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THE BLOW-UP FOR WEAKLY COUPLED REACTION-DIFFUSION SYSTEMS

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Abstract. In this paper we consider a weakly coupled parabolic system with nonnegative exponents in the forcing functions. We find the conditions which result in blow-up in finite time. Also, we obtain the blow-up rate.

1. Introduction

In this paper, we consider the following system:

$$u_t - \Delta u = u^{p_1} v^{p_2} \quad \text{for } x \in \Omega, \quad t > 0, \tag{1.1}$$

$$v_t - \Delta v = u^{p_3} v^{p_4} \quad \text{for } x \in \Omega, \quad t > 0, \tag{1.2}$$

with the initial and boundary conditions

$$u(x, 0) = u_0(x), v(x, 0) = v_0(x), \tag{1.3}$$

$$u|_{\partial\Omega} = 0, \ v|_{\partial\Omega} = 0, \tag{1.4}$$

where Ω is a bounded domain in R^N. p_i for $i = 1, 2, 3, 4$, are nonnegative real numbers. We assume that $p_2 - p_4 + 1 \geq p_3 - p_1 + 1 > 0$.

The study of (1.1), (1.2) is of great interest due to its application. The above models arise in the chemical reaction processes. The density and temperature are governed by a coupled system of reaction diffusion equations in the form of (1.1), (1.2).

In the past years, several authors have studied such a problem. Galaktionov, Kurdyumov and Samarskii obtained the first blow-up results for the semilinear and quasilinear systems of type (1.1), (1.2) in [6], [7]; in particular they established a sufficient condition of global blow-up for general quasilinear systems. Deng [1] studied the blow-up rate for systems like (1.1), (1.2) with $p_1 = p_4 = 0$.

In [13], [14] the authors considered a similar system with different source terms

$$u_t - \Delta u = u f_1(v) \quad \text{for } x \in \Omega, \quad t > 0,$$
$$v_t - \Delta v = v f_2(u) \quad \text{for } x \in \Omega, \quad t > 0,$$

with initial boundary condition. This system can be applied to the special case of (1.1), (1.2) with $p_1 = p_4 = 1$ and $1 < p_2, p_3 < 2$. The authors have proved

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the existence of a blow-up solution and blow-up rate in the case of a single-point blow-up for the initial boundary value problem.

Some early results can also be seen in [4], [5], [11]. In [8], [10], the authors considered the blow-up problem of linear equation with nonlinear boundary condition, and obtained the blow-up rate.

My motivation to study (1.1), (1.2) comes from the results of [3] (in [2] Escobedo and Herrero considered the special case $p_1 = p_4 = 0$). The authors considered this system with the Cauchy problem and obtained Fujita-type global existence and global nonexistence theorems for (1.1), (1.2) analogous to the classical result of Fujita and others for the initial-value problem for $u_t - \Delta u = u^p$, $u(x,0) = u_0(x) \geq 0$. Similar results can be seen in [12].

In this paper we find the necessary and sufficient conditions which result in blow-up and obtain some results about the blow-up rate for (1.1), (1.2).

The paper is organized as follows: In section 2 we prove blow-up and global existence results and in section 3 we present the blow-up rate results.

2. Blow-up and global existence

In this section we consider the system (1.1), (1.2) with the initial and boundary conditions

$$u(x,0) = u_0(x),\ v(x,0) = v_0(x), \tag{2.1}$$

$$u|_{\partial\Omega} = 1, v|_{\partial\Omega} = 1 \tag{2.2}$$

where $u_0(x)$ and $v_0(x)$ are nonnegative functions.

Initially we develop a lemma which will be needed to prove our main results.

Lemma 2.1. *Assume that $C_0 u_0(x) \geq v_0(x)^{\gamma_0}$, where $\gamma_0 = \frac{p_2-p_4+1}{p_3-p_1+1}$ and $C_0 \geq (\gamma_0)^{\frac{1}{p_3-p_1+1}}$. Let (u,v) be a nonnegative solution of the system (1.1), (1.2) with (2.1), (2.2). Then we have*

$$C_0 u(x,t) \geq v(x,t)^{\gamma_0}. \tag{2.3}$$

Proof. Let $J = u^{p_3-p_1+1} - Cv^{p_2-p_4+1}$. Then

$$\begin{aligned} J_t - \Delta J = {} & [(p_3 - p_1 + 1) - C(p_2 - p_4 + 1)]u^{p_3} v^{p_2} \\ & - (p_3 - p_1 + 1)(p_3 - p_1) u^{p_3-p_1-1} |\nabla u|^2 \\ & + C(p_2 - p_4 + 1)(p_2 - p_4) v^{p_2-p_4-1} |\nabla v|^2. \end{aligned}$$

