

## ON KELLEY'S INTERSECTION NUMBERS

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**ABSTRACT.** We introduce a notion of weak intersection number of a collection of sets, modifying the notion of intersection number due to J.L. Kelley, and obtain an analogue of Kelley's characterization of Boolean algebras which support a finitely additive strictly positive measure. We also consider graph-theoretic reformulations of the notions of intersection number and weak intersection number.

Kelley [4] gave a necessary and sufficient condition for a Boolean algebra  $\mathcal{B}$  to have a finitely additive strictly positive measure. A finitely additive nonnegative measure  $\mu$  on  $\mathcal{B}$  is *strictly positive* if, for  $B \in \mathcal{B}$ ,  $\mu(B) = 0$  if and only if  $B$  is the zero element of  $\mathcal{B}$ . Kelley's condition involves the notion of *intersection number* of a collection of elements of  $\mathcal{B}$ , or equivalently, of a collection of sets. In the present note we introduce a related notion of *weak intersection number*, obtain an analogue of Kelley's condition in Theorem 2, and investigate the relationship between these two kinds of intersection numbers (Theorems 2 and 5). We will also reformulate these notions in graph-theoretic terms. In particular, in Theorem 6 we shall establish a connection between the notions of (Kelley's) intersection number and fractional chromatic number for hypergraphs.

Let  $\mathbb{A} = \langle A_1, A_2, \dots, A_n \rangle$ , where  $n > 0$ , be a sequence of elements of  $\mathcal{B}$  and let  $I \subseteq \{1, 2, \dots, n\}$  be of maximum cardinality such that

$$\bigcap_{i \in I} A_i \neq \emptyset.$$

We set

$$\alpha^*(\mathbb{A}) = \frac{|I|}{n}.$$

Let  $\mathcal{S} \subseteq \mathcal{B}$ . The *intersection number* of  $\mathcal{S}$ ,  $\alpha(\mathcal{S})$ , is the infimum of the quantities  $\alpha^*(\mathbb{A})$ , where  $\mathbb{A}$  ranges over all finite sequences of (not necessarily distinct) elements of  $\mathcal{S}$ . Clearly  $\alpha(\mathcal{S})$  is equal to the infimum of  $\alpha(\tilde{\mathcal{S}})$  where  $\tilde{\mathcal{S}}$  ranges over finite subcollections of  $\mathcal{S}$ .

The notion of intersection number is due to Kelley [4]. The notation above is Fremlin's [2].

We will make use of the following reformulation of the definition of intersection number for finite collections. Let  $\mathcal{S} = \{C_1, C_2, \dots, C_p\}$ , and let  $T_p$  be the simplex

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consisting of all those  $t = \langle t_1, \dots, t_p \rangle \in [0, 1]^p$  satisfying

$$\sum_{i=1}^p t_i = 1.$$

Let  $\mathcal{I}$  consist of all those  $I \subseteq \{1, 2, \dots, p\}$  such that

$$\bigcap_{i \in I} C_i \neq \emptyset.$$

Then

$$\alpha(\mathcal{S}) = \min_t \max_{I \in \mathcal{I}} \sum_{i \in I} t_i$$

where the minimum is taken over all  $t$  in  $T_p$ .

This is not difficult to see. Essentially, picking a sequence  $\mathbb{A}$  of sets from  $\mathcal{S}$  is equivalent to specifying the relative frequencies with which the sets from  $\mathcal{S}$  appear in  $\mathbb{A}$ . We omit the details.

We define the *weak intersection number* of  $\mathcal{S}$ ,  $\alpha_w(\mathcal{S})$ , exactly as  $\alpha(\mathcal{S})$ , except that  $A_1, A_2, \dots, A_n$  are now required to be distinct.

Obviously,  $\alpha(\mathcal{S}) \leq \alpha_w(\mathcal{S})$ . It is possible for  $\alpha(\mathcal{S})$  to be strictly smaller than  $\alpha_w(\mathcal{S})$ ; see Fremlin [2], p. 949, where an example of such a collection  $\mathcal{S}$ , consisting of only four sets, is given, and also Theorem 5 below. Note also that, if the zero element of  $\mathcal{B}$  belongs to  $\mathcal{S}$ , then  $\alpha(\mathcal{S}) = \alpha_w(\mathcal{S}) = 0$ . Thus we generally rule the empty set out of consideration at the outset. In particular, let  $\mathcal{B}^+$  be the collection consisting of the nonzero elements of  $\mathcal{B}$ .

