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GO-SPACES WITH σ-CLOSED DISCRETE DENSE SUBSETS

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Dedicated to the memory of F. Burton Jones

Abstract. In this paper we study the question "When does a perfect generalized ordered space have a σ-closed-discrete dense subset?" and we characterize such spaces in terms of their subspace structure, s-mappings to metric spaces, and special open covers. We also give a metrization theorem for generalized ordered spaces that have a σ-closed-discrete dense set and a weak monotone ortho-base. That metrization theorem cannot be proved in ZFC for perfect GO-spaces because if there is a Souslin line, then there is a non-metrizable, perfect, linearly ordered topological space that has a weak monotone ortho-base.

1. Introduction

In this paper we provide necessary and sufficient conditions for a perfect generalized ordered space (GO-space) to have a σ-closed-discrete dense subset (Theorem 2.1), and we prove a new metrization theorem for such spaces using the notion of a weak monotone ortho-base (Theorem 3.1).

To understand the context of our results, recall that any GO-space with a σ-closed-discrete dense subset must be perfect, i.e., closed sets in the space must be G_δ-sets. The converse is consistently false: if there is a Souslin space (a non-separable linearly ordered topological space that has countable cellularity), then there is a perfect GO-space that does not have a σ-closed-discrete dense subset. An old question due to Maurice and van Wouwe [vW] asks whether there is a ZFC example of a perfect GO-space without a σ-closed-discrete dense set.

We now know that several open questions in ordered space theory are closely related to the Maurice-van Wouwe question. The first is due to Heath. After Ponomarev [P] and Bennett [B1] independently proved that if there is a Souslin space, then there is a Souslin space with a point-countable base, Heath asked for a ZFC example of a perfect GO-space that has a point-countable base and yet is non-metrizable. In the light of a result of Bennett and Lutzer [BL1], such a space could not have a σ-closed-discrete dense subset (see 3.1 c), below). The second question, posed by Lutzer [L], asked whether every perfect GO-space can be topologically embedded in a perfect linearly ordered topological space, and recent work of Shi [S] has shown that if there is a counterexample, then it is perfect and

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does not have a σ-closed-discrete dense subset. A third example is the question of Nyikos [N1] asking whether there is a ZFC example of a perfect, non-metrizable, non-archimedean space. Such a space would also be a perfect GO-space not having a σ-closed-discrete dense subset. Readers should consult the recent paper by Qiao and Tall [QT] for further information on this family of inter-related problems.

In Section 2, we will characterize those perfect GO-spaces that have a σ-closed-discrete dense subset by proving the following structure theorem:

1.1 Theorem. *The following properties of a perfect GO-space X are equivalent:*

a) *X has a σ-closed-discrete dense subset;*
b) *there is a sequence $\langle \mathcal{G}_n \rangle$ of open covers of X such that for each $p \in X$, the set $\bigcap\{St(p, \mathcal{G}_n) : n \geq 1\}$ has at most two points;*
c) *there is a sequence $\langle \mathcal{G}_n \rangle$ of open covers of X such that for each $p \in X$, the set $\bigcap\{St(p, \mathcal{G}_n) : n \geq 1\}$ is separable;*
d) *X is the union of two subspaces, each having a G_δ-diagonal in its relative topology;*
e) *X is the union of countably many subspaces, each having a G_δ-diagonal in its relative topology;*
f) *there exist a perfect, paracompact GO-space Y with a G_δ-diagonal and a closed continuous mapping $f : X \to Y$ with $|f^{-1}[y]| \leq 2$ for each $y \in Y$;*
g) *there exist a metrizable GO-space Y and a continuous mapping $f : X \to Y$ with $|f^{-1}[y]| \leq 2$ for each $y \in Y$;*
h) *there exist a topological space Z with a G_δ-diagonal and a continuous s-mapping $g : X \to Z$, i.e., a continuous mapping such that $g^{-1}[z]$ is separable for each $z \in Z$.*