If we let $0 < C < \frac{p_3-p_1+1}{p_2-p_4+1}$, we have

$$[(p_3 - p_1 + 1) - C(p_2 - p_4 + 1)]u^{p_3} v^{p_2} \geq 0.$$

Using the definition of J, we have

$$(p_3 - p_1 + 1)u^{p_3-p_1} \nabla u = \nabla J + C(p_2 - p_4 + 1) v^{p_2-p_4} |\nabla u|^2 = \nabla v;$$

squaring both sides, we have

$$|\nabla u|^2 = [\tfrac{C(p_2-p_4+1)}{p_3-p_1+1}]^2 v^{2(p_2-p_4)} u^{-2(p_3-p_1)} |\nabla v|^2 + b_1 \nabla J,$$

where b_1 is a bounded function. Since

$$Cv^{p_2-p_4+1} = u^{p_3-p_1+1} - J,$$

we have

$$\begin{aligned}&(p_3-p_1+1)(p_3-p_1)u^{p_3-p_1-1}|\nabla u|^2\\&=\frac{(p_3-p_1)(p_2-p_4+1)^2C}{(p_3-p_1+1)}v^{p_2-p_4-1}|\nabla v|^2+b_2J\\&\le C(p_2-p_4+1)(p_2-p_4)v^{p_2-p_4-1}|\nabla v|^2+b_2J,\end{aligned}$$

where b_2 is a bounded function. This proves

$$J_t-\Delta J-b_1\nabla J-b_2J\ge 0.$$

Letting $C\le (C_0)^{p_3-p_1+1}$, we have

$$\begin{aligned}J|_{\partial\Omega}&\ge 0,\\J|_{t=0}&\ge 0.\end{aligned}$$

By the maximum principle, we have

$$J(x,t)\ge 0\quad\text{for } t>0 \text{ and } x\in\Omega.$$

This means $u^{p_3-p_1+1}\ge Cv^{p_2-p_4+1}$. This is

$$C_0u(x,t)\ge v(x,t)^{\gamma_0}.$$

Therefore this completes the proof. □

Remark. The assumption $C_0u_0(x)\ge v_0(x)^{\gamma_0}$ is obviously true when $u_0(x)>0$ for $x\in\overline{\Omega}$. For in this case we can choose a large enough constant C_0 such that the assumption holds. If $u_0(x)\ge 0$, we can choose some $t_0>0$ such that $u(x,t_0)>0$ for $x\in\overline{\Omega}$.

We introduce some notation to develop our next results:

$$P=\begin{pmatrix}p_1 & p_2\\ p_3 & p_4\end{pmatrix},\quad I=\begin{pmatrix}1&0\\0&1\end{pmatrix},$$

and $|P|=\det P$.

Theorem 2.2. *If $|P-I|\ge 0$, then the solution (u,v) of the system (1.1), (1.2) with (2.1), (2.2) is global for any bounded initial data.*

Proof. Using inequality (2.3) and the equation (1.1), we have

$$u_t-\Delta u\le C'u^{p_1}u^{p_2\gamma_0^{-1}}=C'u^{[-|P-I|/(p_2-p_4+1)]+1}.$$

Since $|P-I|\ge 0$, u is the lower solution of the following:

$$\begin{aligned}&w_t-\Delta w=w^\lambda\quad\text{for } t>0, x\in\Omega,\\&w|_{\partial\Omega}=0,\\&w|_{t=0}=u_0(x),\end{aligned}$$

where $0<\lambda=[-|P-I|/(p_2-p_4+1)]+1\le 1$, w exists globally, so u globally exists. By (2.3) so does v. □

Theorem 2.3. *If $|P-I|<0$, and $\Delta u_0+u_0^{p_1}v_0^{p_2}\ge 0,\ \ \Delta v_0+u_0^{p_3}v_0^{p_4}\ge 0$, then the solution of system (1.1), (1.2) with (2.1), (2.2) blows up.*

Proof. Using inequality (2.3) and the equation (1.2), we have

$$v_t - \Delta v \geq C v^{1+\alpha} \quad \text{for } x \in \Omega, \quad t > 0,$$

where $\alpha = \frac{-|P-I|}{p_3-p_1+1} > 0$ and C is a constant. It is easy to obtain that v blows up. Since (2.3), so does u. □

Remark. The system (1.1), (1.2) with (1.3), (1.4) has the same nonglobal results as in section 2 by constructing a lower blow-up solution (w_1, w_2) in smaller domain $\Omega' \subset \Omega$ as follows:

$$w_{1t} - \Delta w_1 = w_1^{p_1} w_2^{p_2} \quad \text{for } x \in \Omega', \quad t > 0,$$
$$w_{2t} - \Delta w_2 = w_1^{p_3} w_2^{p_4} \quad \text{for } x \in \Omega', \quad t > 0,$$

with the initial and boundary conditions

$$w_1(x,0) = u(x,t_0), w_2(x,0) = v(x,t_0),$$
$$w_1|_{\partial\Omega'} = u(x,t_0)|_{\partial\Omega'} > 0, w_2|_{\partial\Omega'} = v(x,t_0)|_{\partial\Omega'} > 0.$$

It is similar for global existence.