It will be convenient to write  $\text{CUP}(\mathcal{S})$ , resp.  $\text{CUPW}(\mathcal{S})$ , for:  $\mathcal{S}$  is the union of countably many collections each of which has a positive intersection number, resp. weak intersection number.

Trivially,  $\text{CUP}(\mathcal{S})$  implies  $\text{CUPW}(\mathcal{S})$ , and  $\text{CUPW}(\mathcal{S})$  implies that the empty set (or the zero element of the Boolean algebra) does not belong to  $\mathcal{S}$ .

Kelley's condition is as follows:

**Theorem 1.** *There is a strictly positive finitely additive measure on a Boolean algebra  $\mathcal{B}$  if and only if  $\text{CUP}(\mathcal{B}^+)$  holds.*

We shall show that the above statement also holds with  $\text{CUPW}(\mathcal{B}^+)$  in place of  $\text{CUP}(\mathcal{B}^+)$ . This is an immediate corollary of the following

**Theorem 2.** *Suppose that  $\mathcal{S}$  is a collection of sets, or elements of a Boolean algebra, such that for every  $X_1$  and  $X_2$  in  $\mathcal{S}$ , if  $X_1 \cap X_2 \neq \emptyset$ , then there is some  $X$  in  $\mathcal{S}$  such that  $X \subseteq X_1 \cap X_2$ . Then  $\text{CUP}(\mathcal{S})$  and  $\text{CUPW}(\mathcal{S})$  are equivalent.*

We begin with some simple lemmas.

**Lemma 1.** *If  $\text{CUPW}(\mathcal{S})$  holds, then  $\mathcal{S}$  satisfies the countable chain condition (c.c.c.), i.e. every collection of pairwise disjoint elements of  $\mathcal{S}$  is at most countable.*

*Proof.* If a collection  $\mathcal{T}$  contains  $n$  pairwise disjoint sets, then clearly  $\alpha_w(\mathcal{T}) \leq \frac{1}{n}$ . Thus if

$$\mathcal{S} = \bigcup_{n=0}^{\infty} \mathcal{S}_n, \quad \alpha_w(\mathcal{S}_n) > 0,$$

then every pairwise disjoint subcollection of  $\mathcal{S}$  can have only finitely many members in each of the  $\mathcal{S}_n$ , and thus must be at most countable.  $\square$

For  $\mathcal{T} \subseteq \mathcal{S}$ , let  $\hat{\mathcal{T}}$  consist of those  $X$  in  $\mathcal{S}$  which contain infinitely many distinct elements of  $\mathcal{T}$ .

**Lemma 2.**  $\alpha(\hat{\mathcal{T}}) \geq \alpha_w(\mathcal{T})$ .

*Proof.* If  $\mathbb{A}$  is a finite sequence of, not necessarily distinct, sets from  $\hat{\mathcal{T}}$ , then there is an equal length sequence  $\mathbb{B}$ , of distinct sets from  $\mathcal{T}$ , such that each  $B_i$  is a subset of (the corresponding)  $A_i$ . This immediately implies the desired conclusion.  $\square$

*Proof of Theorem 2.* Suppose

$$\mathcal{S} = \bigcup_{h=0}^{\infty} \mathcal{S}_h, \quad \alpha_w(\mathcal{S}_h) > 0.$$

Set

$$\mathcal{D} = \mathcal{S} \setminus \bigcup_{n=0}^{\infty} \hat{\mathcal{S}}_n$$

and let  $\mathcal{D}_0$  be a maximal pairwise disjoint subcollection of  $\mathcal{D}$ . Each set in  $\mathcal{D}$  clearly contains at most countably many subsets belonging to  $\mathcal{S}$ , and  $\mathcal{D}_0$  is at most countable by Lemma 1. Hence the collection  $\mathcal{E}$ , consisting of all  $E$  in  $\mathcal{S}$  contained in some  $D$  in  $\mathcal{D}_0$ , is likewise at most countable.

