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THE BRAUER GROUP OF SWEEDLER'S HOPF ALGEBRA H_4

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Abstract. We calculate the Brauer group of the four dimensional Hopf algebra H_4 introduced by M. E. Sweedler. This Brauer group $\mathrm{BM}(k, H_4, R_0)$ is defined with respect to a (quasi-) triangular structure on H_4, given by an element $R_0 \in H_4 \otimes H_4$. In this paper k is a field . The additive group $(k, +)$ of k is embedded in the Brauer group and it fits in the exact and split sequence of groups:

$$1 \longrightarrow (k, +) \longrightarrow \mathrm{BM}(k, H_4, R_0) \longrightarrow \mathrm{BW}(k) \longrightarrow 1$$

where $\mathrm{BW}(k)$ is the well-known Brauer-Wall group of k. The techniques involved are close to the Clifford algebra theory for quaternion or generalized quaternion algebras.

1. Introduction

In their search for general unified theory allowing them to deal with the Brauer group of a quantum group (cf. [2, 3]), the authors have introduced the Brauer group, $\mathrm{Br}(\mathcal{C})$, of a braided monoidal category (see [10]). In loc. cit. it is shown that all known Brauer groups, algebraic as well as geometric ones, appear as examples of $\mathrm{Br}(\mathcal{C})$ for a suitable choice of $\mathcal{C}$ or the braiding on $\mathcal{C}$. The theory of the Brauer group of a quantum group deals mainly with Yetter-Drinfel'd modules (YD modules) and YD module algebras. However, when a Hopf algebra H has a quasi-triangular structure, then the category of left H-modules is itself a braided monoidal category, say ${}_H\mathbf{M}$. The Brauer group $\mathrm{Br}({}_H\mathbf{M})$ is a subgroup of $\mathrm{BQ}(k, H)$ which is the Brauer group of the category YD modules for H, where k is the ground field (or a commutative ring). In [9], the authors have established that the Hopf automorphism group of H may be embedded into $\mathrm{BQ}(k, H)$ (up to factoring out some finite group eventually). Hence $\mathrm{BQ}(k, H)$ need not be a torsion group. The ruling idea was that $\mathrm{BQ}(k, H)$ modulo the image of the Hopf automorphism group would be a torsion group, at least for a finite dimensional H. The calculation in this paper contradicts that idea; even in such a low-dimensional case as H_4, many non-torsion elements do exist in the Brauer group. We provide precise structure and calculation for the Brauer group $\mathrm{BM}(k, H_4, R_0)$ of H_4 with respect to a (quasi-) triangular structure on H_4 given by an element $R_0 \in H_4 \otimes H_4$. Even if the Hopf automorphism group of H_4, the multiplicative group of k, does now not appear, the additive group $(k, +)$ of the field k may be embedded in $\mathrm{BM}(k, H_4, R_0)$. The structure results make it clear that the non-torsion part of $\mathrm{BM}(k, H_4, R_0)$ is represented by H_4-Azumaya

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algebra structures that are non-trivial but that are defined on the trivial element of the Brauer-Wall group. This is contained in the fact that the following sequence of group homomorphism is exact and split:

$$1 \longrightarrow (k,+) \longrightarrow \mathrm{BM}(k, H_4, R_0) \longrightarrow \mathrm{BW}(k) \longrightarrow 1.$$

2. Preliminaries

In this section, we recall the definition of the Brauer group of a Hopf algebra with a bijective antipode and some related notions. Let k be a commutative ring with unit, and H a Hopf algebra over k with a bijective antipode. A Yetter-Drinfel'd H-module (simply, YD H-module) M is a crossed H-bimodule [13]. That is, M is a k-module which is at once a left H-module and a right H-comodule satisfying the following equivalent compatibility conditions [6, 5.1.1]:

(i) $\sum h_{(1)} \cdot m_{(0)} \otimes h_{(2)} m_{(1)} = \sum (h_{(2)} \cdot m)_{(0)} \otimes (h_{(2)} \cdot m)_{(1)} h_{(1)}$,

(ii) $\sum (h \cdot m)_{(0)} \otimes (h \cdot m)_{(1)} = \sum (h_{(2)} \cdot m_{(0)}) \otimes h_{(3)} m_{(1)} S^{-1}(h_{(1)})$

where the sigma notations for comodules and for comultiplications can be found in the standard reference book [11]. An Yetter-Drinfel'd H-module algebra is a YD H-module A such that A is a left H-module algebra and a right H^{op}-comodule algebra. For the details of H-(co)module algebras we refer to [11].

