THE COHOMOLOGY RINGS OF THE ORBIT SPACES
OF FREE TRANSFORMATION GROUPS
OF THE PRODUCT OF TWO SPHERES

RONALD M. DOTZEL, TEJ B. SINGH, AND SATYA P. TRIPATHI

(Communicated by Ralph Cohen)

Abstract. Let $G = \mathbb{Z}_p$, $p$ a prime (resp. $S^1$), act freely on a finistic space $X$ with mod $p$ (resp. rational) cohomology ring isomorphic to that of $S^m \times S^n$. In this paper we determine the possible cohomology algebra of the orbit space $X/G$.

1. Introduction

Let $G = \mathbb{Z}_p$, $p$ a prime (resp. $S^1$, the circle group), act on a space $X$ with mod $p$ (resp. rational) cohomology ring isomorphic to that of $S^m \times S^n$; we abbreviate this as $X \simeq_p S^m \times S^n$ (resp. $X \simeq_Q S^m \times S^n$). There are two spaces associated with the transformation group $(G, X)$; viz. the fixed point set $X^G$ and the orbit space $X/G$. The homological nature of $X^G$ has been studied in detail by Adem [1], Bredon [3], Hsiang [4], Su [6] and Tomter [7]. However, to our knowledge, no one has investigated the homological structure of the space $X/G$. We find here the possibilities for the cohomology algebra $H^*(X/G)$ when the action is free. Throughout this paper, we use Čech cohomology with coefficients in the field $\mathbb{F}_p$ of $p$ elements or $\mathbb{Q}$ of rational numbers, unless otherwise indicated. The mod $p$ Bockstein cohomology operation associated with the coefficient sequence $0 \to \mathbb{F}_p \to \mathbb{F}_p \to \mathbb{F}_p \to \cdots$ will be denoted by $\beta$. We prove the following:

Theorem 1. Let $G = \mathbb{Z}_p$, $p$ an odd prime, act freely on a finistic space $X \simeq_p S^m \times S^n$, $0 < m \leq n$, and assume that $H^*(X; \mathbb{Z})$ is of finite type. Then $H^*(X/G; \mathbb{Z}_p)$ is isomorphic to $\mathbb{Z}_p[x, y, z]/\phi(x, y, z)$ as a graded commutative algebra, where $\phi(x, y, z)$ is one of the following graded ideals:

(i) $(x^2, y^{(m+1)/2}, z^2)$, $m$ odd, $\deg x = 1, y = \beta(x), \deg z = n$;
(ii) $(x^2, y^{(m+n+1)/2}, y^{(n-m+1)/2} - ay^{(n+1)/2}, z^2 - by^m)$, $m$ even, $n$ odd, $\deg x = 1, y = \beta(x), \deg z = m, a, b \in \mathbb{Z}_p$, and $a = 0$ necessarily when $n < 2m$;
(iii) $(x^2, y^{(n+1)/2}, z^2 - by^m)$, $n$ odd, $\deg x = 1, y = \beta(x), \deg z = m, b \in \mathbb{Z}_p, b \neq 0$ only when $m$ is even and $2m < n$.

Theorem 2. Let $G = \mathbb{Z}_2$ act freely on a finistic space $X \simeq_2 S^m \times S^n$, $0 < m \leq n$. Then $H^*(X/G; \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2[y, z]/\psi(y, z)$ as a graded algebra, where $\psi(y, z)$ is one of the following graded ideals:

Received by the editors September 4, 1998 and, in revised form, June 3, 1999.
2000 Mathematics Subject Classification. Primary 57S17; Secondary 57S25.
(i) \((y^{m+2}, z^2), \deg y = 1, \deg z = n;\)
(ii) \((y^{m+n+1}, y^{n-m+1}, z, z^2 - ay_mz - by^{2m}), \deg y = 1, \deg z = m, a, b \in \mathbb{Z}_2\) and \(a = 0\) necessarily when \(n < 2m;\)
(iii) \((y^{n+1}, z^2 - ay_mz - by^{2m}), \deg y = 1, \deg z = m, a, b \in \mathbb{Z}_2\) and \(b = 0\) necessarily when \(m = n\) or \(n < 2m.\)

