$, and define $h_g(y) = \alpha(1)$. Define $h_g$ to be the identity on $\phi^{-1}\{x_1, \ldots, x_n\}$. Then $h_g : \Sigma \rightarrow \Sigma$ is a homeomorphism and $g \circ \phi = \phi \circ h_g$. Furthermore, if $r \geq 3$, then, since $\pi_1(G_0) = 0$, any two such paths $g_t$ would lead to homotopic paths in $\mathbb{P}^1$. Hence, for $r \geq 3$, $g \mapsto h_g$ is a well-defined homomorphism $G_0 \rightarrow \text{Homeo}(\Sigma)$ making $\phi$ equivariant. This completes the proof of Lemma 3.
For the rest of this section assume that the branched cover \( \phi : \Sigma \to \mathbb{P}^1 \) is completely non-Galois; It has no non-trivial deck transformations. This is equivalent to the algebraic assumption that the monodromy group of \( \phi \) has trivial centralizer in \( S_n \). From this, we have a well-defined group homomorphism \( G \to \text{Homeo}(\Sigma) \) given by \( g \mapsto h_g \), where \( h_g \) is defined as in the definition of \( G \).

Next, we construct some useful covering maps. Let \( \Pi = ((\mathbb{P}^1)^r - \Delta)/S_r \), which is homeomorphic to \( \text{Homeo}(\mathbb{P}^1)/G \) and let \( Q = \text{Homeo}(\mathbb{P}^1)/G_0 \). Because \( G_0 \) is the identity component of \( G \) and of \( G \), if follows that the natural quotient maps \( Q \to H \to \Pi \) are both covering maps. Furthermore, since \( G_0 \) is a normal subgroup of \( G \), the covering map \( Q \to H \) is regular (Galois), with deck group equal to \( G_0 \). This deck group acts on \( Q \) from the right in the obvious manner, with quotient \( H \). Note that \( Q \) is almost, but not quite, the universal cover of \( H \); \( \pi_1(Q) = \pi_1(\text{Homeo}(\mathbb{P}^1)) = \mathbb{Z}_2 \), since \( G_0 \) is contractible and \( SO(3) \to \text{Homeo}(\mathbb{P}^1) \) is a homotopy equivalence (a fact dating back to Kneser [K] in 1926).

We now remind the reader of some basic Teichmüller theory. Given the closed oriented (topological) surface \( \Sigma \), define the Teichmüller space \( T_\Sigma \) by
\[
T_\Sigma = \{ (\Sigma_0, [q_0]) : \Sigma_0 \text{ is a Riemann surface and } [q_0] \text{ is an isotopy class of homeomorphisms } \Sigma \to \Sigma_0 \}/\sim
\]
where we define \((\Sigma_0, q_0) \sim (\Sigma_1, q_1) \) if there is an analytic isomorphism \( h : \Sigma_0 \to \Sigma_1 \) such that \( q_1 \circ h \) is isotopic to \( q_0 \).

The mapping class group of \( \Sigma \), defined by \( \Gamma_\Sigma = \text{Homeo}(\Sigma)/\text{isotopy} \), acts on \( T_\Sigma \) from the right by
\[
(\Sigma_0, [q_0]) \cdot [h] = (\Sigma_0, [q_0 \circ h]).
\]
The quotient of \( T_\Sigma \) under this action is the moduli space of \( \Sigma \), defined by \( M_\Sigma = \{ \text{Riemann surfaces } \Sigma_0 \text{ homeomorphic to } \Sigma \}/\text{analytic isomorphism} \).

Let \( p : \Sigma \to \mathbb{P}^1 \) be any branched cover; define \( \Sigma_p \) to be the Riemann surface with underlying space \( \Sigma \) and with the unique complex structure making \( p \) analytic. We now define maps \( \Psi : H \to M_\Sigma \) and \( \tilde{\Psi} : Q \to T_\Sigma \) by \( \Psi(fG) = \Sigma_{f\phi} \) and \( \tilde{\Psi}(fG_0) = (\Sigma_{f\phi}, \text{id}_\Sigma) \). It is immediately clear that the following diagram commutes:
\[
\begin{array}{ccc}
\text{Homeo}(\mathbb{P}^1)/G_0 & \to & Q \\
\downarrow & & \downarrow \tilde{\Psi} \\
\text{Homeo}(\mathbb{P}^1)/G & \to & H \\
& & \downarrow \Psi \\
& & M_\Sigma
\end{array}
\]
The vertical arrows in this diagram are simply quotient maps involving the right action of \( G/G_0 \) on \( Q \) and the right action of \( \Gamma_\Sigma \) on \( T_\Sigma \). Define a group homomorphism \( R : G/G_0 \to \Gamma_\Sigma \) by \( gG_0 \mapsto [h_g] \). The fact that \( R \) is well-defined follows from the proof of Lemma 3, which actually shows that if \( g \in G_0 \), then \( h_g \) is homotopic (hence isotopic) to the identity. In [KIKO], we give a general algorithm for computing the composition of \( R \) with the natural homomorphism \( \Gamma_\Sigma \to SL(2g, \mathbb{Z}) \) (defined by action on \( H_1(\Sigma) \)). In the genus one case, this gives \( R \) precisely, since \( \Gamma_\Sigma \to SL(2, \mathbb{Z}) \) is an isomorphism. In the current paper, instead of using this general method, we get the information we need from a specific geometric observation in the next section.