Setting  $\mathcal{D}_E = \{D \in \mathcal{D} : E \subseteq D\}$ , we have

$$\mathcal{D} = \bigcup_{E \in \mathcal{E}} \mathcal{D}_E.$$

For if  $D$  belongs to  $\mathcal{D}$ , then, for some  $D_0$  in  $\mathcal{D}_0$ , we have  $D \cap D_0 \neq \emptyset$ ; hence by the hypothesis of Theorem 2, there is some  $X$  in  $\mathcal{S}$  such that  $X \subseteq D \cap D_0$ . Such  $X$  clearly belongs to  $\mathcal{E}$ .

By CUPW( $\mathcal{S}$ ), each  $E$  in  $\mathcal{S}$  is nonempty, hence  $\alpha(\mathcal{D}_E) = 1$ , and  $\alpha(\hat{\mathcal{S}}_n) > 0$  by Lemma 2. Thus CUP( $\mathcal{S}$ ) is established.

The other direction being trivial, Theorem 2 follows.  $\square$

Kelly [4] also introduced the notion of *covering number* of a collection  $\mathcal{T}$  of elements of  $\mathcal{B}$ . This notion is dual to that of intersection number. It is simpler to work in the setting of an algebra of sets; thus in case of an abstract Boolean algebra one should pass to the Stone space.

Hence let  $\mathcal{B}$  be an algebra of subsets of a set  $S$ . For  $\mathbb{B} = \langle B_1, B_2, \dots, B_n \rangle$  consisting of elements of  $\mathcal{B}$ , and  $n > 0$ , let  $k$  be the minimum, over all  $x$  in  $S$ , of the cardinality of the set of those  $i$  such that  $1 \leq i \leq n$  and  $x$  belongs to  $B_i$ . The *covering number*,  $\gamma(\mathcal{T})$ , of a collection  $\mathcal{T} \subseteq \mathcal{B}$ , is the supremum of the quantities  $\frac{k}{n}$  as  $\mathbb{B}$  ranges over all finite sequences of (not necessarily distinct) elements of  $\mathcal{T}$ . We obtain the *weak covering number*,  $\gamma_w(\mathcal{T})$ , if we require that  $B_1, B_2, \dots, B_n$  be distinct.

For  $\mathcal{S} \subseteq \mathcal{B}$ , let  $\mathcal{T}$  be the collection of the complements of members of  $\mathcal{S}$ . Then we have the following relations:

$$\gamma(\mathcal{T}) = 1 - \alpha(\mathcal{S}), \quad \gamma_w(\mathcal{T}) = 1 - \alpha_w(\mathcal{S}).$$

The following result of Kelley [4] is dual to Theorem 1.

**Theorem 3.** *There is a strictly positive finitely additive measure on a Boolean algebra  $\mathcal{B}$  if and only if the collection of nonunit elements of  $\mathcal{B}$  is the union of countably many collections each of which has a covering number less than one.*

Again, the analogue of Theorem 3 for the weak covering number remains valid. For the rest of this note we shall restrict ourselves to working with intersection numbers and weak intersection numbers, the translation to the setting of covering numbers being straightforward.

Our next result, given in Theorem 5, is also motivated by a theorem of Kelley [4], namely,

**Theorem 4** ([4]). *Let  $\mathcal{S}$  be a nonempty subcollection of a Boolean algebra  $\mathcal{B}$ . Then there is a probability measure  $\mu$  on  $\mathcal{B}$  such that the infimum of  $\mu(B)$ , over  $B$  in  $\mathcal{S}$ , is equal to the intersection number,  $\alpha(\mathcal{S})$ , of  $\mathcal{S}$ .*

It should be remarked that Kelley's criterion, stated here as Theorem 1, is a corollary of Theorem 4 (also due to him).