3. Blow-up rate

In this section, we assume that $p_i \geq 1$ for $i = 1,2,3,4$ are nonnegative real numbers. Also $u_0(x)$ and $v_0(x)$ are nonnegative functions, increase in $(0, \frac{l}{2})$ and decrease in $(\frac{l}{2}, l)$.

Lemma 3.1. *Assume that* $C_0 u_0(x) \geq v_0(x)^{\gamma_0}$, *where* $\gamma_0 = \frac{p_2-p_4+1}{p_3-p_1+1}$ *and* $C_0 \geq (\gamma_0)^{\frac{1}{p_3-p_1+1}}$. *Let* (u,v) *be a nonnegative solution of the system (1.1), (1.2). Then we have*

$$C_0(u(x,t)+1) \geq v(x,t)^{\gamma_0}. \tag{3.1}$$

The proof is similar to that of Lemma 2.1.

Theorem 3.2. *For the solutions* $u(x,t)$, $v(x,t)$ *of system (1.1), (1.2), let* $U(t) = \max_{x\in\Omega} u(x,t)$ *and* $V(t) = \max_{x\in\Omega} v(x,t)$. *Then the functions are Lipschitz continuous and*

$$U_t \leq U^{p_1} V^{p_2} \quad \textit{for } t > 0 \textit{ a.e.}, \tag{3.2}$$
$$V_t \leq U^{p_3} V^{p_4} \quad \textit{for } t > 0 \quad \textit{a.e.} \tag{3.3}$$

The proof is very similar to that of Theorem 4.5 in [5], hence we omit it.

Theorem 3.3. *For the solutions* $u(x,t)$, $v(x,t)$ *of system (1.1), (1.2), and initial functions* $u_0(x,t)$ *and* $v_0(x,t)$ *are symmetric for* $x = \frac{l}{2}$. *Further, set* $u_0(x)$ *and* $v_0(x)$ *to be increasing functions for* $0 < x < \frac{l}{2}$. *Also there exists a constant* $0 < \delta < 1$ *such that* $\Delta u_0 + (1-\delta) u_0^{p_1} v_0^{p_2} \geq 0$, $\Delta v_0 + (1-\delta) u_0^{p_3} v_0^{p_4} \geq 0$. *Then*

$$u_t \geq \delta u^{p_1} v^{p_2} \quad \textit{for } x \in \Omega, \quad t > 0, \tag{3.4}$$
$$v_t \geq \delta u^{p_3} v^{p_4} \quad \textit{for } x \in \Omega, \quad t > 0. \tag{3.5}$$

Proof. Since $u_0(x)$ and $v_0(x)$ increase for $0 < x < \frac{l}{2}$, we have that $u(x,t)$ and $v(x,t)$ increase for $0 < x < \frac{l}{2}$ (see [4, 5]). Letting $F(x,t) = u_t + \frac{\delta}{1-\delta} u_{xx}$ and $G(x,t) = v_t + \frac{\delta}{1-\delta} v_{xx}$, we have

$$F_t - F_{xx} \geq p_1 u^{p_1-1} v^{p_2} F + p_2 u^{p_1} v^{p_2-1} G,$$
$$G_t - G_{xx} \geq p_3 u^{p_3-1} v^{p_4} F + p_4 u^{p_3} v^{p_4-1} G,$$

and

$$F(x,t)|_{x=0,l} = 0, G(x,t)|_{x=0,l} = 0,$$

$$F(x,0) \geq 0, G(x,0) \geq 0.$$

By the maximum principle, one can see that both F and G are nonnegative in $(0,l) \times (0,T)$. From equation (1.1), (1.2), we have

$$\frac{1}{\delta} u_t \geq u^{p_1} v^{p_2},$$

$$\frac{1}{\delta} v_t \geq u^{p_3} v^{p_4}.$$

This completes the proof. □

Theorem 3.4. *If* $|P - I| < 0$, $\Delta u_0 + (1-\delta) u_0^{p_1} v_0^{p_2} \geq 0$, $\Delta v_0 + (1-\delta) u_0^{p_3} v_0^{p_4} \geq 0$, $u(x,t)$ *and* $v(x,t)$ *are the solutions of the system (1.1), (1.2), and* T *is the blowup time, then*