In [2] we defined the Brauer group of a Hopf algebra H by considering isomorphism classes of H-Azumaya algebras. A YD H-module algebra A is called *H-Azumaya* if it is faithfully projective as a k-module and if the following YD H-module algebra are isomorphisms:

$$F: \quad A\#\overline{A} \longrightarrow \mathrm{End}(A), \qquad F(a\#\overline{b})(c) = \sum a c_{(0)} (c_{(1)} \cdot b),$$

$$G: \quad \overline{A}\#A \longrightarrow \mathrm{End}(A)^{op}, \quad G(\overline{a}\#b)(c) = \sum a_{(0)} (a_{(1)} \cdot c) b,$$

where $\overline{A}$ is the H-opposite YD H-module algebra of A (cf. [2]). For a faithfully projective YD H-module M, the endomorphism ring $\mathrm{End}_k(M)$ is a YD H-module algebra with H-structures given by

$$(h \cdot f)(m) = \sum h_{(1)} \cdot f(S(h_{(2)}) \cdot m),$$
$$\sum f_{(0)}(m) \otimes f_{(1)} = \sum f(m_{(0)})_{(0)} \otimes S^{-1}(m_{(1)}) f(m_{(0)})_{(1)}.$$

Two H-Azumaya algebras A and B are Brauer equivalent (denoted by $A \sim B$) if there exist two faithfully projective YD H-modules M and N such that $A\#\mathrm{End}(M) \cong B\#\mathrm{End}(N)$. Note that $A \sim B$ if and only if A is H-Morita equivalent to B (cf. [3, Theorem 2.10]). The Brauer group of the Hopf algebra H is denoted by $\mathrm{BQ}(k, H)$. An element in $\mathrm{BQ}(k, H)$ represented by an H-Azumaya algebra A is indicated by $[A]$. The unit in $\mathrm{BQ}(k, H)$ is represented by $\mathrm{End}(M)$ for any faithfully projective YD H-module M.

If A is an H-Azumaya algebra, the left and right H-centers defined by

$$\{a \in A \mid ax = \sum x_{(0)}(x_{(1)} \cdot a),\ \forall x \in A\},$$

resp.

$$\{a \in A \mid xa = \sum a_{(0)}(a_{(1)} \cdot x),\ \forall x \in A\},$$

are trivial.

Now let H be a quasi-triangular Hopf algebra, that is, H is a Hopf algebra with an invertible element $R = \sum R^{(1)} \otimes R^{(2)}$ in $H \otimes H$ satisfying several axioms (e.g.

see [7]). If A is a left H-module algebra, then A is simultaneously a Yetter-Drinfel'd H-module algebra with the right H^{op}-comodule structure given by

$$A \longrightarrow A \otimes H^{op}, \quad a \mapsto \sum R^2 \cdot a \otimes R^1$$

for $a \in A$. The subset of $\mathrm{BQ}(k, H)$ consisting of the elements represented by the H-Azumaya algebras with right H^{op}-comodule structures stemming from left H-module structures in the above way, turns out to be a subgroup of $\mathrm{BQ}(k, H)$, denoted by $\mathrm{BM}(k, H, R)$. It is obvious that $\mathrm{BM}(k, H, R)$ contains the Brauer group $\mathrm{Br}(k)$.

