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DIAGONAL TYPE CONDITIONS ON GROUP C*-ALGEBRAS

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Abstract. Let G be a locally compact group with $\mathbf{C}^*(G)$ and $\mathbf{C}^*_r(G)$ its enveloping and reduced C*-algebras respectively. We show that if $\mathbf{C}^*(G)$ is residually finite dimensional, then G is maximally almost periodic, and $\mathbf{C}^*_r(G)$ is residually finite dimensional if and only if G is both amenable and maximally almost periodic. Letting λ_G be the left regular representation of G, we show that a certain quasidiagonality condition on $\{\lambda_G(s) : s \in G\}$ implies that G is amenable.

This paper summarizes a few results relating the structures of locally compact groups to diagonal type structures for some of their C*-algebras, namely *residual finite dimensionality* and *quasidiagonality*.

The main results in the first section regard such a group G having a residually finite dimensional enveloping C*-algebra, $\mathbf{C}^*(G)$. We show that if $\mathbf{C}^*(G)$ is residually finite dimensional, then G is maximally almost periodic, and mention a case where the converse does not hold. We obtain a partial converse, though: G is maximally almost periodic exactly when its reduced C*-algebra $\mathbf{C}^*_r(G)$ is the quotient of a residually finite dimensional C*-algebra which is a quotient of $\mathbf{C}^*(G)$. We then obtain the result that G is maximally almost periodic and amenable exactly when $\mathbf{C}^*_r(G)$ is residually finite dimensional. We conclude the section by exploring the class of groups having residually finite dimensional enveloping C*-algebras. In particular, we examine the stability of this class under free products and amalgams.

The second section records the result that a certain quasidiagonality condition for the unitaries of the left regular representation of G implies that G is amenable. We also extend, to a larger class than discrete groups, Rosenberg's result that if $\mathbf{C}^*_r(G)$ is quasidiagonal, then G is amenable.

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1. Maximal almost periodicity and residual finite dimensionality

A locally compact group G is called *maximally almost periodic* if there exists a separating family of continuous finite dimensional unitary representations of G. A C*-algebra $\mathcal{A}$ is called *residually finite dimensional* if there is a separating family of finite dimensional representations of $\mathcal{A}$. If $\mathcal{A}$ admits a countable separating family of finite dimensional representations, then we could say that $\mathcal{A}$ may be faithfully represented as a block diagonal C*-algebra.

We recall that there is a bijective correspondence between the collection of equivalence classes of continuous unitary representations of a locally compact group G and the collection of equivalence classes of non-degenerate involutive representations of its enveloping C*-algebra $\mathbf{C}^*(G)$. If $\sigma : \mathbf{C}^*(G) \to \mathcal{B}(\mathcal{H})$ is an involutive representation on a Hilbert space $\mathcal{H}$, the associated continuous unitary representation of G will be denoted by σ_G; if σ_G is a continuous unitary representation of G, then denote the corresponding representation of $\mathbf{C}^*(G)$ by σ. Note that $\mathbf{C}^*(G)$ may be viewed as the completion of $\mathcal{C}_c(G)$, the convolution algebra of continuous functions with compact support on G, in an appropriate norm. If for s in G and f in $\mathcal{C}_c(G)$ we write $s{\cdot}f(t) = f(s^{-1}t)$ for t in G, then $\sigma(s{\cdot}f) = \sigma_G(s)\sigma(f)$.

Let Σ_G denote the collection of equivalence classes of continuous unitary representations of G. By abuse of notation we will treat subsets of Σ_G as sets of representatives of elements of Σ_G. Given σ_G, τ_G in Σ_G, we say that σ_G is *weakly contained* in τ_G, denoted $\sigma_G \prec \tau_G$, if σ is weakly contained in τ as a representation of $\mathbf{C}^*(G)$, i.e. $\ker\sigma \supset \ker\tau$. If $\sigma_G \in \Sigma_G$ and $\mathcal{S}$ is a family in Σ_G, write $\sigma_G \prec \mathcal{S}$ if $\ker\sigma \supset \bigcap_{\pi_G\in\mathcal{S}} \ker\pi$. Let $\widehat{G}$ denote the set of all irreducible representations in Σ_G, and $\widehat{G}_f$ denote the set of all finite dimensional representations in $\widehat{G}$. Then G maximally almost periodic if and only if $\widehat{G}_f$ is a separating family for G.