Complementary to Theorem 4, Kelley [4] pointed out that if  $\mathcal{B}$  and  $\mathcal{S}$  are as above, then for every probability measure  $\mu$  on  $\mathcal{B}$ , the infimum of  $\mu(B)$ , over  $B$  in  $\mathcal{S}$ , is less than, or equal to,  $\alpha(\mathcal{S})$ . Thus since  $\alpha(\mathcal{S})$  may be strictly smaller than  $\alpha_w(\mathcal{S})$  (see Fremlin [2], p. 949, and Theorem 5 below), the analogue of Theorem 4 for weak intersection numbers cannot hold. The next theorem sheds further light on the relationship between  $\alpha(\mathcal{S})$  and  $\alpha_w(\mathcal{S})$ .

**Theorem 5.** *Given  $r < 1$ , there is a collection  $\mathcal{S}$  such that  $\alpha(\mathcal{S}) = 0$  but  $\alpha_w(\mathcal{S}) > r$ .*

Theorem 5 is immediate from Lemmas 3 and 4 below.

As usual, let  $\mathbb{N}$  be the set of positive integers. Let  $r < 1$  and let  $k$  in  $\mathbb{N}$  be such that  $\frac{k}{k+1} > r$ .

Let  $S$  consist of those nonempty  $I \subseteq \mathbb{N}$  such that  $|I| \leq k \cdot \min(I)$ . For  $i$  in  $\mathbb{N}$  let  $C_i$ , contained in  $S$ , consist of those  $I$  in  $S$  which contain  $i$  as a member, and let  $\mathcal{S}$  be the family of these  $C_i$ , as  $i$  varies over  $\mathbb{N}$ .

**Fact 1.**  $S$  is hereditary, i.e. if  $I$  belongs to  $S$  and  $J$  is a nonempty subset of  $I$ , then  $J$  belongs to  $S$  as well.

Using Fact 1 we obtain

**Fact 2.** For a nonempty set  $I \subseteq \mathbb{N}$ ,

$$\bigcap_{i \in I} C_i \neq \emptyset$$

if and only if  $I$  belongs to  $S$ .

**Lemma 3.** *The weak intersection number of  $\mathcal{S}$ ,  $\alpha_w(\mathcal{S})$ , equals  $\frac{k}{k+1}$ .*

*Proof.* Let  $F$  be a finite subset of  $\mathbb{N}$ ; thus  $|F| = m(k+1) + \ell$ , for some nonnegative integers  $m$  and  $\ell$ , where  $\ell \leq k$ . Then the set  $I$ , consisting of the  $mk + \ell$  largest elements of  $F$  belongs to  $S$ , since  $\min(I) \geq m+1$  and thus  $k \cdot \min(I) \geq mk + \ell$ . Thus

$$\bigcap_{i \in I} C_i \neq \emptyset$$

and

$$\frac{|I|}{|F|} = \frac{mk + \ell}{m(k+1) + \ell} \geq \frac{k}{k+1}.$$

On the other hand, let  $F$  be the set of positive integers less than or equal to  $k+1$ . Then the set  $I$  consisting of the  $k$  largest elements of  $F$  is a maximum cardinality subset of  $F$  belonging to  $S$ . Thus Lemma 3 is established.  $\square$

**Lemma 4.** *The intersection number of  $\mathcal{S}$ ,  $\alpha(\mathcal{S})$ , equals zero.*

*Proof.* For  $n$  in  $\mathbb{N}$ , let  $A_1, A_2, \dots, A_{p_n}$ , for suitable  $p_n$ , be the sequence of members of  $\mathcal{S}$ , consisting of the sets  $C_i$ , for  $1 \leq i \leq n$ , with each  $C_i$  repeated  $\ell_i$  times, where

$$\ell_i = \frac{n!}{i}.$$

The maximum over  $I$  in  $\mathcal{S}$  (see Fact 2) of the sums

$$\sum_{i \in I} \ell_i$$

is reached when  $I$  is an interval of the form  $[\ell, \ell + k\ell - 1]$ , and this maximum is of the order  $(n!) \log k$ . Thus  $\alpha^*(A_1, A_2, \dots, A_{p_n})$  is of the order  $\frac{\log k}{\log n}$ ; hence  $\alpha(\mathcal{S}) = 0$ .  $\square$

The rest of the paper is devoted to some remarks on the following problem related to Theorem 5, and also to a possible alternate proof of Theorem 2.