The Brauer group $\mathrm{BQ}(k, H)$ is a special case of the Brauer group $\mathrm{Br}(\mathcal{C})$ of a braided monoidal category $\mathcal{C}$ as introduced in [10]. The fact that $\mathrm{BM}(k, H, R)$ is a subgroup of $\mathrm{BQ}(k, H)$ when (H, R) is a quasi-triangular Hopf algebra can be explained in a categorical way. If $\mathcal{D}$ is a closed braided monoidal subcategory of a braided monoidal category $\mathcal{C}$, then the Brauer group $\mathrm{Br}(\mathcal{D})$ is a subgroup of $\mathrm{Br}(\mathcal{C})$. This fact allows us to consider various subgroups of the Brauer group $\mathrm{Br}(\mathcal{C})$ of a braided monoidal category $\mathcal{C}$ whenever $\mathcal{C}$ contains certain closed braided subcategories. For example, if (H, R) is a quasi-triangular Hopf algebra, then the category ${}_H\mathbf{M}$ of left H-modules is a closed braided subcategory of the braided category of YD H-modules with braiding ϕ given by:

$$M \otimes N \longrightarrow N \otimes M, \ m \otimes n \mapsto \sum R^2 \cdot n \otimes R^1 \cdot m,$$

where $m \in M, n \in N$. The Brauer group $\mathrm{Br}({}_H\mathbf{M})$ of ${}_H\mathbf{M}$ is indeed $\mathrm{BM}(k, H)$. When H is a finite dimensional Hopf algebra, it is well-known that the category of YD H-modules is equivalent to the category of left $D(H)$-modules, where $D(H)$ is the Drinfel'd double of H. So we have that $\mathrm{BQ}(k, H)$ is equal to $\mathrm{BM}(k, D(H), R)$, where R is the canonical quasi-triangular structure on $D(H)$.

3. Actions on Azumaya algebras

In this section, we will consider actions of H_4 on Azumaya algebras. Throughout k is a field with characteristic different from two. Let H_4 be the Sweedler 4-dimensional Hopf algebra over k. That is, H_4 is generated by two elements g and h such that

$$g^2 = 1, \quad h^2 = 0, \quad gh + hg = 0.$$

The comultiplication, the antipode and the counit of H_4 are given by

$$\begin{array}{ll} \Delta(g) = g \otimes g, & \Delta(h) = 1 \otimes h + h \otimes g, \\ S(g) = g, & S(h) = gh, \\ \epsilon(g) = 1, & \epsilon(h) = 0. \end{array}$$

It is well-known that H_4 is a quasi-triangular Hopf algebra with a family of quasi-triangular structures with respect to parameter t varying over k:

$$R_t = \frac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g) + \frac{t}{2}(1 \otimes 1 + g \otimes g + 1 \otimes g - g \otimes 1)(h \otimes h).$$

When t is zero, then (H_4, R_0) is a triangular Hopf algebra. In this paper we will mainly concentrate on the case (H_4, R_0).

An algebra A is called a *generalized quaternion algebra* if A is a four dimensional algebra generated by two elements u, v subject to the relations:

$$u^2 = \alpha, \quad v^2 = \beta, \quad uv + vu = \gamma \tag{1}$$

for some α, β and γ in k. A is said to be *non-singular* if $\gamma^2 - 4\alpha\beta \neq 0$ (in this case, it is isomorphic to a quaternion algebra). Otherwise A is *singular*. $d = \gamma^2 - 4\alpha\beta$ is called the *determinant* of the generalized quaternion algebra A. We denote A by $\left[\frac{\alpha,\beta,\gamma}{k}\right]$.

There is a natural action of H_4 on $\left[\frac{\alpha,\beta,\gamma}{k}\right]$ given by

$$\begin{array}{lll} g \rightharpoonup u = -u, & g \rightharpoonup v = -v, & g \rightharpoonup (uv) = uv, \\ h \rightharpoonup u = 0, & h \rightharpoonup v = 1, & h \rightharpoonup (uv) = u \end{array} \tag{2}$$

such that $\left[\frac{\alpha,\beta,\gamma}{k}\right]$ is a (left) H_4-module algebra (cf. [4]). We call the action (2) the *standard* action of H_4 on $\left[\frac{\alpha,\beta,\gamma}{k}\right]$. This standard action makes $\left[\frac{\alpha,\beta,\gamma}{k}\right]$ into an H_4^*-Galois extension of k (cf. [4]).

Now let us recall from [1] that an action of a Hopf algebra H on an algebra A is called *inner* if there is an invertible element $\pi \in \mathrm{Hom}(H, A)$, the convolution algebra, such that

$$h \rightharpoonup a = \sum \pi(h_{(1)}) a \pi^{-1}(h_{(2)})$$

whenever h is in H and a is in A. If, in addition, π is an algebra map, then the action is called a *strongly inner* action.