Theorem 1.1. *For a locally compact group G, if $\mathbf{C}^*(G)$ is residually finite dimensional, then G is maximally almost periodic.*

Proof. Let $\mathcal{F}$ be a separating family of finite dimensional non-degenerate involutive representations of $\mathbf{C}^*(G)$. Then $\mathcal{F}_G$ is separating for G. Indeed, if there were $s \in \bigcap_{\pi\in\mathcal{F}} \ker\pi_G$, then for all f in $\mathcal{C}_c(G)$ we would have $s{\cdot}f - f \in \bigcap_{\pi\in\mathcal{F}} \ker\pi$, a contradiction, unless $s = e$. □

Although categorically suggestive, the converse to Theorem 1.1 is false. In fact, it fails for a residually finite discrete group! In [8, p. 137] it is shown that for the residually finite group $G = \mathrm{SL}_2(\mathbb{Z}[1/p])$, $\mathbf{C}^*(G)$ is not residually finite dimensional.

We can answer the question posed in [8, p. 137] of whether for a discrete group G, $\mathbf{C}^*(G)$ being residually finite dimensional implies that G is residually finite.

Corollary 1.2. *If G is a finitely generated (hence discrete) group and $\mathbf{C}^*(G)$ is residually finite dimensional, then G is residually finite. However, there exist countable groups for which $\mathbf{C}^*(G)$ is residually finite dimensional but G is not residually finite.*

Proof. If $\mathbf{C}^*(G)$ is residually finite dimensional, then G is maximally almost periodic. If $\pi \in \widehat{G}_f$, then $\pi(G)$ is a finitely generated linear group and so is residually finite by [1, Cor. 1]. This shows that G is residually finite.

If G is a non-trivial divisible group, i.e. G is Abelian and $G^n = \{s^n : s \in G\}$ is all of G for n in $\mathbb{N}$, then any quotient of G is also divisible, so can be finite only if

it is trivial. Hence G cannot be residually finite. However, any divisible group G has that its commutative C*-algebra $\mathbf{C}^*(G)$ is residually finite dimensional, since its characters separate points. In particular, the additive group of rationals $\mathbb{Q}$ has a residually finite dimensional enveloping C*-algebra, but is not residually finite.

Much of this proof is contained in [9, Prop. 4]. □

For a locally compact group G, let λ_G denote the left regular representation of G on $\mathbf{L}^2(G)$, given for s in G and ξ in $\mathbf{L}^2(G)$ by $\lambda_G(s)\xi(t) = \xi(s^{-1}t)$ for almost all t in G. Let $\mathbf{C}^*_r(G) = \lambda(\mathbf{C}^*(G))$ denote the *reduced C*-algebra* of G.

Theorem 1.3. *If G is a locally compact group, then the following are equivalent:*

(i) *G is maximally almost periodic,*

(ii) $\lambda_G \prec \widehat{G}_f$,

(iii) $\lambda : \mathbf{C}^*(G) \to \mathbf{C}^*_r(G)$ *factors through a residually finite dimensional C*-algebra.*

Proof. (i)⇒(ii) This is [4, Example 1.11 (ii)].