In Section 3, we prove a new metrization theorem for GO-spaces, using the notion of a weak monotone ortho-base. Recall ([N2]) that a base $\mathcal{B}$ for a space is a *weak monotone ortho-base* if, for any monotonic subcollection $\mathcal{M} \subseteq \mathcal{B}$ (i.e., where $\mathcal{M}$ is linearly ordered by $\subseteq$), either the set $\bigcap \mathcal{M}$ is open or else $\bigcap \mathcal{M}$ consists of a single point q and $\mathcal{M}$ is a local base at q. We will prove:

1.2 Theorem. *A GO-space is metrizable if and only if it has a σ-closed-discrete dense subset and a weak monotone ortho-base.*

That theorem is consistently false for perfect GO-spaces without a σ-closed-discrete dense subset, because if there is a Souslin space, then there is a Souslin space with a weak monotone ortho-base. See (3.2), below.

Recall that a *generalized ordered space* (GO-space) is a triple $(X, \mathcal{T}, <)$ where $\mathcal{T}$ is a Hausdorff topology on X and $<$ is a linear ordering of X such that $\mathcal{T}$ has a base consisting of open, convex sets. In case $\mathcal{T}$ is the usual open interval topology of the ordering $<$, we say that $(X, \mathcal{T}, <)$ is a *linearly ordered topological space* (LOTS). It is known that the GO-spaces are precisely the topological subspaces of LOTS [L].

In what follows, it will be important to distinguish between *relatively discrete* subsets of a space (i.e., those subsets that contain no limit points of themselves) and subsets that are both closed and discrete (i.e., subsets that have no limit points in the entire space). This leads us to use the somewhat cumbersome term "σ-closed-discrete" to describe a set that is the countable union of subspaces that are both closed and discrete.

We want to thank a group of colleagues for helpful conversations concerning Section 3 of this paper. At the George Mason Topology and Dynamics Conference

in March, 1998, Dennis Burke, Joe Mashburn, Steven Purisch, and Adrian Stanley met with the three authors of this paper for an informal seminar on ordered spaces, and the resulting discussions were enlightening.

2. Perfect spaces with σ-closed-discrete dense subsets

As noted above, any GO-space with a σ-closed-discrete dense subset is perfect, while the converse is consistently false, and it not known whether there is a ZFC example of a perfect GO-space without a σ-closed-discrete dense subset. The goal of this section is to characterize (in ZFC) those perfect GO-spaces that *do* have a σ-closed-discrete dense set.

2.1 Theorem. *Let X be a perfect GO-space. Then the following are equivalent:*

a) *X has a σ-closed-discrete dense subset;*
b) *there are open covers $\mathcal{G}_n$ of X such that for each $p \in X$, the set $\bigcap\{St(p, \mathcal{G}_n) : n \geq 1\}$ has at most two points;*
c) *there are open covers $\mathcal{G}_n$ of X such that for each $p \in X$ the set $\bigcap\{St(p, \mathcal{G}_n) : n \geq 1\}$ is countable;*
d) *there are open covers $\mathcal{G}_n$ of X such that for each $p \in X$ the set $\bigcap\{St(p, \mathcal{G}_n) : n \geq 1\}$ is a separable subspace of X;*
e) *$X = X_1 \cup X_2$ where each X_i has a G_δ-diagonal for its relative topology;*
f) *$X = \bigcup\{X_n : n \geq 1\}$ where each X_n has a G_δ-diagonal for its relative topology;*
g) *X has a dense subspace $Y = \bigcup\{Y_n : n \geq 1\}$ where each Y_n has a G_δ-diagonal for its relative topology;*
h) *there exist a perfect, paracompact GO-space Y with a G_δ-diagonal and a closed continuous mapping $f : X \to Y$ such that $|f^{-1}[y]| \leq 2$ for each $y \in Y$;*
i) *there exist a metrizable GO-space Y and a continuous mapping $g : X \to Y$ such that for each $y \in Y$, $|g^{-1}[y]| \leq 2$;*
j) *there exist a topological space Z with a G_δ-diagonal and a continuous $g : X \to Z$ such that $g^{-1}[z]$ is separable for each $z \in Z$.*