**Theorem 3.** Let \(G = S^1\) act freely on a finitistic space \(X \sim_{Q} S^m \times S^n, 0 < m \leq n.\) Then \(H^*(X/G; Q)\) is isomorphic to \(Q[y, z]/\psi(y, z)\) as a graded algebra, where \(\psi(y, z)\) is one of the following graded ideals:

(i) \((y^{(m+1)/2}, z^2), m \text{ odd, } \deg y = 2, \deg z = n.\)
(ii) \((y^{(m+n+1)/2}, zy^{(n-m+1)/2} - ay_n^{(n+1)/2}, z^2 - by^m), m \text{ even, } \deg y = 2, \deg z = m\) and \(a = 0\) necessarily when \(n < 2m.\)
(iii) \((y^{(n+1)/2}, z^2 - by^m), n \text{ odd, } \deg y = 2, \deg z = m, b \neq 0\) only when \(m\) is even and \(2m < n.\)

The main gadget employed in our proofs is the Leray-Serre spectral sequence of the map \(X \rightarrow X_G \xrightarrow{\pi} B_G,\) which has \(E_2^{k,l} = H^k(B_G; H^l(X))\) as its \(E_2-\text{term and converges to } H^{k+l}(X_G),\) in the sense of Bredon \([2,\text{ VII}, 1.6].\)

2. **Some known results**

Suppose \(G = Z_p,\) \(p\) a prime, acts on a finitistic space \(X \sim_p S^m \times S^n.\) The following facts can be easily deduced.

**Proposition 1.** If \(G\) acts trivially on \(H^*(X)\) and the spectral sequence of the map \(\pi: X_G \rightarrow B_G,\) which has \(E_2^{k,l} = H^k(B_G; H^l(X))\) as its \(E_2-\text{term and converges to } H^{k+l}(X_G),\) in the sense of Bredon \([2,\text{ VII}, 1.6].\)

**Proposition 2.** If \(m\) and \(n\) are even and \(p > 2,\) then \(X_G \neq \Phi [2,\text{ III}, 7.10].\)

**Proposition 3.** If \(H^*(X; Z)\) is of finite type, \(p > 2\) and \(G\) acts nontrivially on \(H^*(X),\) then \(p = 3\) and \(X_G \neq \Phi [6].\)

**Proposition 4.** If \(p = 2\) and \(G\) acts nontrivially on \(H^*(X),\) then \(X_G \neq \Phi\) and \(m = n [2,\text{ VII}, 7.5].\)

We recall that for \(G = Z_p,\)

\[H^*(B_G; Z_p) = \left\{ \begin{array}{ll} Z_p[t], & \deg t = 1, p = 2, \\
Z_p[t] \otimes Z_p[t], & \deg s = 1, t = \beta(s), p > 2, \end{array} \right.\]
and for \(G = S^1,\)

\[H^*(B_G; Q) = Q[t], \quad \deg t = 2.\]

3. **Proofs**

**Proof of Theorem 7.** Since there are no fixed points, it follows from Propositions 1 and 3 that \(m\) and \(n\) cannot both be even and that \(Z_p\) acts trivially on \(H^*(X).\) Hence the Leray-Serre spectral sequence of the map \(\pi: X_G \rightarrow B_G,\) which has \(E_2^{k,l} = H^k(B_G; H^l(X))\) as its \(E_2-\text{term and converges to } H^{k+l}(X_G),\) does not collapse at the \(E_2-\text{term and } E_2^{r,l} = H^k(B_G) \otimes H^l(X).\) Let \(r \geq 2\) be the smallest integer such that \(d_r \neq 0.\) By the multiplicative properties of the spectral sequence, we have \(d_r(1 \otimes v_1) \neq 0\) or \(d_r(1 \otimes v_2) \neq 0.\) Suppose, first, that \(d_r(1 \otimes v_1) \neq 0.\) Then \(r = m + 1\)
There are elements $z_1, z_2$ in $H^1(X_G)$ such that $\iota(z_1) = z_2$ (resp. $z_2 = \iota z_1$) if $m \neq n$ (resp. $m = n$). Then $xz \neq 0 \neq yz$ and $z^2 = 0$. The multiplication

$$y \cup (\cdot) : H^k(X_G) \to H^{k+2}(X_G)$$

is an isomorphism in degrees $k$ such that $0 \leq k < m-2$ and $n < k < m + n - 2$. So

$$H^*(X_G) \cong Z_p[x, y, z]/(x^2, y^{(m+1)/2}, z^2).$$

For $m = 1$, we have

$$H^*(X_G) \cong Z_p[x, z]/(x^2, z^2).$$
Since $G$ acts freely on $X$, $H^*(X_G)$ is isomorphic to $H^*(X/G)$ as a ring and we are in case (1).