If A is an Azumaya algebra, then any action of H_4 on A is an inner action by the Skolem-Noether theorem (cf. [8, 5]). If the inner action is induced by an invertible element $\pi \in \mathrm{Hom}(H_4, A)$, then the two elements $u = \pi^{-1}(g)$ and $v = \pi^{-1}(h)$ in A satisfy the relation (1) for some $\alpha \neq 0, \beta$ and γ in k. One may check that, in this case, the action of H_4 on A is determined by u and v and reads as follows:

$$g(a) = u^{-1}au, \quad h(a) = av - vg(a) \tag{3}$$

for any $a \in A$. We call the subalgebra generated by the elements u and v *the induced subalgebra with respect to the action*. The induced subalgebra is unique though the elements u and v are not.

Lemma 1. *Let H_4 act non-trivially on an Azumaya algebra A. Let u, v be the generators of the induced subalgebra of A such that u and v satisfy* (3) *and the relations in* (1) *for some $\alpha \neq 0, \beta$ and γ in k. Then the action is strongly inner if and only if $d = \gamma^2 - 4\alpha\beta$ is zero and α is a square of some non-zero element in k.*

Proof. Assume that the action of H_4 on A is strongly inner. Then there is an algebra map $\delta : H_4 \longrightarrow A$ such that

$$\begin{aligned} &g \rightharpoonup a = \delta(g) a \delta(g), \\ &h \rightharpoonup a = a\delta(gh) + \delta(h) a \delta(g) = au'w + wau' \end{aligned}$$

for any $a \in A$, where $\delta(g) = u'$, $\delta(h) = w$ and

$$u'^2 = 1, \quad w^2 = 0, \quad u'w + wu' = 0.$$

On the other hand, we have that

$$g \rightharpoonup a = u^{-1}au, \; h \rightharpoonup a = av - v(g \rightharpoonup a)$$

(see (3)) for any $a \in A$. Comparing these two actions, we obtain that

$$u = su', \quad w = \frac{1}{2}(u'v - vu') \tag{4}$$

where $s \neq 0$ is in k. It follows that $\alpha = s^2$ and that

$$0 = w^2 = \frac{1}{4s^2}(uv - vu)^2 = \frac{1}{4s^2}(\gamma^2 - 4\alpha\beta).$$

Therefore $d = \gamma^2 - 4\alpha\beta = 0$.

Now one may easily see that if $d = 0$ and α is a square of some non-zero element in k, then the elements u' and w given by (4) give a strongly inner action of H_4 on A. □

As a consequence of Lemma 1, we obtain that when an inner action of H_4 on an Azumaya algebra A is not strong and the action of g is strongly inner, then the induced subalgebra is a non-singular generalized quaternion algebra.

Corollary 2. *If an action of H_4 on an Azumaya algebra A is not strongly inner and the action of g is strongly inner, then there are an Azumaya subalgebra B and a non-singular generalized quaternion algebra $\left[\frac{\alpha,\beta,\gamma}{k}\right]$ in A such that*

$$A = \left[\frac{\alpha, \beta, \gamma}{k}\right] \otimes B$$

as H_4-module algebras, where H_4 acts in a non-strongly inner way on $\left[\frac{\alpha,\beta,\gamma}{k}\right]$ and acts trivially on B.

Proof. By assumption and Lemma 1, A contains a generalized quaternion algebra $\left[\frac{\alpha,\beta,\gamma}{k}\right]$ which is non-singular. Since $\left[\frac{\alpha,\beta,\gamma}{k}\right]$ is an Azumaya algebra, by the commutator theorem there is an Azumaya subalgebra B in A such that

$$A = \left[\frac{\alpha, \beta, \gamma}{k}\right] \otimes B.$$

Since B commutes with $\left[\frac{\alpha,\beta,\gamma}{k}\right]$ and the inner action of H_4 on A is induced by the two generators u, v of $\left[\frac{\alpha,\beta,\gamma}{k}\right]$, H_4 acts trivially on B. Thus the action on A is uniquely determined by the restricted action on the subalgebra $\left[\frac{\alpha,\beta,\gamma}{k}\right]$. □

4. Quaternion algebras

In this section, we work with the non-singular generalized quaternion algebras which, over a field k, turn out to be quaternion algebras.