Outline of Proof. We will use a sequence of lemmas to show that

Step 1: a) $\Rightarrow$ b) $\Rightarrow$ c) $\Rightarrow$ d) $\Rightarrow$ g) $\Rightarrow$ a);
Step 2: a) $\Rightarrow$ b) $\Rightarrow$ e) $\Rightarrow$ f) $\Rightarrow$ g) $\Rightarrow$ a);
Step 3: b) $\Rightarrow$ h) $\Rightarrow$ i) $\Rightarrow$ j) $\Rightarrow$ b).

Because it is immediate that b) $\Rightarrow$ c) $\Rightarrow$ d), e) $\Rightarrow$ f) $\Rightarrow$ g) and i) $\Rightarrow$ j), it will be enough to show:

a) $\Rightarrow$ b) in Lemma 2.2;
d) $\Rightarrow$ g) in Lemma 2.3;
g) $\Rightarrow$ a) in Lemma 2.4; and
b) $\Rightarrow$ e) in Lemma 2.3;
b) $\Rightarrow$ h) $\Rightarrow$ i) $\Rightarrow$ j) $\Rightarrow$ b) in Lemma 2.5.

2.2 Lemma. *In Theorem 2.1,* a) $\Rightarrow$ b).

Proof. Suppose that $D = \bigcup\{D_n : n \geq 1\}$ is dense in X, where each D_n is a closed discrete subset of X. We may assume that $D_n \subseteq D_{n+1}$ for each n. Because X is collectionwise normal, for each $x \in D_n$ there is a convex open set $U(x, n)$ such that

1) the collection $\{U(x, n) : x \in D_n\}$ is pairwise disjoint and $U(x, n) \cap D_n = \{x\}$.

Because X is perfect, X is first countable, so that we may assume

2) if $x \in D_n$, then $\{U(x, n + k) : k \geq 1\}$ is a local base at x.

For each $p \in X - D_n$, let $V(p,n)$ be the convex component of $X - D_n$ to which p belongs and let $\mathcal{H}_n = \{U(x,n) : x \in D_n\} \cup \{V(p,n) : p \in X - D_n\}$. Observe that

3) if $p \in D_n$, then $St(p, \mathcal{H}_n) = U(p,n)$, and if $p \in X - D_n$, then either $St(p, \mathcal{H}_n) = V(p,n)$ or else there is a unique $x \in D_n$ such that $p \in U(x,n)$ and $St(p, \mathcal{H}_n) = V(p,n) \cup U(x,n)$.

Let $S(p) = \bigcap\{St(p, \mathcal{H}_n) : n \geq 1\}$ and note that

4) $S(p)$ is a convex subset of X.

Observe that if $p \in D$, then, for large enough values of n, $St(p, \mathcal{H}_n) = U(p,n)$ by 3). Therefore assertion 2) yields

5) if $p \in D$, then $S(p) = \{p\}$.

We claim that

6) for each $p \in X$, the set $S(p)$ is finite.

To verify 6), recall from 5) that if $p \in D$, then $S(p)$ is a singleton. Hence suppose $p \in X - D$. If $S(p)$ is not finite, then we may choose points a, b, c, d in $S(p)$ with either $p < a < b < c < d$ or $d < c < b < a < p$. The cases are similar, so we consider only the first. Because $D_n \subseteq D_{n+1}$ we can find n so that D_n meets each of the non-empty open sets $]p, b[$ and $]b, d[$. Choose $y \in D_n \cap]p, b[$ and $z \in D_n \cap]b, d[$. Then $V(p,n) \subseteq]\leftarrow, b[$ so that $d \in St(p, \mathcal{H}_n)$ combines with 3) to yield $St(p, \mathcal{H}_n) = V(p,n) \cup U(x,n)$ for some unique $x \in D_n$ with $\{p, d\} \subseteq U(x,n)$. But then convexity of $U(p,n)$ gives $y, z \in [p, d] \subseteq U(x,n)$ so that 1) yields $\{y, z\} \subseteq U(x,n) \cap D_n \subseteq \{x\}$ which is impossible. That contradiction establishes assertion 6).