**Problem 1.** Is there a collection  $\mathcal{S}$  with a positive *weak* intersection number but such that  $\text{CUP}(\mathcal{S})$  fails?

A negative answer would give the equivalence of  $\text{CUP}(\mathcal{S})$  and  $\text{CUPW}(\mathcal{S})$ , and thus also an alternate proof of Theorem 2, as immediate corollaries. Furthermore, the referee commented that “A problem *DU* from Fremlin’s list of problems (unpublished) can be formulated as a question about the weak intersection number. A negative answer to Problem 1 from this paper would also solve Fremlin’s problem”.

We will reformulate Problem 1 in graph-theoretic terms. The main part of this reformulation involves establishing a connection between the notion of intersection number and the notion of fractional chromatic number for hypergraphs. This result, which, we hope, is of independent interest, is stated in Theorem 6 below. The notion of weak intersection number is likewise reformulated graph-theoretically in Theorem 6.

We begin with some definitions and two constructions, or transformations, the first of which associates with each hypergraph a certain collection of sets, and the second goes in the opposite direction.

A hypergraph  $H = (V, \mathcal{E})$  has a nonempty set  $V$  of vertices and a collection  $\mathcal{E}$  of edges, where

$$\mathcal{E} \subseteq \{X \subseteq V : 2 \leq |X| < \aleph_0\}.$$

A subset  $I$ , of  $V$ , is independent if  $I$  has no subset belonging to  $\mathcal{E}$ . Let  $\mathcal{I}$  be the collection of all *finite* independent sets of  $H$ , and, for  $v$  in  $V$ , let  $\mathcal{I}_v$  consist of those  $I$  in  $\mathcal{I}$  which have  $v$  as a member. We further set

$$\mathcal{S}_H = \{\mathcal{I}_v : v \in V\}.$$

The sets  $\mathcal{I}_v$  are nonempty and distinct, since  $\{v\} \in \mathcal{I}_v$ , and, for a finite subset  $I$  of  $V$ ,

$$\bigcap_{v \in I} \mathcal{I}_v \neq \emptyset$$

if and only if  $I$  is independent.

Conversely, let  $\mathcal{S} = \{C_v : v \in V\}$  be any family of distinct nonempty sets. Let  $\mathcal{E}$  be the collection of all finite subsets  $X$  of  $V$  such that

$$\bigcap_{x \in X} C_x = \emptyset$$

and let  $H_{\mathcal{S}} = (V, \mathcal{E})$  be the hypergraph associated with the collection  $\mathcal{S}$ .

By definition of  $\mathcal{E}$ , we clearly have, for a finite subset  $I$  of  $V$ :  $I$  is independent if and only if

$$\bigcap_{v \in I} C_v \neq \emptyset.$$

Beginning with a collection  $\mathcal{S} = \{C_v : v \in V\}$  as above, let's apply the two transformations in succession, first obtaining the hypergraph  $H = H_{\mathcal{S}} = (V, \mathcal{E})$  and then the collection  $\mathcal{S}_H = \{\mathcal{I}_v : v \in V\}$  associated to the hypergraph  $H$ . The collection  $\mathcal{S}_H$  has the same intersection number, and weak intersection number, as the initial collection  $\mathcal{S}$ ; in fact, the two collections have identical "intersection patterns". This will permit us to link the notions of intersection number and fractional chromatic number in Theorem 6 below, and also obtain a graph-theoretic reformulation of Problem 1. More precisely, we have the following lemma, which is immediate:

**Lemma 5.** *The sets  $\mathcal{I}_v$ , for  $v \in V$ , are distinct nonempty sets, and, for all finite subsets  $I$  of  $V$ , the sets  $\mathcal{I}_v (v \in I)$ , have nonempty intersection if and only if the same is true of the sets  $C_v (v \in I)$ , if and only if  $I$  is independent. Thus for all subsets  $W$  of  $V$ , the collections  $C_v (v \in W)$ , and  $\mathcal{I}_v (v \in W)$ , have equal intersection numbers, and weak intersection numbers. In particular  $\alpha(\mathcal{S}) = \alpha(\mathcal{S}_H)$  and similarly for  $\alpha_w$ .*