(ii)⇒(iii) Let $\rho_G = \bigoplus_{\pi_G \in \widehat{G}_f} \pi_G$ and $\mathbf{C}^*_f(G) = \rho(\mathbf{C}^*(G))$. Let $\sigma : \mathbf{C}^*_f(G) \to \mathbf{C}^*_r(G)$ be given by $\sigma(\rho(a)) = \lambda(a)$ for a in $\mathbf{C}^*(G)$. Since $\lambda_G \prec \rho_G$, σ is well-defined, and is clearly an involutive representation of $\mathbf{C}^*_f(G)$. Hence the following diagram commutes:

$$\begin{tikzcd}
\mathbf{C}^*(G) \arrow[rr, "\lambda"] \arrow[dr, "\rho"'] & & \mathbf{C}^*_r(G) \\
& \mathbf{C}^*_f(G) \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

(iii)⇒(i) Let $\rho : \mathbf{C}^*(G) \to \mathcal{B}(\mathcal{H})$ be any representation factoring λ, where $\rho(\mathbf{C}^*(G))$ is residually finite dimensional with separating family $\mathcal{F}$ of finite dimensional representations. If $s \in \ker \rho_G$, then for all $f \in \mathcal{C}_c(G)$, $s{\cdot}f - f \in \ker \rho \subset \ker \lambda$. Hence $s{\cdot}f - f = 0$ for all f in $\mathcal{C}_c(G)$; since $\lambda|_{\mathcal{C}_c(G)}$ is injective, and thus $s = e$. This means that $\mathcal{F}_G$ separates points of G so G is maximally almost periodic. □

We can abstractly characterize all locally compact groups G for which $\mathbf{C}^*_r(G)$ is residually finite dimensional.

Theorem 1.4. *A locally compact group G is maximally almost periodic and amenable if and only if $\mathbf{C}^*_r(G)$ is residually finite dimensional.*

Proof. If G is maximally almost periodic and amenable, then it is immediate that $\mathbf{C}^*(G) \cong \mathbf{C}^*_r(G) \cong \mathbf{C}^*_f(G)$ by [16, Theo. 4.21] and Theorem 1.3 (iii) above, so $\mathbf{C}^*_r(G)$ is residually finite dimensional.

If $\mathbf{C}^*_r(G)$ is residually finite dimensional, let $\{\pi_\alpha\}_{\alpha \in A}$ be a separating family of finite dimensional involutive representations of $\mathbf{C}^*_r(G)$. For α in A, $\pi_\alpha \circ \lambda$ is a finite dimensional representation on $\mathbf{C}^*(G)$ for which $(\pi_\alpha \circ \lambda)_G \prec \lambda_G$. Hence G is amenable by [13, Theo. 3]. Then $\mathbf{C}^*(G) \cong \mathbf{C}^*_r(G)$ is residually finite dimensional, so G is maximally almost periodic by Theorem 1.1. □

For countable groups the above theorem follows from [6, Cor. 4].

We have some interest in characterising the class of locally compact groups G for which $\mathbf{C}^*(G)$ is residually finite dimensional. We know that all such amenable groups are the maximally almost periodic ones. There are, however, non-amenable groups which are in this class.

If G and H are two discrete groups, denote their *free product* (or *coproduct* in the category of groups) by $G \star H$. Provided that at least one of G or H is not a 2 element group and both are non-trivial, it is known that $G \star H$ contains a copy of the free group F_2, and hence is not amenable.

Proposition 1.5. *If G and H are discrete groups such that $\mathbf{C}^*(G)$ and $\mathbf{C}^*(H)$ are residually finite dimensional, then $\mathbf{C}^*(G \star H)$ is residually finite dimensional.*

Proof. There is an obvious isomorphism $\mathbf{C}^*(G \star H) \cong \mathbf{C}^*(G) \star_{\mathbb{C}} \mathbf{C}^*(H)$, where $\mathbf{C}^*(G) \star_{\mathbb{C}} \mathbf{C}^*(H)$ is the unital C*-algebra free product of $\mathbf{C}^*(G)$ and $\mathbf{C}^*(H)$. This C*-algebra is residually finite dimensional by [11, Theo. 3.2]. □

We remark that for $F_2 \cong \mathbb{Z} \star \mathbb{Z}$, the above result is due to Choi [5] (also see [7, VII.6.1]).