Recall that X is perfect and that any perfect GO space is paracompact [L]. Therefore, for each n we can find open covers $\mathcal{G}_n$ of X such that

7) each member of $\mathcal{G}_n$ is convex, $\mathcal{G}_n$ refines $\mathcal{H}_n$, and $\mathcal{G}_{n+1}$ star refines $\mathcal{G}_n$.

Then for each n and $p \in X$, $St(p, \mathcal{G}_n) \subseteq St(p, \mathcal{H}_n)$ so that the set $T(p) = \bigcap\{St(p, \mathcal{H}_n) : n \geq 1\}$ is a subset of $S(p)$. From 6) we conclude

8) for each $p \in X$, $T(p)$ is a finite convex subset of X.

We claim that

9) if $q \in T(p)$, then $T(q) = T(p)$.

To verify 9), suppose $q \in T(p)$ and $x \in T(q)$. Fix $n \geq 1$. Because $q \in T(p) \subseteq St(p, \mathcal{G}_{n+1})$, some $G' \in \mathcal{G}_n$ has $\{p, q\} \subseteq G'$. Because $x \in S(q)$, some $G'' \in \mathcal{G}_{n+1}$ has $\{x, q\} \subseteq G''$. Because $\mathcal{G}_{n+1}$ star refines $\mathcal{G}_n$, we know that some $G \in \mathcal{G}_n$ has $G' \cup G'' \subseteq St(q, \mathcal{G}_{n+1}) \subseteq G$. But then $\{p, q, x\} \subseteq G$ so that $x \in St(p, \mathcal{G}_n)$. Thus, $x \in T(p)$ so that $T(q) \subseteq T(p)$. A similar argument gives $T(p) \subseteq T(q)$.

Next we claim that

10) for each $p \in X$, $|T(p)| \leq 2$.

To prove 10), recall from 8) that each $T(p)$ is a finite convex subset of X. If some p has $|T(p)| \geq 3$, then $T(p)$ must contain an isolated point q of X. From 9), $T(p) = T(q)$. Being isolated, $q \in D$ so that 5) yields $T(q) \subseteq S(q) = \{q\}$ which is impossible because $|T(q)| = |T(p)| \geq 3$. □

2.3 Lemma. *In Theorem 2.1,* d) ⇒ g) *and* b) ⇒ e).

Proof. First consider d) ⇒ g). Because X is a perfect GO-space, X is paracompact [L]. As in the proof of Lemma 2.2, we can refine the open covers $\mathcal{G}_n$ given by d) to obtain open covers $\mathcal{H}_n$ such that

1) for each $n \geq 1$, $\mathcal{H}_n$ is a convex open cover of X that refines $\mathcal{G}_n$, and $\mathcal{H}_{n+1}$ star-refines $\mathcal{H}_n$.

Then $\bigcap\{St(p, \mathcal{H}_n) : n \geq 1\} \subseteq \bigcap\{St(p, \mathcal{G}_n) : n \geq 1\}$ for each $p \in X$ so that, because any subspace of a separable GO-space is again separable, we have

2) for each $p \in X$, the set $S(p) = \bigcap\{St(p, \mathcal{H}_n) : n \geq 1\}$ is a separable subset of X.

As in the proof of Lemma 2.2 we have

3) if $q \in S(p)$, then $S(q) = S(p)$

so that the collection $\mathcal{S} = \{S(p) : p \in X\}$ is a partition of X. Choose an indexing $\mathcal{S} = \{S(p_\alpha) : \alpha \in A\}$ such that

4) if $\alpha \neq \beta$ are in A, then $S(p_\alpha) \neq S(p_\beta)$.