We shall now define the notions of fractional coloring, and fractional chromatic number, for finite hypergraphs only. We note in passing that, for infinite hypergraphs, the definitions are formally the same, except that  $\mathcal{I}$  should then consist of all independent subsets of  $V$ , not just the finite ones. (The cardinality of a hypergraph  $H$ ,  $|H|$ , is the cardinality  $|V|$  of its set of vertices  $V$ .) We follow Leader [5] where these notions are investigated for ordinary graphs.

A different, but equivalent approach is adopted in [7], p. 41.

For a mapping  $f : \mathcal{I} \rightarrow [0, 1]$  we define  $\hat{f} : V \rightarrow \mathbb{R}$  by

$$\hat{f}(v) = \sum_{I \in \mathcal{I}_v} f(I).$$

An  $f$  as above is a fractional coloring of  $H$  if  $\hat{f}(v) \geq 1$  for every  $v$  in  $V$ . The weight,  $w(f)$ , of a fractional coloring  $f$ , is

$$w(f) = \sum_{I \in \mathcal{I}} f(I),$$

and the fractional chromatic number of  $H$ ,  $\chi^*(H)$ , is the infimum of  $w(f)$ , over all fractional colorings  $f$ . Clearly  $\chi^*(H) \geq 1$ . Thus, in particular,  $\chi^*(H)$  is never zero.

Denoting by  $\chi(H)$  the (ordinary) chromatic number of  $H$ , one has  $\chi^*(H) \leq \chi(H)$  (see e.g. Johnson [3], pp. 99 and 100).

By a subhypergraph of  $H = (V, \mathcal{E})$  we mean an induced subhypergraph, i.e. one of the form  $(W, \mathcal{E} \cap \mathcal{P}(W))$ , where  $W$  is a non-empty subset of  $V$ . If  $G$  is a subhypergraph of  $H$ , then  $\chi^*(G) \leq \chi^*(H)$ .

We can now state our result linking the notions of intersection number of a collection of sets and fractional chromatic number for hypergraphs. Assuming the notation of Lemma 5, we have

**Theorem 6.** *If  $H$  is a hypergraph and  $\mathcal{S}_H$  is the collection associated to  $H$ , then the intersection number of the collection  $\mathcal{S}_H$  is equal to the reciprocal of the quantity  $\sup \chi^*(G)$ , where the supremum is taken over all finite subhypergraphs of  $H$ . Furthermore, the weak intersection number of the collection  $\mathcal{S}_H$  is equal to*

$$\inf_G \max_I \frac{|I|}{|G|},$$

where the infimum is taken over all finite subhypergraphs  $G$  of  $H$ , and the maximum over all independent subsets  $I$  of  $G$ .

*Remark.* For finite  $H$ , the supremum,  $\sup \chi^*(G)(= \max \chi^*(G))$ , in Theorem 6, is equal to  $\chi^*(H)$ , since  $\chi^*(G) \leq \chi^*(H)$  for any subhypergraph  $G$  of  $H$ . Leader [5], Theorem 1, p. 413, gives an example of an infinite graph whose fractional chromatic number is strictly greater than the supremum of the fractional chromatic numbers of its finite subgraphs. Clearly,  $H$  is finite if and only if  $\mathcal{S}_H$  is finite.

*Proof of Theorem 6.* The second part of the theorem, dealing with weak intersection numbers, is immediate from the definitions. Furthermore, the general case of the first part, dealing with the (ordinary) intersection numbers, follows easily from the corresponding result for finite hypergraphs. Thus the rest of the proof is devoted to proving the first part of the theorem for finite hypergraphs only, and  $H$  is therefore assumed to be finite for the rest of the proof.