Lemma 3. *Let A be an H_4-module quaternion algebra. If the induced subalgebra is equal to A, then we can choose a basis for A such that the action of H_4 is the standard action on A.*

Proof. Let u, v be the two generators of the induced subalgebra $\left[\frac{\alpha,\beta,\gamma}{k}\right]$ which is equal to A by assumption. Set $u' = uv - vu$ and $v' = -\alpha d^{-1}u^{-1}u'$, where $d = \gamma^2 - 4\alpha\beta$. Then

$$u'^2 = d, \quad v'^2 = -\alpha d^{-1}, \quad u'v' + v'u' = 0.$$

It follows that $A \cong \left[\frac{d,-\alpha d^{-1},0}{k}\right]$. Now one may check that, with respect to the basis $\{1, u', v', u'v'\}$ of A, the action of H_4 is the standard action, that is,

$$g(u') = -u', \quad g(v') = -v', \quad h(u') = 0, \quad h(v') = 1.$$ □

Lemma 4. *Let A, B be two H_4-module algebras. Then the braided product $A\#B$ is the same as the graded product of the graded algebras A and B.*

Proof. The braided product is determined by the (quasi-) triangular structure R_0 in which the skew derivation h disappears. If we write $A = A_0 + A_1$ and $B = B_0 + B_1$, where $g(x_0) = x_0, g(x_1) = -x_1$ if $x_i \in A_i, B_i$, $i = 0, 1$, then $R_0(a_i \otimes b_j) = (-1)^{ij} b_j \otimes a_i$. By definition, R_0 exactly induces the classic graded product $A\hat{\otimes}B$ (see [12]). $\square$

In the sequel, we use $\left\langle \frac{\alpha,\beta,\gamma}{k} \right\rangle$ to indicate the generalized quaternion algebra $\left[\frac{\alpha,\beta,\gamma}{k}\right]$ with the standard H_4-action. The following proposition says that a generalized quaternion algebra with the standard H_4-action is H_4-Azumaya only if it is a quaternion algebra.

Proposition 5. *Let $A = \left\langle \frac{\alpha,\beta,\gamma}{k} \right\rangle$ be a generalized quaternion algebra. Then A is an H_4-Azumaya algebra if and only if the determinant $d = \gamma^2 - 4\alpha\beta$ is not zero.*

Proof. Assume that d is not equal to zero. Then A is an Azumaya algebra in the usual sense. Consider the H_4-opposite algebra $\overline{A}$ which is generated by $\overline{u}$ and $\overline{v}$ subject to the relations:

$$\overline{u}^2 = -u^2 = -\alpha, \quad \overline{v}^2 = -v^2 = -\beta, \quad \overline{uv} + \overline{vu} = -\gamma.$$

It turns out that $\overline{A}$ is equal to $\left\langle \frac{-\alpha,-\beta,-\gamma}{k} \right\rangle$. Since A and $\overline{A}$ are graded Clifford algebras and $A\#\overline{A}$ is the graded product, $A\#\overline{A}$ is still a graded Clifford algebra and hence an Azumaya algebra in the usual sense. Now the canonical H_4-linear map

$$F: \quad A\#\overline{A} \longrightarrow \mathrm{End}(A)$$

given by $F(a\#\overline{b})(x) = \sum a(R^2 \cdot x)(R^1 \cdot b)$, where $a, x \in A$ and $\overline{b} \in \overline{A}$, is an isomorphism because F sends the center k of $A\#\overline{A}$ onto the center of $\mathrm{End}(A)$. Similarly, the other canonical H_4-linear map

$$G: \ \overline{A}\#A \longrightarrow \mathrm{End}(A)^{op}$$

is an isomorphism too. So we have proved that $A = \left\langle \frac{\alpha,\beta,\gamma}{k} \right\rangle$ is an H_4-Azumaya algebra.