Then for each $\alpha \in A$, fix a countable set $T(p_\alpha)$ that is dense in $S(p_\alpha)$, and fix an indexing $T(p_\alpha) = \{t(\alpha, n) : n \geq 1\}$, with repetitions being allowed. Let $Y_n = \{t(\alpha, n) : \alpha \in A\}$ and note that

5) for each $\alpha \in A$, $T(p_\alpha) \cap Y_n = \{t(\alpha, n)\}$.

Let $Y = \bigcup\{Y_n : n \geq 1\}$. Then Y is a dense subspace of X. Fix $n \geq 1$ and let $\mathcal{L}_m = \{H \cap Y_n : H \in \mathcal{H}_m\}$. Then each $\mathcal{L}_m$ is a relatively open cover of Y_n and for each $p \in Y_n$ we have $\bigcap\{St(p, \mathcal{L}_m) : m \geq 1\} = \{p\}$. Thus, Y_n has a G_δ-diagonal for its relative topology, and we see that d) $\Rightarrow$ g).

The proof that b) $\Rightarrow$ e) in Theorem 2.1 is analogous. Suppose that $\langle \mathcal{G}_n \rangle$ is the sequence of open covers given by b). As in the first part of this proof, use paracompactness of X to refine $\mathcal{G}_n$ to a convex open cover $\mathcal{H}_n$ in such a way that $\mathcal{H}_{n+1}$ star-refines $\mathcal{H}_n$. Then the set $T(p) = \bigcap\{St(p, \mathcal{H}_n) : n \geq 1\}$ is a convex subset of X with at most two points, and $\{T(p) : p \in X\}$ is a partition of X. Define $X_1 = \{\min(T(p)) : p \in X\}$ and let let $X_2 = \{\max(T(p)) : p \in X\}$. Then $X = X_1 \cup X_2$, and both X_1 and X_2 have G_δ-diagonals in their relative topologies. □

2.4 Lemma. *In Theorem 2.1,* g) $\Rightarrow$ a).

Proof. Let $Y = \bigcup\{Y_n : n \geq 1\}$ be the dense subspace given by g), where each Y_n has a G_δ-diagonal for its relative topology. According to [BLP], because the GO-space Y_n has a G_δ-diagonal, Y_n contains a dense metrizable subspace Z_n. (Alternatively, use the set of isolated points of the GO-space Y_n and a sigma disjoint open convex refinement of the G_δ-diagonal sequence for Y_n to find a σ-disjoint π-base for Y_n and then invoke H.E. White's theorem [W] that any regular, first-countable space with such a π-base must have a dense metrizable subspace.) Being metrizable, each Z_n has a dense subspace $D_n = \bigcup\{D(n, k) : k \geq 1\}$ where each $D(n, k)$ is discrete in its relative topology. But X is perfect, so that each $D(n, k)$ is a union of countably many closed, discrete subspaces of X. Then $D = \bigcup\{D(n, k) : n \geq 1, k \geq 1\}$ is the required σ-closed-discrete dense subset of X. □

2.5 Lemma. *In Theorem 2.1,* b) $\Rightarrow$ h) $\Rightarrow$ i) $\Rightarrow$ j) $\Rightarrow$ b).

Proof. First, b) $\Rightarrow$ h). In light of b), there is a sequence $\langle \mathcal{G}_n \rangle$ of open covers of X with the property that $\mathcal{G}_{n+1}$ star-refines $\mathcal{G}_n$ and for each $p \in X$, the set $S(p) = \bigcap\{St(p, \mathcal{G}_n) : n \geq 1\}$ has at most two points. As in the proof of Lemma 2.3, the collection $\{S(p) : p \in X\}$ partitions X, and each $S(p)$ is convex. According to [vW, Proposition 1.2.3], if Y is the quotient space that results from collapsing

each $S(p)$ to a single point, then Y, with its natural ordering, is a GO-space and the quotient mapping $f : X \to Y$ is closed and continuous.