Normalizing fractional colorings, i.e. dividing by the weight  $w = w(f)$ , gives those probability distributions  $\rho$  on the collection of independent sets,  $\mathcal{I}$ , which satisfy

$$\hat{\rho}(v) = \sum_{I \in \mathcal{I}_v} \rho(I) > 0, \quad (v \in V),$$

where  $\hat{\rho}$  is defined by analogy with  $\hat{f}$ . When  $\rho = \frac{f}{w}$ , we have  $\hat{\rho} = \frac{\hat{f}}{w}$ .

Let  $r = r_\rho$  be the minimum of  $\hat{\rho}(v)$  over  $v$  in  $V$ . For  $\rho$  in the above class, we have  $r > 0$  and we shall define a fractional coloring  $f_\rho$  by

$$f_\rho(I) = \min \left\{ 1, \frac{1}{r} \rho(I) \right\}.$$

Then the weight of  $f_\rho$  is less than, or equal to  $\frac{1}{r}$ , and thus  $\frac{1}{r} \geq \chi^*(H)$ . (For finite hypergraphs,)  $\chi^*(H)$  is clearly attained as  $w_0 = w(f_0)$  for some fractional coloring  $f_0$  of  $H$ . For such an  $f_0$  there must exist some  $v_0$  in  $V$  such that  $\hat{f}_0(v_0) = 1$ . Otherwise, dividing  $f_0$  by the minimum (greater than one) over  $v$  in  $V$ , of  $\hat{f}_0(v)$ , would yield a fractional coloring of weight less than  $w_0$ , contradicting the definition of  $\chi^*(H)$ .

Let  $\rho_0 = \frac{f_0}{w_0}$ , where  $f_0$  is as above. Then,

$$r_{\rho_0} = \hat{\rho}_0(v_0) = \frac{\hat{f}_0(v_0)}{w_0} = \frac{1}{w_0} = \frac{1}{\chi^*(H)}.$$

We thus obtain

$$\frac{1}{\chi^*(H)} = \max_{\rho} \min_{v \in V} \hat{\rho}(v),$$

where the maximum is taken over all probability distributions on  $\mathcal{I}$ .

As Edward R. Scheinerman, [6], pointed out, applying the Min-Max Theorem (with the pay-off matrix being the incidence matrix of the collection  $\mathcal{I}$ ), we finally obtain

$$\frac{1}{\chi^*(H)} = \min_t \max_{I \in \mathcal{I}} \sum_{v \in I} t(v),$$

where the minimum is taken over all probability distributions  $t$  on  $V$ . However, as pointed out in introductory paragraphs, the quantity above is the intersection number of the collection  $C_v$ , ( $v \in V$ ). This completes the proof of Theorem 6.  $\square$

The statement that  $\mathcal{S}_H$  has a positive weak intersection number can also be formulated in graph-theoretic terms by extending the terminology of Bollobás [1], p. 258, from the setting of ordinary finite graphs to that of arbitrary (finite or infinite) hypergraphs. For a finite hypergraph  $H$  let  $\beta_0(H)$  be the maximum cardinality of an independent set of vertices. For a number  $\epsilon \in [0, 1]$ , let us say that an arbitrary (finite or infinite) hypergraph  $H'$  has property  $P(\beta_0, \epsilon)$  if  $\beta_0(H) \geq \epsilon|H|$  for every finite subhypergraph  $H$  of  $H'$ . Clearly,  $\mathcal{S}_H$  has positive weak intersection number if and only if  $H$  has property  $P(\beta_0, \epsilon)$  for some  $\epsilon > 0$ .

Using the notation above and Theorem 6, we have the following graph-theoretic reformulation of Problem 1:

**Problem 2.** Is there a hypergraph  $H = (V, \mathcal{E})$  such that

- (i)  $H$  has the property  $P(\beta_0, \epsilon)$  for some  $\epsilon > 0$ , and
- (ii) for any partition of  $V$  into countably many sets  $V_n$ , ( $n \in \mathbb{N}$ ), there is some  $n$  in  $\mathbb{N}$  such that the induced subhypergraph on  $V_n$  contains finite subhypergraphs with arbitrarily large fractional chromatic number?

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We have learned from Johnson's survey article [8] that the identity in the last paragraph of the proof of Theorem 6 is due to Clarke and Jamison [9].

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