Suppose, now, that $d_r(1 \otimes v_1) = 0$ and $d_r(1 \otimes v_2) \neq 0$. We then have either $r = n - m + 1$ and $d_r(1 \otimes v_2) = A \otimes v_1$ or $r = n + 1$ and $d_r(1 \otimes v_2) = A \otimes 1$, $0 \neq A \in H^*(BG)$. In the former case if $n$ were even, then we would have

$$0 = d_r(1 \otimes v_2^2) = A \otimes v_1 v_2 + (-1)^{n(r+1)} A \otimes v_2 v_1 = 2A \otimes v_3 \neq 0.$$ 

Hence $n$ is odd. We then observe that $m$ must be even. Assume the contrary and consider the spectral sequence of the map $\pi$ with coefficients in $\mathbb{Z}$, the ring of integers. Since $H^*(X; \mathbb{Z})$ is finitely generated, it has no $p$-torsion elements; consequently, we have $\tilde{E}_2^{k,l} = H^k(BG; H^l(X; \mathbb{Z})) = 0$, for all $k$ odd. Thus $\tilde{E}_2^{k,l} = 0$, for all $k$ odd and $r \geq 2$. The coefficients homomorphism $q : \mathbb{Z} \to \mathbb{Z}_p$ gives the commutative diagram:

$$
\begin{array}{c}
\tilde{E}_{n-m+1}^{0,n} \\
\downarrow q^* \\
\tilde{E}_{n-m+1}^{0,n} \\
\downarrow q^* \\
\tilde{E}_{n-m+1}^{0,n} \\
\end{array}
\begin{array}{c}
d_{n-m+1} \\
\longrightarrow d_{n-m+1} \\
\longrightarrow d_{n-m+1} \\
\end{array}
\begin{array}{c}
\tilde{E}_{n-m+1}^{n-m+1,m} \\
\tilde{E}_{n-m+1}^{n-m+1,m} \\
\tilde{E}_{n-m+1}^{n-m+1,m} \\
\end{array}
$$

The composition $d_{n-m+1} \circ q^*$ is the trivial homomorphism, for $n - m + 1$ is odd. Since $q^*$ in the left is surjective, the bottom $d_{n-m+1}$ is trivial. But this is not the case; hence our assertion. Thus we must have $m$ even and $n$ odd so that we can write $d_{n-m+1}(1 \otimes v_2) = t(n-m+1)/2 \otimes v_1$. It follows that the differential

$$d_{n-m+1} : E_{n-m+1}^{n,m} \to E_{n-m+1}^{*}$$

is an isomorphism and $d_{n-m+1}(E_{n-m+1}^{*,m}) = 0 = d_{n-m+1}(E_{n-m+1}^{*,m+n})$. So we have $E_{n-m+1}^{k,n} = 0 = E_{n-m+1}^{k+n-m+1,m}$, $E_{n-m+1}^{k,m+n} = E_{n-m+1}^{2,2+n}$ and $E_{n-m+1}^{k,0} = E_{n-m+1}^{2,0}$ for all $k \geq 0$ and $r = n - m + 2$. It is easily seen that the differential

$$d_{m+1} : E_{m+1}^{k,m} \to E_{m+1}^{k+m+1,0}$$

is trivial for $0 \leq k \leq n - m$, since $E_{m+1}^{k,m} = E_{n-m+1}^{2,2+n}$. Because there are no fixed points, the differential

$$d_{n+m+1} : E_{n+m+1}^{0,m+n} \to E_{n+m+1}^{n+m+1,0}$$

must be nontrivial so that we can assume $d_{n+m+1}(1 \otimes v_3) = t(n+m+1)/2 \otimes 1$. Then, the differential

$$d_{n+m+1} : E_{n+m+1}^{n,m+n} \to E_{n+m+1}^{n,0}$$

is an isomorphism. Consequently, $E_\infty = E_{m+n+2}$ and the only nonzero vector spaces in the $E_\infty$-term are $E_{k,m}^{\infty} = \mathbb{Z}_p$ for $0 \leq k \leq n - m$ and $E_{k,0}^{\infty} = \mathbb{Z}_p$ for $0 \leq k \leq m + n$. It follows that

$$H^k(X_G) = \begin{cases} 
\mathbb{Z}_p, & 0 \leq k < m \text{ and } n < k \leq m + n; \\
\mathbb{Z}_p \oplus \mathbb{Z}_p, & m \leq k \leq n; \\
0, & m + n < k.
\end{cases}$$