For each n and each $G \in \mathcal{G}_n$, let $G^* = \bigcup\{S(p) : p \in X \text{ and } S(p) \subseteq G\}$. Let $\mathcal{L}_n = \{f[G^*] : G \in \mathcal{G}_n\}$. Because f is a closed mapping, each member of $\mathcal{L}_n$ is open in Y. It is easy to check that $\langle \mathcal{L}_n \rangle$ is a G_δ-diagonal sequence of open covers of Y. Because f is continuous and closed, Y is paracompact and perfect.

Next, h) $\Rightarrow$ i). Suppose that we have a perfect mapping $f : X \to Y$ where $(Y, \mathcal{T}, <)$ is a perfect GO-space that has a G_δ-diagonal. But then, according to a result of Przymusinski (see [A, Theorem 2.1]), there is a metrizable GO-topology $\mathcal{S}$ on $(Y, <)$ such that $\mathcal{S} \subseteq \mathcal{T}$. The the mapping $f : X \to (Y, \mathcal{S}, <)$ is the mapping required by assertion i).

Clearly i) $\Rightarrow$ j), so it remains only to prove that j) $\Rightarrow$ b). So suppose that Z is a topological space and that $\langle \mathcal{G}_n \rangle$ is a G_δ-diagonal sequence of open covers of Z, and that $g : X \to Z$ is a continuous mapping such that for each $z \in Z$, the set $g^{-1}[z]$ is separable. Letting $\mathcal{H}_n = \{g^{-1}[G] : G \in \mathcal{G}_n\}$, we obtain open covers of X that satisfy d). Because d) $\Rightarrow$ b) by Step 1 of Theorem 2.1, the proof is complete. $\square$

Theorem 2.1 can be read in a different way: it describes situations in which a perfect GO-space must have a σ-closed-discrete dense subset. For example:

2.6 Corollary. *Suppose X is a perfect GO-space. If X has any of the following properties, then X must have a σ-closed-discrete dense subset:*

a) *X is the union of countably many subspaces, each having a G_δ-diagonal in its relative topology;*
b) *there is a continuous s-mapping from X onto a topological space with a G_δ-diagonal;*
c) *there is a sequence $\langle \mathcal{G}_n \rangle$ of open covers of X such that for each $p \in X$, $\bigcap\{St(p, \mathcal{G}_n) : n \geq 1\}$ is a separable subspace of X.* $\square$

3. Metrization of GO-space with σ-closed-discrete dense sets

There are metrization theorems for GO-spaces with a σ-closed-discrete dense set that are known to be consistently false for GO-spaces that are merely perfect. The purpose of this section is to add a new metrization theorem to that list, using the notion of a *weak monotone ortho-base* defined in the introduction.

3.1 Theorem. *The following properties of a GO-space $(X, \mathcal{T}, <)$ are equivalent:*

a) *$(X, \mathcal{T})$ is metrizable;*
b) *X has a σ-closed-discrete dense subset D such that the set $\{x \in X : [x, \to[$ is open or $]\leftarrow, x]$ is open$\}$ is a subset of D;*
c) *X has a σ-closed-discrete dense subset and a point-countable base;*
d) *X has a σ-closed-discrete dense subset and a weak monotone ortho-base.*

Proof. The equivalence of a) and b) is due to Faber [F] and the equivalence of a) and c) was proved in [BL1]. Because any metric space satisfies d), it will be enough to prove that d) implies a) for GO-spaces. Suppose $\mathcal{B}$ is a weakly monotone ortho-base for X and D is a σ-closed discrete dense subset of X. (To help readers understand why certain steps in the following argument are necessary, we note that members of $\mathcal{B}$ might not be convex sets.)

Let I be the set of all isolated points of X. Let $R = \{x \in X - I : [x, \to[\text{ is open in } X\}$ and let $L = \{x \in X - I :]\leftarrow, x] \text{ is open in } X\}$. In light of Faber's metrization

theorem b), it will be enough to prove that R and L are each σ-closed-discrete, because then $D \cup R \cup L$ will be the σ-closed-discrete dense set required in b), where D is the dense set given in d).