$$C_1 (T-t)^{\alpha_1} \leq \max_{x \in \Omega} u(x,t), \tag{3.6}$$

$$\max_{x \in \Omega} v(x,t) \leq C_2 (T-t)^{\alpha_2}, \tag{3.7}$$

where $\alpha_1 = \frac{p_2 - p_4 + 1}{|P-I|}$, $\alpha_2 = \frac{p_3 - p_1 + 1}{|P-I|}$ *and* C_i *are positive constants.*

Proof. Since $U(t) \to \infty$ as $t \to T$, there exists a t_0 such that $U(t) > 1$ for any $t > t_0$. Hence we have

$$2C_0 U(t) \geq V(t)^{\gamma_0}. \tag{3.8}$$

Letting $(x(t), t)$ be the points at which $v(x,t)$ attains its maximum, for any $t_2 > t_1 > t_0$, we have

$$\frac{V(t_2) - V(t_1)}{t_2 - t_1} \geq \frac{v(x(t_1), t_2) - v(x(t_1), t_1)}{t_2 - t_1} = v_t(t_1) + o(1). \tag{3.9}$$

From (3.5), (3.8) and (3.9), we have

$$V_t \geq C V(t)^{1+\alpha}, \tag{3.10}$$

where $\alpha = \frac{-|P-I|}{p_3 - p_1 + 1}$ and C is a constant. Integrating (3.10) from t to T yields

$$V(t) \leq C_2 (T-t)^{\alpha_2}. \tag{3.11}$$

Combining (3.8) and (3.2), we have

$$U_t \leq C U(t)^{1+\alpha}, \tag{3.12}$$

where $\alpha = \frac{-|P-I|}{p_2 - p_4 + 1}$ and C is a constant. Integrating (3.12) from t to T, we obtain

$$U(t) \geq C_1 (T-t)^{\alpha_1}. \tag{3.13}$$

This completes the proof. □

Theorem 3.5. *Assume the conditions of Theorem 3.4 are true and*

$$\max_{x\in\Omega} u(x,t) \le C_3(T-t)^{\alpha_1}; \tag{3.14}$$

then

$$\max_{x\in\Omega} v(x,t) \ge C_4(T-t)^{\alpha_2}. \tag{3.15}$$

Proof. We claim that $V(t)(T-t)^{-\alpha_1}$ is bounded from below by a positive constant C_4. Otherwise, we assume that $\max\inf_{t\to T} V(t)(T-t)^{-\alpha_1} = 0$. Then there exist a sequence $\{t_k\} \in (0,T)$ with $t_k \to T$ and a sequence $\{\epsilon_k\}$ with $\epsilon_k \to 0$ such that

$$V(t)(T-t)^{-\alpha_1} \le \epsilon_k \quad \text{for any } k = 1,2,3,\cdots. \tag{3.16}$$

Now we choose a positive integer m such that $C_3(m+1)^{\alpha_2} \le \frac{C_1}{2}$. For such a choice of m, when t_k is close to T we can select a corresponding sequence $\{\tau_k\}$ such that $T-\tau_k = (m+1)(T-t_k)$. Now using (3.2), (3.14) and (3.11), we would obtain that

$$\begin{aligned} U(t_k) &\le U(\tau_k) + \int_{\tau_k}^{t_k} U^{p_1}(s)V^{p_2}(s)ds \\ &\le C_3(T-\tau_k)^{\alpha_1} + [C_3(T-t_k)^{\alpha_1}]^{p_1} V^{p_2}(t_k)(t_k-\tau_k) \\ &\le C_3(m+1)^{\alpha_1}(T-t_k)^{\alpha_2} + C_3^{p_2}(T-t_k)^{\alpha_1 p_2+1}V^{p_2}(t_k) \\ &\le \frac{C_1}{2}(T-t_k)^{\alpha_1} + C_3^{p_2}(T-t_k)^{\alpha_1 p_2+1-\alpha_1 p_1}\epsilon^{p_1} \\ &\le \frac{C_1}{2}(T-t_k)^{\alpha_1} + C_3^{p_2}(T-t_k)^{\alpha_1}\epsilon^{p_1}, \end{aligned}$$

if k is sufficiently large; this contradicts (3.13). Thus this completes our proof. □

Remark. Actually, the vector (α_1, α_2) satisfies

$$(P-I)\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \begin{pmatrix}-1\\ -1\end{pmatrix}.$$

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