We note that $1 \otimes v_1 \in E_2^{0,m}$ is a permanent cocycle and determines an element $w \in E_\infty$. We have

$$\text{Tot} E^{*,*}_\infty \cong \mathbb{Z}_p[x,y,w]/(x^2, w^2, y^{(m+n+1)/2}, y^{(n-m+1)/2}w)$$
as graded commutative algebras, where \( x \) and \( y \) satisfy \( \pi^*(s) = x \) and \( \pi^*(t) = y \). The multiplication

\[
y \cup (\cdot) : H^k(X_G) \to H^{k+2}(X_G)
\]

is an isomorphism in degrees \( k \) for \( 0 \leq k \leq n - 2 \) and \( n < k \leq m + n - 2 \). We choose an element \( z \in H^m(X/G) \) such that \( \iota^*(z) = v_1 \). Then \( y^*z \) and \( y^{(m+2r)/2} \) are linearly independent over \( Z_p \) for \( r \leq (n - m - 1)/2 \). It is possible to change \( z \) suitably so that \( y^{(n-m+1)/2}z = 0 \) and \( z^2 = b y^m \), \( b \in Z_p \), when \( n < 2m \) and \( y^{(n-m+1)/2}z = ay^{(n+1)/2} \) and \( z^2 = b y^m \), \( a,b \in Z_p \), when \( 2m < n \). Therefore,

\[
H^*(X_G) \cong Z_p[x,y,z]/\langle x^2, y^{(m+1)/2}, y^{(n-m+1)/2}z - ay^{(n+1)/2}, z^2 - by^m \rangle
\]

and we are in case (ii).

Finally, consider the possibility \( r = n + 1 \), \( d_r(1 \otimes v_1) = 0 \) and \( d_r(1 \otimes v_2) = A \otimes 1 \), \( 0 \neq A \in H^*(B_G) \). Then \( n \) must be odd and we can set \( d_{n+1}(1 \otimes v_2) = 0 \). So \( d_{n+1}(1 \otimes v_3) = \pm t^{(n+1)/2} \otimes v_1 \); consequently the differentials

\[
d_{n+1} : E_{n+1}^{k,n} \to E_{n+1}^{k+n+1,0},
\]

\[
d_{n+1} : E_{n+1}^{k,n+m} \to E_{n+1}^{k+n+1,m}
\]

are isomorphisms. We obtain \( E_{\infty} = E_{n+2} \) and the only nonzero vector spaces in the \( E_{\infty} \)-term are \( E_{\infty}^{k,m} = Z_p = E_{\infty}^{*,0} \) for \( 0 \leq k \leq n \). Thus

\[
H^k(X_G) = \begin{cases} 
Z_p, & 0 \leq k < m \text{ and } n < k \leq m + n; \\
Z_p \oplus Z_p, & m \leq k \leq n; \\
0, & m + n < k.
\end{cases}
\]

We note that \( 1 \otimes v_1 \in E_{n+2}^{0,m} \) is, again, a permanent cocycle and gives an element \( w \in E_{n+2}^{0,m} \). Choosing \( x \in H^1(X_G) \) and \( y \in H^2(X_G) \) such that \( \pi^*(s) = x \) and \( \pi^*(t) = y \), we obtain

\[
\text{Tot } E^*_{\infty} = Z_p[x,y,w]/\langle x^2, y^{(n+1)/2}, w^2 \rangle
\]
as graded commutative algebras. Now, we choose \( z \in H^m(X_G) \) such that \( \iota^*(z) = v_1 \). Then \( z \) represents \( w \) and the multiplication

\[
y \cup (\cdot) : H^k(X_G) \to H^{k+2}(X_G)
\]