We begin with the given dense set $D = \bigcup\{D_n : n \geq 1\}$ where each D_n is closed and discrete. We may assume that $D_n \subseteq D_{n+1}$ for each n. Because X has a σ-closed-discrete dense set, X is perfect and hence first-countable. Let $X_n = X - D_n$ and let $\mathcal{B}_n = \{B \in \mathcal{B} : B$ is contained in some convex component of $X_n\}$. Note that $\mathcal{B}_n$ is a base at every point of X_n. Now suppose, for contradiction, that

(*) the set R is not σ-closed-discrete.

Let $S_n = \{p \in R \cap X_n :$ for each $B \in \mathcal{B}_n$, if $p \in B$ then $conv(p, B) \subset [p, \rightarrow [\}$, where for any set U, $conv(p, U)$ denotes the convex component of U that contains p. We claim that each S_n is relatively discrete. If not, then there is a sequence $\langle p_k \rangle$ in S_n that converges to a point q of S_n. Because $q \in R$, the set $[q, \rightarrow [$ is open, so we may assume that $p_k > p_{k+1}$ for each k. Because $q \in S_n \subseteq X_n$, we may choose $B_q \in \mathcal{B}$ with $q \in B_q$, and where B_q is a subset of $conv(q, X_n)$. Hence $B_q \in \mathcal{B}_n$. Next, because $q \in S_n$, we know that $conv(q, B_q) \subseteq [q, \rightarrow [$. Choose k so large that $p_k \in conv(q, B_q)$. Because $p_k > q$, we are forced to conclude that $conv(p_k, B_q) = conv(q, B_q) \nsubseteq [p_k, \rightarrow [$, contradicting $p_k \in S_n$. Therefore, each S_n is relatively discrete. Because X is perfect, each S_n is σ-closed-discrete, whence so is $S = \bigcup\{S_n : n \geq 1\}$.

Because $D \cup S$ is σ-closed-discrete while, by (*), R is not, we may choose a point $r \in R - (D \cup S)$. Let $n(1) = 1$ and note that $r \in X_{n(1)}$ and $r \notin S_{n(1)}$. Hence there is a $B_{n(1)} \in \mathcal{B}_{n(1)}$ with $r \in B_{n(1)}$ such that $conv(r, B_{n(1)})$ contains points on both sides of r. For induction hypothesis, suppose $1 = n(1) < \cdots < n(k)$ and we have chosen $B_{n(j)} \in \mathcal{B}_{n(j)}$ with

(i) $r \in B_{n(j)}$;
(ii) $B_{n(j+1)} \subseteq conv(r, B_{n(j)})$ whenever $1 \leq j < k$;
(iii) $conv(r, B_{n(j)})$ contains points on both sides of r for $1 \leq j \leq k$.

Consider $conv(r, B_{n(k)})$. Both $conv(r, B_{n(k)}) \cap] \leftarrow, r[$ and $conv(r, B_{n(k)}) \cap]r, \rightarrow [$ are nonempty open sets, so we can find $n(k+1) > n(k)$ such that $D_{n(k+1)}$ meets both sets. Choose $d \in D_{n(k+1)} \cap conv(r, B_{n(k)}) \cap] \leftarrow, r[$ and $e \in D_{n(k+1)} \cap conv(r, B_{n(k)}) \cap]r, \rightarrow [$. Then $conv(r, X_{n(k+1)})$ is an open neighborhood of r that is contained in $]d, e[\subseteq B_{n(k)}$. Because $r \notin S_{n(k+1)}$, we can find $B_{n(k+1)} \in \mathcal{B}_{n(k+1)}$ such that $r \in B_{n(k+1)}$ and such that $conv(r, B_{n(k+1)})$ contains points on both sides of r. Because $B_{n(k+1)} \in \mathcal{B}_{n(k+1)}$, we know that $B_{n(k+1)}$ is contained in some convex component of $X_{n(k+1)}$. Because $r \in B_{n(k+1)}$, we have $B_{n(k+1)} \subseteq conv(r, X_{n(k+1)}) \subseteq]d, e[\subseteq B_{n(k)}$. Thus the induction continues, giving $B_{n(k)}$ for each k.