**Lemma 4.** \( \tilde{\Psi} \) is equivariant with respect to the homomorphism \( R : G/G_0 \to \Gamma_\Sigma \).
Proof. We need to show that if $f \in \text{Homeo}(\mathbb{P}^1)$ and $g \in G$, then $\bar{\Psi}(f \mathcal{G}_0 \cdot g) = (\bar{\Psi}(f \mathcal{G}_0)) \cdot [h_g]$. Restating using the definitions, we need to show that $(\Sigma_{f \mathcal{G}_0}, [id]) \sim (\Sigma_{f \mathcal{G}_0}, [h_g])$. In other words, we need to show that the diagram

\[
\begin{array}{ccc}
\Sigma_{f \mathcal{G}_0} & \xrightarrow{id} & \Sigma_{f \mathcal{G}_0} \\
\downarrow h_g & & \downarrow h_g \\
\Sigma_{f \mathcal{G}_0} & \xrightarrow{f \mathcal{G}_0} & \Sigma_{f \mathcal{G}_0}
\end{array}
\]

commutes up to homotopy (which is obvious!), and that $h_g : \Sigma_{f \mathcal{G}_0} \to \Sigma_{f \mathcal{G}_0}$ is analytic. To prove this second fact, consider the diagram

\[
\begin{array}{ccc}
\Sigma_{f \mathcal{G}_0} & \xrightarrow{h_g} & \Sigma_{f \mathcal{G}_0} \\
\downarrow f \mathcal{G}_0 & & \downarrow f \mathcal{G}_0 \\
\mathbb{P}^1 & \xrightarrow{f} & \mathbb{P}^1
\end{array}
\]

which commutes by definition of $h_g$. The two vertical branched cover maps are analytic by definition of the complex structures on the $\Sigma$'s. Conclude the homeomorphism $h_g$ is analytic as well. This completes the proof of Lemma 4.

4. Statement and proof of Theorem 1

**Theorem 1.** Let $n \geq 4$ be an integer. There exists a finite subset $Y \subset \mathcal{M}_1$ (where $\mathcal{M}_1$ is the moduli space of genus one Riemann surfaces) with the following property. If $\Sigma_0$ is a Riemann surface of genus one, and $[\Sigma_0] \notin Y$, then there exists a holomorphic function $f : \Sigma_0 \to \mathbb{P}^1$ of degree $n$ such that all branch points of $f$ have multiplicity 3, no two branch points of $f$ map to the same point in $\mathbb{P}^1$, and the monodromy group of $f$ is the full alternating group $A_n$.

**Proof.** Fix $n$. Let $\phi : \Sigma \to \mathbb{P}^1$ be the topological branched cover with monodromy group $A_n$ constructed in Section 2 using Lemma 1. In building this cover, we may choose our branch points $x_1, \ldots, x_n$ and our basepoint $x_0$ arbitrarily in $\mathbb{P}^1$. Since $A_n$ has trivial centralizer in $S_n$, the branched cover $\phi : \Sigma \to \mathbb{P}^1$ is completely non-Galois, and hence we can use $\phi$ to make all the constructions of Section 3 involving Hurwitz spaces, Teichmüller theory, etc. Express $\mathbb{P}^1$ as the union of two discs $B_1$ and $B_2$ whose intersection and common boundary is a smooth circle $C$. Choose these discs so that $B_1$ contains $D_1 \cup D_2$, $B_2$ contains $D_3 \cup \cdots \cup D_n$, and, for $i = 1, \ldots, n$, $C \cap D_i = x_0$. See Figure 2.

![Diagram](image_url)
We now wish to visualize the topology of $\phi^{-1}(B_1)$ and $\phi^{-1}(B_2)$. The monodromy along the curve $C$ is trivial: $\rho_1 \rho_2$ is the identity. Conclude that $\phi^{-1}(C)$ consists of $n$ disjoint circles, each mapped homeomorphically to $C$ by $\phi$. Since we numbered the points of $\phi^{-1}(x_0)$ using $\{1, \ldots, n\}$, this allows labelling components of $\phi^{-1}(C)$ as $C_1, \ldots, C_n$ according to which point of $\phi^{-1}(x_0)$ they contain. Using the algebraic properties of $\rho_1, \ldots, \rho_n$ in Lemma 1, conclude the following: $\phi^{-1}(B_1)$ consists of one component with boundary $C_1 \cup C_2 \cup C_3$ and $n-3$ other components; each of these other components has as its boundary one of the remaining $C_i$’s (for $i > 3$), and is mapped homeomorphically onto $B_1$. On the other hand, $\phi^{-1}(B_2)$ consists of only two components: the first maps homeomorphically to $B_2$ and has as its boundary $C_1$ and the second has as its boundary $C_2 \cup \cdots \cup C_n$. We illustrate this situation in Figure 3, with $\mathbb{P}^1$ and $\Sigma$ shown split in two along $C$ and $\phi^{-1}(C)$.