Conversely, if $\left\langle \frac{\alpha,\beta,\gamma}{k} \right\rangle$ is an H_4-Azumaya algebra, then its left (or right) H_4-center is trivial. Let $x = s_1 + s_2 u + s_3 v + s_4 w$ be an element in the H_4-center of A, where $w = uv, s_i \in k$. By definition, we have

$$xy = \sum (R^2 \cdot y)(R^1 \cdot x)$$

for any $y \in A$. Choose y to be the element u. We obtain

$$(\alpha s_2 + \gamma s_3) + (s_1 + \gamma s_4)u - \alpha s_4 v - s_3 w = -\alpha s_2 + s_1 u + s_4 v - s_3 w.$$

This amounts to the linear equations:

$$\begin{cases} 2\alpha s_2 + \gamma s_3 = 0, \\ \gamma s_4 = 0, \\ \alpha s_4 = 0. \end{cases}$$

Since α and γ cannot be zero at the same time, we have $s_4 = 0$. Similarly, $xv = \sum(R^2 \cdot v)(R^1 \cdot x)$ implies that

$$\gamma s_2 + 2\beta s_3 = 0.$$

Now the uniqueness of the solution for s_2 and s_3 yields that the determinant $d = \gamma^2 - 4\alpha\beta$ is not equal to zero. □

As a consequence of Corollary 2 and Proposition 5, we obtain the following:

Corollary 6. *If A is an Azumaya algebra on which H_4 acts in a non-strongly inner way and g acts in a strongly inner way, then A is an H_4-Azumaya algebra.*

5. The Brauer group $\mathrm{BM}(k, H_4, R_0)$

In this section, we calculate the Brauer group $\mathrm{BM}(H_4, R_0, k)$. We first consider the 2×2 matrix algebra M_2 which is isomorphic to the quaternion algebra $\left[\frac{\alpha,-\alpha,0}{k}\right]$ for any non-zero element $\alpha \in k$. However, when we endow $\left[\frac{\alpha,-\alpha,0}{k}\right]$ with the standard H_4-action, the H_4-module algebra $\left\langle\frac{\alpha,-\alpha,0}{k}\right\rangle$ is no longer isomorphic to $\left\langle\frac{\beta,-\beta,0}{k}\right\rangle$ as H_4-module algebras whenever α is not equal to β. This is the case when $\left\langle\frac{\alpha,-\alpha,0}{k}\right\rangle$ and $\left\langle\frac{\beta,-\beta,0}{k}\right\rangle$ represent different elements in the Brauer group $\mathrm{BM}(H_4, R_0, k)$. Actually, we have a group embedding of the abelian group $(k, +)$ into $\mathrm{BM}(k, H_4, R_0)$. We let M_2 stand for the 2×2 matrix algebra with the trivial H_4-action; in other words, $[M_2]$ represents the unit in $\mathrm{BM}(k, H_4, R_0)$.

Let Φ be a map from $(k, +)$ to $\mathrm{BM}(k, H_4, R_0)$ sending 0 to $[M_2]$ and $\alpha \neq 0$ to $[\left\langle\frac{\alpha^{-1},-\alpha^{-1},0}{k}\right\rangle]$.

Proposition 7. *Φ is a group monomorphism from $(k, +)$ into $\mathrm{BM}(k, H_4, R_0)$.*

Proof. We first compute the action of H_4 on the braided product $\left\langle\frac{\alpha^{-1},-\alpha^{-1},0}{k}\right\rangle \# \left\langle\frac{\beta^{-1},-\beta^{-1},0}{k}\right\rangle$. Let $\{u, v\}$ and $\{u', v'\}$ be the generators of $\left\langle\frac{\alpha^{-1},-\alpha^{-1},0}{k}\right\rangle$ and of $\left\langle\frac{\beta^{-1},-\beta^{-1},0}{k}\right\rangle$ respectively. One may take a while to check that the inner action of H_4 on $\left\langle\frac{\alpha^{-1},-\alpha^{-1},0}{k}\right\rangle \# \left\langle\frac{\beta^{-1},-\beta^{-1},0}{k}\right\rangle$ is induced by two elements:

$$U = sw\#w', \quad V = -\frac{\alpha}{2}v\#1 - \frac{\beta}{2}1\#v' + tw\#w'$$

where $w = uv, w' = u'v'$, s $(\neq 0)$ and t vary over k. Choose $s = \alpha\beta$ and $t = 0$. Then $U^2 = 1$ and $V^2 = -\frac{1}{4}(\alpha + \beta)$. We may assume that $\alpha \neq -\beta$. Otherwise, we have that $\left\langle\frac{\beta^{-1},-\beta^{-1},0}{k}\right\rangle = \overline{\left\langle\frac{\alpha^{-1},-\alpha^{-1},0}{k}\right\rangle}$ and $\left\langle\frac{\alpha^{-1},-\alpha^{-1},0}{k}\right\rangle \# \left\langle\frac{\beta^{-1},-\beta^{-1},0}{k}\right\rangle \cong \mathrm{End}(\left\langle\frac{\alpha^{-1},-\alpha^{-1},0}{k}\right\rangle)$ which represents the unit in $\mathrm{BM}(k, H_4, R_0)$.

Assume that $\alpha + \beta \neq 0$. Let $U' = 2UV$ and $V' = -2(\alpha + \beta)^{-1}V$. Then U' and V' generate the quaternion algebra $\left\langle\frac{\sigma^{-1},-\sigma^{-1},0}{k}\right\rangle$ with the standard action, where $\sigma = \alpha + \beta$. By Corollary 2, we obtain

$$\left\langle\frac{\alpha^{-1},-\alpha^{-1},0}{k}\right\rangle \# \left\langle\frac{\beta^{-1},-\beta^{-1},0}{k}\right\rangle \cong \left\langle\frac{\sigma^{-1},-\sigma^{-1},0}{k}\right\rangle \otimes M_2$$

where $\sigma = \alpha + \beta$. It follows that

$$\left\langle \frac{\alpha^{-1}, -\alpha^{-1}, 0}{k} \right\rangle \# \left\langle \frac{\beta^{-1}, -\beta^{-1}, 0}{k} \right\rangle] = [\left\langle \frac{(\alpha+\beta)^{-1}, -(\alpha+\beta)^{-1}, 0}{k} \right\rangle]$$

in the Brauer group $\mathrm{BM}(k, H_4, R_0)$. So we have proved that Φ is a group morphism. Finally, since the standard H_4-action on a quaternion algebra is not strongly inner, we have $[\left\langle \frac{\alpha^{-1}, -\alpha^{-1}, 0}{k} \right\rangle] \neq 1$ whenever α is not equal to zero. □

Now we are ready to compute the Brauer group $\mathrm{BM}(k, H_4, R_0)$. Let A be an H_4-module algebra. As mentioned in the proof of Lemma 4, A has a natural $\mathbb{Z}_2$-gradation:

$$A_0 = \{x \in A \mid g(x) = x\}, \quad A_1 = \{x \in A \mid g(x) = -x\}.$$

If A is an H_4-Azumaya algebra, $A = A_0 + A_1$ is $\mathbb{Z}_2$-graded Azumaya algebra in the sense of Wall. This is so because the canonical isomorphism

$$F: \quad A\#\overline{A} \longrightarrow \mathrm{End}(A)$$

is automatically a $\mathbb{Z}_2$-graded isomorphism. Thus the forgetful map

$$\Psi: \quad [A] \longrightarrow [A]$$

yields a homomorphism from $\mathrm{BM}(k, H_4, R_0)$ to the Brauer-Wall group $\mathrm{BW}(k)$ (for details on $\mathrm{BW}(k)$, we refer to [12]). Here Ψ is a homomorphism because of Lemma 4.

On the other hand, if B is a $\mathbb{Z}_2$-graded Azumaya algebra in the sense of Wall, then we can endow B with a natural H_4-module structure in the following way:

$$g \rightharpoonup x_i = (-1)^i x_i, \quad h \rightharpoonup x_i = 0$$

where x_i is a homogeneous element in B_i, $i = 0, 1$. One may easily check that B is an H_4-Azumaya algebra. So we obtain that $\mathrm{BW}(k)$ is a subgroup of $\mathrm{BM}(k, H_4, R_0)$. In fact, the split map Ψ can be easily explained in a categorical way as follows. Since R_0 is a (quasi-) triangular structure of $k\mathbb{Z}_2$, we have (quasi-) triangular Hopf algebra maps:

$$(k\mathbb{Z}_2, R_0) \xrightarrow{\text{incl.}} (H_4, R_0) \xrightarrow{\text{proj.}} (k\mathbb{Z}_2, R_0)$$

which induce braided monoidal functors:

$$_{k\mathbb{Z}_2}\mathbf{M} \longleftarrow \ _{H_4}\mathbf{M} \longleftarrow \ _{k\mathbb{Z}_2}\mathbf{M} \tag{5}$$

whose composite functor is the identity functor. Since the Brauer groups are functorial (see [10]), the monoidal functors of (5) induce group morphisms:

$$\mathrm{BM}(k, k\mathbb{Z}_2, R_0) \longleftarrow \mathrm{BM}(k, H_4, R_0) \longleftarrow \mathrm{BM}(k, k\mathbb{Z}_2, R_0)$$

whose composite map is the identity map. One may identify $\mathrm{BM}(k, k\mathbb{Z}_2, R_0)$ with $\mathrm{BW}(k)$ by Lemma 4. Now we are able to state our main theorem.

Theorem 8.

$$1 \longrightarrow (k, +) \xrightarrow{\Phi} \mathrm{BM}(k, H_4, R_0) \xrightarrow{\Psi} \mathrm{BW}(k) \longrightarrow 1 \tag{6}$$

is a split and exact sequence of group morphisms.

Proof. It suffices to prove that $\Phi(k,+)$ is the kernel of Ψ. Since a $\mathbb{Z}_2$-graded matrix algebra represents the unit in the Brauer-Wall group, we have that $\Phi(k,+) \subseteq \mathrm{Ker}(\Psi)$. Conversely, assume that A is an H_4-Azumaya algebra such that $\Psi([A]) = 1$ in $\mathrm{BW}(k)$ and $[A] \neq 1$ in $\mathrm{BM}(k, H_4, R_0)$. Since $\Psi([A]) = 1$, the action of g on A is strongly inner, and we may identify A with a graded matrix algebra when we forget the action of the skew-derivation h, say, $\langle M_n \rangle$. Since $[A] \neq 1$ in $\mathrm{BM}(k, H_4, R_0)$, we know that H_4 acts in a non-strongly inner way on A. Thus we may apply Corollary 2 to obtain:

$$\langle M_n \rangle = \left[\frac{\alpha, \beta, \gamma}{k}\right] \otimes B$$

for some Azumaya algebra B with the trivial H_4-action. This implies that $\left[\frac{\alpha,\beta,\gamma}{k}\right]$ is not a non-trivial quaternion algebra because α is a square of some non-zero element in k. So $\left[\frac{\alpha,\beta,\gamma}{k}\right]$ must be the 2×2-matrix algebra. Hence B is a matrix algebra as well and represents the unit in $\mathrm{BM}(k, H_4, R_0)$.

Now we may choose α equal to d^2. By applying Lemma 3 we obtain that

$$\left[\frac{1, \beta, \gamma}{k}\right] \cong \left\langle \frac{d, -d, 0}{k} \right\rangle$$

where d is the determinant. So we get $[A] = [\left\langle \frac{d,-d,0}{k} \right\rangle] = \Phi(d^{-1})$ for some non-zero element d. □

From the exact sequence (6) we get the whole Brauer group $\mathrm{BM}(k, H_4, R_0)$ which is the direct product of $(k,+)$ with $\mathrm{BW}(k)$ that is easily computed (see [12]). Finally we remark that the Brauer group $\mathrm{BM}(k, H_4, R_t)$, $t \neq 0$, is much more complicated than $\mathrm{BM}(k, H_4, R_0)$. The full Brauer group $\mathrm{BQ}(k, H_4)$ is a very rich group containing subgroups like $\mathrm{BM}(k, H_4, R_0)$, $\mathrm{BM}(k, H_4, R_1)$, $BC(k, H_4, R'_t)$ and $k^*/\mathbb{Z}_2$, where $\{R'_t\}$ is a family of coquasi-triangular structures of H_4. The investigation of $BQ(k, H_4)$ will be carried out in the forthcoming paper.

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