If G and H are groups and there is a group Γ which is embedded in each of G and H via homomorphisms i and j respectively, then the *amalgam* of G and H over Γ is $G \star_\Gamma H = (G \star H)/N$, where N is the smallest normal subgroup in $G \star H$ containing $\{i(a)j(a)^{-1} : a \in \Gamma\}$. It is interesting to try to determine if the class of discrete groups having residually finite dimensional C*-algebras is closed under amalgams over common subgroups. We can conclude that this is not the case in general. Note that from [19, p. 35], $\mathrm{SL}_2(\mathbb{Z}) \cong (\mathbb{Z}/4\mathbb{Z}) \star_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}/6\mathbb{Z})$. We do not know if $\mathbf{C}^*(\mathrm{SL}_2(\mathbb{Z}))$ is residually finite dimensional. However, by [19, p. 80], for a prime p, there is a subgroup $\Gamma(p)$ of $\mathrm{SL}_2(\mathbb{Z})$ such that $\mathrm{SL}_2(\mathbb{Z}[1/p]) \cong \mathrm{SL}_2(\mathbb{Z}) \star_{\Gamma(p)} \mathrm{SL}_2(\mathbb{Z})$. Hence, either $\mathrm{SL}_2(\mathbb{Z})$ or $\mathrm{SL}_2(\mathbb{Z}[1/p])$ shows that this class is not closed under amalgams by arbitrary subgroups. We are unaware of whether this class is closed for amalgams over finite subgroups.

If G is a locally compact group for which $\mathbf{C}^*(G)$ is residually finite dimensional, then the same is true for any open subgroup H by the fact that $\mathbf{C}^*(H)$ may be realized as a subalgebra of $\mathbf{C}^*(G)$ by [17, Prop. 1.2]. We are not aware if this property holds for closed subgroups in general.

2. Quasidiagonality and amenability

For our definitions and preliminary results, we follow [14], but suitably generalize them to non-separable Hilbert spaces. A set $\mathcal{S}$ of bounded operators on a Hilbert space $\mathcal{H}$ is called *quasidiagonal* if for any set $\mathcal{M} \subset \mathcal{H}$ which spans a dense subspace of $\mathcal{H}$, any finite sets $\mathcal{S}_0 \subset \mathcal{S}$, $\mathcal{M}_0 \subset \mathcal{M}$ and $\varepsilon > 0$, there is a finite rank projection p on $\mathcal{H}$ such that

$$(*) \qquad \|pa - ap\| \leq \varepsilon \text{ for } a \text{ in } \mathcal{S}_0 \text{ and } \|p\xi - \xi\| < \varepsilon \text{ for } \xi \text{ in } \mathcal{M}_0.$$

The following is the non-separable analogue of [14, Lem. 1 (4)].

Proposition 2.1. *A set $\mathcal{S}$ of bounded operators on a Hilbert space $\mathcal{H}$ is quasidiagonal if and only if there exists an increasing net of finite rank projections $\{p_\alpha\}_{\alpha \in A}$ tending strongly to the identity operator such that $\lim_{\alpha \in A} \|p_\alpha a - a p_\alpha\| = 0$ for a in $\mathcal{S}$.*

Proof. If $\mathcal{S}$ is quasidiagonal, let $\mathcal{M}$ be any subset of $\mathcal{H}$ which spans a dense subspace and let $A = \{(\mathcal{S}_0, \mathcal{M}_0, \varepsilon) : \mathcal{S}_0 \subset \mathcal{S}$ and $\mathcal{M}_0 \subset \mathcal{M}$ are finite, and $\varepsilon > 0\}$. There is a natural ordering under which A is a directed set. For $\alpha = (\mathcal{S}_0, \mathcal{M}_0, \varepsilon)$ in A, let p_α be the projection satisfying $(*)$ for the triple α. The rest of the proof is straightfoward. □

A representation $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of a C*-algebra $\mathcal{A}$ is *quasidiagonal* if $\rho(\mathcal{A})$ is quasidiagonal. We say that a C*-algebra $\mathcal{A}$ is *weakly quasidiagonal* if it has a faithful quasidiagonal representation.

It is clear from the above proposition that a residually finite dimensional C*-algebra is weakly quasidiagonal.

