NO \( n \)-POINT SET IS \( \sigma \)-COMPACT

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Abstract. Let \( n \) be an integer greater than 1. We prove that there exist no \( F_\sigma \)-subsets of the plane that intersect every line in precisely \( n \) points.

Let \( n \geq 2 \) be some fixed integer. A subset of the plane \( \mathbb{R}^2 \) is called an \( n \)-point set if every line in the plane meets the set in precisely \( n \) points. The question of whether \( n \)-point sets can be Borel sets is a long standing open problem (see e.g. Mauldin [6] for details). Sierpinski [7, p. 447] has given a simple example of a closed set that meets every line in \( \aleph_0 \) points. It was shown by Baston and Bostock [1] and by Bouhjar, Dijkstra, and van Mill [2] that 2-point sets, respectively 3-point sets, cannot be \( F_\sigma \) in the plane. Both papers use a method suggested by Larman [5] for the case \( n = 2 \) which consists of proving on the one hand that 2-point sets cannot contain arcs and on the other hand that 2-point sets that are \( F_\sigma \) must contain arcs. Observe that to prove the result that is the subject of this note Larman’s program cannot be followed because it was shown in [2] that \( n \)-point sets can contain arcs whenever \( n \geq 4 \).

Theorem. Let \( n \geq 2 \). No \( n \)-point set is an \( F_\sigma \)-subset of the plane.

The three authors of this note each, independently of each other, found a proof for this theorem. We decided to publish the shortest proof jointly.

Proof. Assume that \( A \) is an \( n \)-point set that is an \( F_\sigma \)-subset of the plane. Let \( xy \) be an arbitrary rectangular coordinate system for the plane and let \( \lambda \) be the Lebesgue measure on \( \mathbb{R} \). According to [2] Proposition 3.2] there exists a nondegenerate interval \( [a, b] \) on the \( x \)-axis and continuous functions \( f_1 < f_2 < \cdots < f_n \) from \( [a, b] \) into \( \mathbb{R} \) such that \( A \) contains the graph of each \( f_i \). Consider an \( f_i \) and its graph \( G_i \). Since \( A \) is an \( n \)-point set each horizontal line intersects \( G_i \) in at most \( n \) points. So every fibre of \( f_i \) has cardinality at most \( n \). Consequently, according to Banach [4] Exercise 17.34], the variation of \( f_i \) is bounded by \( n(M - m) \), where \( m \) and \( M \) are the minimum and maximum values of \( f_i \). According to Lebesgue [4] Theorem 17.17] the derivative of a function of bounded variation such as \( f_i \) exists almost everywhere. Select a Borel set \( B \subset [a, b] \) such that \( \lambda(B) = b - a \) and every \( f_i \) is differentiable at every point of \( B \). By the Whitney Extension Theorem for \( C^1 \) functions [3] Theorem 3.1.16] there exists a set \( C \subset B \) such that \( \lambda(C) > 0 \) and
continuously differentiable functions $g_i : [a, b] \to \mathbb{R}$ with $g_i(C) = f_i(C)$ for $1 \leq i \leq n$. The functions $g_i$ satisfy the premises of Theorem 7 in [6] so we may conclude that the set $A$ is bounded or intersects some line in $n + 1$ points. Either way, the result is inconsistent with $A$ being an $n$-point set.

References