Because $r \in R$, we know that r is not an isolated point of X. Hence r has no immediate successor in X. Consequently, some sequence in the dense set D converges monotonically to r from above. Therefore, the set $\bigcap\{conv(r, B_{n(k)}) : k \geq 1\}$ cannot contain any point of $]r, \rightarrow [$ so that $r \in \bigcap\{conv(r, B_{n(k)}) : k \geq 1\} \subseteq] \leftarrow, r]$. However, as noted in the inductive construction, $B_{n(k+1)} \subseteq conv(r, B_{n(k)}) \subseteq B_{n(k)}$ so that $r \in \bigcap\{B_{n(k)} : k \geq 1\} \subseteq] \leftarrow, r]$. Thus, $\bigcap\{B_{n(k)} : k \geq 1\}$ is not open, because $r \in R$ yields $r \notin L \cup I$. Because $\mathcal{B}$ is a weak monotone ortho-base for X, the collection $\{B_{n(k)} : k \geq 1\}$ must be a local base at the point r, and that is impossible because each $B_{n(k)}$ contains points on both sides of r while the open set $[r, \rightarrow [$ does not. That contradiction shows that the set R is σ-closed-discrete.

Analogously, the set L is also σ-closed-discrete. According to Faber's theorem, X is metrizable. □

3.2 Example. If there is a Souslin space, then there is a Souslin space with a weak monotone ortho-base.

Proof. Starting with any Souslin space, construct a compact connected Souslin space with no separable nondegenerate intervals, and discard the end points of that space to obtain a LOTS X. In X, construct a tree (with respect to $\subseteq$) $\mathcal{J}$ of closed non-degenerate subintervals of X in such a way that:

(a) if I is a closed interval at level α of $\mathcal{J}$, then the set of intervals in level $\alpha+1$ that are contained in I is ordered like the set of all integers by the natural ordering inherited from X;
(b) if $\mathcal{L}$ is any subcollection of $\mathcal{J}$, then $\bigcap\mathcal{L}$ is either empty, or is a point, or is a member of $\mathcal{J}$;
(c) if I belongs to level α of $\mathcal{J}$, then the interior of I is the union of all members of level $\alpha+1$ that are contained in I;
(d) if I and J are distinct members of $\mathcal{J}$, then either $I\cap J=\emptyset$ or else one of I and J is contained in the interior of the other.

Let $\mathcal{J}_\alpha$ be the αth level of the tree $\mathcal{J}$. For each limit ordinal λ, let E_λ be the set of all endpoints of non-degenerate members of $\mathcal{J}_\lambda$ and let $E=\bigcup\{E_\lambda : \lambda \text{ is a limit ordinal}\}$. Let Y be the set of all points $p\in X-E$ such that some nested family $\mathcal{L}\subseteq\mathcal{J}$ has $\{p\}=\bigcap\mathcal{L}$. Then Y is a dense subspace of X, so Y is a LOTS in its relative topology and order, and Y is perfect because X is. Finally, $\{\mathrm{Int}_X(J)\cap Y : J\in\mathcal{J}\}$ is a weak monotone ortho-base for Y. □

3.3 *Remark.* Even for first countable LOTS, the existence of a weak monotone ortho-base is very different from the existence of a point-countable base. Burke and Purisch pointed out that the usual space $\Omega=[0,\omega_1[$ of countable ordinals has a weak monotone orthobase, and Ω does not have a point-countable base. For an example of a paracompact, first countable LOTS that has a weak monotone ortho-base but not a point-countable base, one can use the extended Big Bush appearing in Section 3 of [BL2].

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