Let $A \subset B_1$ be a thin collar of $C = \partial B_1$, i.e., an annulus in $B_1$ one of whose boundary components is $C$. Define $g \in \text{Homeo}(\mathbb{P}^1)$ to be a single Dehn twist along $A$. More precisely, the Dehn twist $g$ is defined as follows. Identify $A$ with $S^1 \times [0,1]$ and define $g: A \to A$ by $g(z,t) = (e^{2\pi i t}z, t)$. Clearly, $g$ is a homeomorphism of $A$ which is the identity on $\partial A$. Extend $g$ to all of $\mathbb{P}^1$ by defining it to be the identity outside of $A$. If we define $h_0 \in \text{Homeo}(\Sigma)$ to consist of simultaneous Dehn twists along all $n$ components of $\phi^{-1}(A)$, then $\phi \circ h_0 = g \circ \phi$. We conclude that $g \in G$ and $R(g G_0) = [h_0]$. Referring to Figure 3, note that all the $C_i$’s except $C_2$ and $C_3$ bound discs in $\Sigma$ (e.g., $C_1$ bounds a disc in $\phi^{-1}(B_2)$ while $C_4, \ldots, C_n$ bound discs}
in $\phi^{-1}(B_1)$; hence the corresponding Dehn twists are trivial in the mapping class group $\Gamma$. The curves $C_2$ and $C_3$ are isotopic to each other in $\Sigma$ (by inspection of Figure 3); hence their Dehn twists are equal to each other in $\Gamma$. We conclude that $[h_g]$ is a double Dehn twist along the essential curve $C_2$ in the torus $\Sigma$. Hence $[h_g]$ is of infinite order in $\Gamma$: That a Dehn twist along an essential curve in a closed orientable surface has infinite order in the mapping class group follows easily by considering its action on the fundamental group. Since each point in $T$ has finite stabilizer in $\Gamma$, it follows that $\Psi : Q \to T$ and, hence, $\Psi : H \to M_1$ are non-constant functions. Since $H$ and $M_1$ both have the structure of quasiprojective varieties (see [M], p. 25, for $M_2$ and [F1], p. 53, for $H$), $\Psi$ is an algebraic map which extends to the compactification of $H$ (see [Gr], p. 247). When $M_1$ has dimension 1 ($\Sigma$ has genus one), conclude that the image of $H$ in $M_1$ is a quasiprojective subvariety of dimension one. Hence $M_1 - \Psi(H)$ consists of at most a finite number of points. This finishes the proof of Theorem 1.

Comment 0. $\Psi$ shows, for each $n \geq 1$, $H$ has exactly two components. Further, the Riemann surfaces parametrized by each have branch cycles satisfying the conditions of Lemma 1. In particular, $\Psi$ restricted to each component is non-constant.

Comment 1. We originally conceived of Theorem 1 as an application of Fried’s Theorem 3.6 in [F2], which states that if a certain representation of $\pi_1(H)$ on $H_1(\Sigma; Z)$ has infinite image, then $\Psi$ is non-constant. However, one gets a similar result by considering our homomorphism $R$ instead, which is a natural lift of Fried’s representation. In addition, $R$ has infinite image by the pictorial argument involving Dehn twists given here, rather than by the more algebraic computations involving $H_1(\Sigma; Z)$ (see for example [F2] and [KlKo]). We present this view for the sake of variety, and because we think it may appeal to the more geometrically-minded reader.

Comment 2. Having proved that, for each $n \geq 4$, the map $\Psi : H \to M_1$ misses at most a finite number of points of $M_1$, it is natural to ask, for each such $n$, whether the map does in fact miss some points or whether it might actually be surjective. Mark van Hoeij, using very nice computations involving $J$-invariants, has shown that in the case $n = 5$ the map $\Psi$ is actually surjective. For higher values of $n$, it seems likely that it remains surjective but someone needs to prove it! For $n = 4$, we don’t have a conjecture.

Comment 3. The preprint [F3] makes further applications of Dehn twists in order to compute explicitly the monodromy action of $\pi_1 Q$ on the cohomology of a Riemann surface corresponding to a point on a Hurwitz space. (For other examples of this, see [F2] and [KlKo].) As a result, the map $\Psi$ is shown in [F3] to have higher dimensional image on other components of Hurwitz spaces constructed from $r$-tuples of 3-cycles corresponding to higher genus covers of $P^1$. We don’t, however, know this dimension as a precise function of $r$ and $n$.

References

[F1] Fried, M., Fields of definition of function fields and Hurwitz families, Comm. in Alg. 5(1) 1977, 17-82. MR 56:12000


Department of Mathematics, University of California at Irvine, Irvine, California 92717
E-mail address: mfried@math.uci.edu

Department of Mathematics, Florida State University, Tallahassee, Florida 32306
E-mail address: klassen@math.fsu.edu

Unigraphics Solutions, 100824 Hope St., Cypress, California 90630
E-mail address: YKopeliovich@email101.webango.com