is an isomorphism for \( 0 \leq k \leq n - 2 \) and \( n < k \leq m + n - 2 \), so that \( y^2 z \neq 0 \) for \( r \leq (n - 1)/2 \). If \( m \) is even and \( n < 2m \), then \( z^2 = by^m/2 \), \( b \in Z_p \). We replace \( z \) by \( z - (b/2)y^{m/2} \) and have \( z^2 = 0 \). If \( m \) is even and \( 2m < n \), then \( z^2 = by^m/2 + cy^m \), \( b, c \in Z_p \). Replacing \( z \) by \( z - (b/2)y^{m/2} \) we obtain the relation \( z^2 = b'y^m \). Thus

\[
H^*(X_G) \cong Z_p[x,y,z]/\langle x^2, y^{(n+1)/2}, z^2 - by^m \rangle
\]
as a graded commutative algebra and we obtain case (iii). This completes the proof of the theorem.

**Proof of Theorem** The proof is analogous to that of Theorem [11] we describe it rather briefly. The spectral sequence of the map \( \pi \) is nontrivial. Let \( r \geq 2 \) be the least integer such that \( d_r \neq 0 \). Then, we must have \( d_r(1 \otimes v_1) \neq 0 \) or
$d_r (1 \otimes v_2) \neq 0$. Suppose that $r = m + 1, d_{m+1} (1 \otimes v_1) = t^{m+1} \otimes 1$ and $m \neq n$. Then $d_{m+1} (1 \otimes v_2) = 0, d_{m+1} (1 \otimes v_3) = t^{m+1} \otimes v_2$ so that $E_\infty = E_{m+2}$ and we obtain

$$H^k(X_G) = \begin{cases} Z_2, & 0 \leq k \leq m \text{ and } n \leq k \leq m + n; \\ 0, & \text{otherwise.} \end{cases}$$

The element $1 \otimes v_4 \in E_2^{0,n}$, being a permanent cocycle, yields an element $w \in E_2^{0,n}$. So, the total complex $\text{Tot} E_\infty$ is the graded algebra $Z_2[y, w]/(y^1, w^2)$, where $y = \pi^* (1), t \in H^1(B_G)$. Choose $z \in H^n(X_G)$ such that $t^*(z) = v_2$. Then $z$ determines $w$ and satisfies $z^2 = 0$. Since the multiplication

$$y \cup (\cdot) : H^k(X_G) \to H^{k+1}(X_G)$$

is an isomorphism for $0 \leq k < m$ and $n \leq k < m + n$, we have $y^r z \neq 0, 1 \leq r \leq m$. Therefore

$$H^*(X_G) \cong Z_2[y, z]/(y^{m+1}, z^2).$$

As the action of $G$ is free, $H^*(X_G) \cong H^*(X/G)$ and we have the case (i). If $m = n, d_{m+1} (1 \otimes v_1) = t^{m+1} \otimes 1$ and $d_{m+1} (1 \otimes v_2) = c t^{m+1} \otimes 1$, $c \in Z_2$, then $d_{m+1} (1 \otimes v_3) = t^{m+1} \otimes (v_2 + cv_1)$. We obtain $E_\infty = E_{m+2}$ and

$$H^k(X_G) = \begin{cases} Z_2, & 0 \leq k \leq 2m \text{ and } k \neq m; \\ Z_2 \oplus Z_2, & k = m; \\ 0, & 2m < k. \end{cases}$$

In this case, the element $1 \otimes (v_2 + cv_1) \in E_2^{0,m}$ is a permanent cocycle and determines an element $w \in E_2^{0,m}$. Let $y \in H^1(X_G)$ and $z \in H^m(X_G)$ be such that $\pi^*(1) = y$ and $t^*(z) = v_2 + cv_1$. Then $z$ represents $w$ in $E_2^{0,m}$ and satisfies $z^2 = ay^m z$ for some $a \in Z_2$. The element $yz$ represents $0 \neq tw \in E_\infty^{1,m}$ and the multiplication

$$y \cup (\cdot) : H^k(X_G) \to H^{k+1}(X_G)$$

is an isomorphism for $m < k < 2m$ so that $y^r z \neq 0, 1 \leq r \leq m$. Thus

$$H^*(X_G) \cong Z_2[y, z]/(y^{m+1}, z^2 - ay^m z)$$

and we are in case (iii) with $b = 0$.