We now define two algebras associated with the left regular representation. Let $\mathbf{C}^*_\delta(G) = \overline{\mathrm{span}}\{\lambda_G(s) : s \in G\}$ and $\mathbf{VN}(G)$ be the weak operator closure of $\mathbf{C}^*_\delta(G)$, or equivalently the weak operator closure of $\mathbf{C}^*_r(G)$, in $\mathcal{B}(\mathbf{L}^2(G))$. Note that if G is discrete, then $\mathbf{C}^*_\delta(G) \cong \mathbf{C}^*_r(G)$, and if G is Abelian, then $\mathbf{C}^*_\delta(G) \cong \mathcal{AP}(\widehat{G})$, where $\mathcal{AP}(\widehat{G})$ is the commutative C*-algebra of almost periodic functions on the *dual group* $\widehat{G}$. It is well known that $\mathbf{VN}(G)$ is the commutant of $\{\lambda'_G(s) : s \in G\}$, where λ'_G is the right regular representation of G on $\mathbf{L}^2(G)$, given for s in G and ξ in $\mathbf{L}^2(G)$ by $\lambda'_G(s)\xi(t) = \Delta(s)^{1/2}\xi(ts)$ for almost all t in G.

Lemma 2.2. *If G is not compact, then $\mathbf{VN}(G)$ contains no non-trivial compact operators.*

Proof. Fix a in $\mathbf{VN}(G) \setminus \{0\}$. Find ξ_a in $\mathcal{C}_c(G)$ such that $\|\xi_a\|_2 = 1$ and $\langle a\xi_a \,|\xi_a\rangle \neq 0$. (By a numerical range argument we may obtain that $|\langle a\xi_a \,|\xi_a\rangle| > (1-\varepsilon)\frac{1}{2}\|a\|$, for any $\varepsilon > 0$.) Let $K = \mathrm{supp}(\xi_a)$. Then we may inductively find a sequence $\{t_n\}_{n\in\mathbb{N}}$ in G such that $Kt_1 \dots t_n$ is disjoint from K and $Kt_1 \dots t_k$ for any $k = 1, \dots, n-1$. Put $\xi_{a,0} = \xi_a$ and for n in $\mathbb{N}$, $\xi_{a,n} = \lambda'_G(t_1 \dots t_n)^*\xi_a$. Then $\{\xi_{a,n}\}_{n\in\mathbb{N}\cup\{0\}}$ is an orthonormal sequence in $\mathbf{L}^2(G)$ and

$$\langle a\xi_{a,n} \,|\xi_{a,n}\rangle = \langle a\lambda'_G(t_1 \dots t_n)^*\xi_{a,0} \,|\lambda'_G(t_1 \dots t_n)^*\xi_{a,0}\rangle = \langle a\xi_{a,0} \,|\xi_{a,0}\rangle \neq 0$$

for all n, so a is not compact; for if a were compact, then $\lim_{n\to\infty} \|a\xi_{a,n}\| = 0$. □

The problem of knowing the conditions under which $\mathbf{C}^*_r(G)$ is (weakly) quasidiagonal has attracted some attention ([14], [18], [21], [22]). It is proved in [18] that for a discrete group G, $\mathbf{C}^*_r(G)$ is quasidiagonal (as a concrete family of operators) only if G is amenable. We have found that the most natural extension of this result is not an extension to $\mathbf{C}^*_r(G)$ for locally compact G, but to any space of operators containing $\mathbf{C}^*_\delta(G)$.

Theorem 2.3. *If G is a locally compact group for which $\mathbf{C}^*_\delta(G)$ is weakly quasidiagonal, then G is amenable.*

Proof. We may assume that G is not compact. Let H be a countable subgroup of G and $\mathbf{C}^*_{\lambda_G}(H) = \overline{\mathrm{span}}\{\lambda_G(s) : s \in H\}$. Then $\mathbf{C}^*_{\lambda_G}(H)$ is weakly quasidiagonal. Let $D \subset \mathbf{C}^*_{\lambda_G}(H)$ be a countable dense subset, and for d in $D\setminus\{0\}$, create $\{\xi_{d,n}\}_{n\in\mathbb{N}\cup\{0\}}$ as in the lemma above. Then for any separable, $\mathbf{C}^*_{\lambda_G}(H)$-invariant subspace $\mathcal{H}$ of $\mathbf{L}^2(G)$ containing $\{\xi_{d,n} : n \in \mathbb{N}\cup\{0\}$ and $d \in D\}$, $\mathbf{C}^*_{\lambda_G}(H)|_{\mathcal{H}}$ intersects the compacts on $\mathcal{H}$ trivially. If ρ is a faithful quasidiagonal *-representation of $\mathbf{C}^*_{\lambda_G}(H)$ on a separable Hilbert space $\mathcal{K}$ (i.e. take ρ to be the restriction of some faithful

quasidiagonal representation to a separable invariant subspace), then so too is $\rho^{(\infty)} : \mathbf{C}^*_{\lambda_G}(H) \to \mathcal{B}(\mathcal{K}^{(\infty)})$, and

$$\ker \mathrm{id}|_{\mathcal{H}} = \ker \pi \circ \mathrm{id}|_{\mathcal{H}} = \ker \pi \circ \rho^{(\infty)} = \ker \rho^{(\infty)}$$

where π is the map onto the Calkin algebra. Hence $\mathrm{id}|_{\mathcal{H}}$ and $\rho^{(\infty)}$ are approximately unitarily equivalent by Voiculescu's "Weyl-von Neumann" Theorem [20] (see [7, Cor. II.5.6]), so $\mathrm{id}|_{\mathcal{H}}$ is quasidiagonal by [14, Cor. 3]. Hence $\mathbf{C}^*_{\lambda_G}(H)|_{\mathcal{H}}$ is quasidiagonal as a set of operators on $\mathcal{H}$. Since $\mathcal{H}$ is an arbitrary separable subspace of $\mathbf{L}^2(G)$ satisfying the conditions above, it follows that $\mathbf{C}^*_{\lambda_G}(H)$ is quasidiagonal as a set of operators on $\mathbf{L}^2(G)$.

Let $\{p_\alpha\}_{\alpha \in A}$ be a net of finite rank projections on $\mathbf{L}^2(G)$ strongly increasing to the identity such that $\lim_{\alpha \in A} \|p_\alpha a - a p_\alpha\| = 0$ for a in $\mathbf{C}^*_{\lambda_G}(H)$. Let $\mathcal{U}$ be any ultrafilter on A containing the sets $\{\alpha \in A : \alpha \geq \alpha_0\}$ for each α_0 in A. For any f in $\mathbf{L}^\infty(G)$ let M_f be the multiplication operator by f on $\mathbf{L}^2(G)$. Let tr be the canonical trace defined on finite rank operators on $\mathbf{L}^2(G)$. Then we may define a mean (state) on $\mathbf{L}^\infty(G)$ by setting

$$m(f) = \lim_{\alpha \in U,\, U \in \mathcal{U}} \frac{1}{\mathrm{tr}(p_\alpha)} \mathrm{tr}(p_\alpha M_f p_\alpha)$$

for f in $\mathbf{L}^\infty(G)$. We can now follow [22, Prop. 4.2] (see also [7, Prop. VII.7.8]) to see that m is invariant for the left action of H on $\mathbf{L}^\infty(G)$. First see that for t in G and f in $\mathbf{L}^\infty(G)$, $\lambda_G(t) M_f \lambda_G(t)^* = M_{t \cdot f}$. Then for such f and t in H we have

$$\begin{aligned}
|m(t \cdot f) - m(f)| &= \lim_{\alpha \in U,\, U \in \mathcal{U}} \frac{1}{\mathrm{tr}(p_\alpha)} |\mathrm{tr}(p_\alpha \lambda_G(t) M_f \lambda_G(t)^* p_\alpha) \\
&\qquad - \mathrm{tr}(\lambda_G(t) p_\alpha M_f p_\alpha \lambda_G(t)^*)| \\
&= \lim_{\alpha \in U,\, U \in \mathcal{U}} \frac{1}{\mathrm{tr}(p_\alpha)} |\mathrm{tr}([p_\alpha \lambda_G(t) - \lambda_G(t) p_\alpha] M_f \lambda_G(t)^* p_\alpha) \\
&\qquad + \mathrm{tr}(\lambda_G(t) p_\alpha M_f [\lambda_G(t)^* p_\alpha - p_\alpha \lambda_G(t)^*])| \\
&\leq \lim_{\alpha \in U,\, U \in \mathcal{U}} 2 \|p_\alpha \lambda_G(t) - \lambda_G(t) p_\alpha\| \|f\|_\infty = 0.
\end{aligned}$$

For each finite subset F in G, let m_F be the mean defined above which is invariant for the action of $\langle F \rangle$ on $\mathbf{L}^\infty(G)$. Then any weak* cluster point of the net $\{m_F\}_{F \in \mathcal{F}}$, indexed over the collection of all finite subsets of G, is a left invariant mean for the action of G on $\mathbf{L}^\infty(G)$. Hence G is amenable. □

We note that the method used above, of reducing to countable subgroups and applying Voiculescu's theorem, gives a generalization of [18, Theo. A1].

Corollary 2.4. *For a discrete group G, if $\mathbf{C}^*_r(G)$ is weakly quasidiagonal, then G is amenable.*

For a locally compact group G, let G_d denote the same group but with the discrete topology. Note that in general $\mathbf{C}^*_r(G_d)$ is a quotient of $\mathbf{C}^*_\delta(G)$ by [10, Theo. 2.5] (see also [2, Lem. 2]).

Corollary 2.5. *Suppose G is a locally compact group which contains an open subgroup H such that H_d is amenable (for example, if H is solvable) and $\mathbf{C}^*_\delta(G)$ is weakly quasidiagonal. Then G_d is amenable.*

Proof. The condition that G has an open subgroup H for which H_d is amenable is equivalent to the existence of an isomorphism $\mathbf{C}^*_\delta(G) \cong \mathbf{C}^*_r(G_d)$ by [3, Theo. 2]. Then $\mathbf{C}^*_r(G_d)$ is weakly quasidiagonal so G_d is amenable. □

We have another extension of Rosenberg's result of a slightly different nature.

Theorem 2.6. *If G is a locally compact group with an open normal compact subgroup K and $\mathbf{C}^*_r(G)$ is weakly quasidiagonal, then G is amenable.*

Proof. Let $\mathbf{C}^*_{r,K}(G) = \overline{\mathrm{span}}\{\lambda(\chi_{tK}) : t \in G\}$, where χ_{tK} is the characteristic function of tK. As in [12, p. 217] let $\mathbf{L}^2_K(G)$ be the closed subspace of elements of $\mathbf{L}^2(G)$ which are almost everywhere constant on (left) cosets of K, and let $j : \ell^2(G/K) \to \mathbf{L}^2_K(G)$ be the unitary given by $j\xi = \xi \circ q$, where $q : G \to G/K$ is the quotient map. Note that we may assume $\mu(K) = 1$, where μ is the Haar measure on G, so that j is indeed a unitary. Then the map $a \mapsto j^*aj$ is an isomorphism identifying $\mathbf{C}^*_{r,K}(G)$ with $\mathbf{C}^*_r(G/K)$. Since $\mathbf{C}^*_{r,K}(G)$ is weakly quasidiagonal, so too is $\mathbf{C}^*_r(G/K)$. Since G/K is discrete, this implies that G/K is amenable; hence so too is G, by [16, Prop. 1.13]. □

We remark that the class of groups having a compact open normal subgroup includes all [IN] groups that have a compact connected component of the identity. (See [15] for a description of [IN] groups.) This class also includes all Lie groups which have a compact connected component of the identity.

Added in proof

A.-T. Lau has kindly pointed out to us that Lemma 2.2 can be found in [23].

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