Next, when $d_r (1 \otimes v_1) = 0, r = n - m + 1$ and $d_r (1 \otimes v_2) = t^r \otimes v_1$, then $d_r (1 \otimes v) = Q$. So, we have $E_{r+1}^{k,m+n} = E_2^{k,m+n}, E_{r+1}^{k,0} = E_2^{k,0}$, for $k \geq 0$, and $E_{r+1}^{k,m}$ is trivial for $k > n - m$ and $E_2^{k,m}$ for $k \leq n - m$. It is easily seen that the differential

$$d_{m+1} : E_\infty^{k,m} \to E_\infty^{k,m+1,0}$$

is also trivial for $0 \leq k \leq n - m$. Then the differential

$$d_{n+m+1} : E_\infty^{k,m} \to E_\infty^{k,m+n+1,0}$$

must be an isomorphism for all $k$, because the action of $G$ on $X$ is free. Thus, the only nonzero vector spaces in the $E_\infty$-term are $E_\infty^{k,m} = Z_2$ for $0 \leq k \leq n - m$ and $E_\infty^{k,0} = Z_2$, for $0 \leq k \leq m + n$. Consequently

$$H^k(X_G) = \begin{cases} Z_2, & 0 \leq k < m \text{ and } n < k \leq m + n; \\ Z_2 \oplus Z_2, & m \leq k \leq n; \\ 0, & m + n < k. \end{cases}$$
The element \( 1 \otimes v_1 \in E^{0,m}_2 \) is a permanent cocycle and gives an element \( w \in E^{0,m}_\infty \). We obtain

\[
\text{Tot } E^{*,*}_\infty \cong Z_2[y, w]/(y^{m+n+1}, w^2, y^{n-m+1}w)
\]
as graded algebras. The multiplication by \( \pi^*(t) = y \),

\[
y \cup (\cdot) : H^k(X_G) \to H^{k+1}(X_G),
\]
is an isomorphism for \( 0 \leq k \leq n - 1 \) and \( n < k < m + n \). Since the composition \( \iota^*\pi^* \) is trivial in positive degrees, it is possible to choose \( z \in H^m(X_G) \) such that \( \iota^*(z) = v_1, y^{n-m+1}z = 0 \) and \( z^2 = ay^mz + by^{2m} \), where \( a = 0 \) when \( 2m > n \). Thus we have

\[
H^*(X_G) \cong Z_2[y, z]/(y^{m+n+1}, y^{n-m+1}z, z^2 - ay^mz - by^{2m})
\]
as graded algebras and we are in case (ii).

Finally, consider the case \( d_r(1 \otimes v_1) = 0, r = n + 1 \) and \( d_{n+1}(1 \otimes v_2) = t^{n+1} \). Then \( d_{n+1}(1 \otimes v_3) = t^{n+1} \otimes v_1 \). So the differentials

\[
d_{n+1} : E^{k,n}_{n+1} \to E^{k+1,n+1}_{n+1}, \quad \text{and}
\]

\[
d_{n+1} : E^{k,m+n}_{n+1} \to E^{k,m+n+1}_{n+1}
\]
are isomorphisms. We have \( E^\infty_\infty = E^\infty_{n+2} \) and the only nonzero vector spaces in the \( E^\infty_\infty \)-term are \( E^{k,m}_{\infty} = Z_2 = E^{k,0} \), for \( 0 \leq k \leq n \). Therefore

\[
H^k(X_G) = \begin{cases} 
Z_2, & 0 \leq k < m \text{ and } n < k \leq m + n; \\
Z_2 \oplus Z_2, & m \leq k \leq n; \\
0, & m + n < k.
\end{cases}
\]

Taking \( y \in H^1(X_G) \) and \( z \in H^m(X_G) \) with \( \pi^*(t) = y \) and \( \iota^*(z) = v_1 \), it can be easily seen that

\[
H^*(X_G) \cong Z_2[y, z]/(y^{n+1}, z^2 - ay^mz - by^{2m}),
\]
as graded algebras, where \( b = 0 \), if \( n < 2m \). This completes the proof. \( \square \)

**Proof of Theorem** Though arguments given herein are different, the technique of the proof remains the same. Since there are no fixed points, the spectral sequence of the map \( \pi : X_G \to B_G \) does not collapse at the \( E_2 \)-term and \( E^{k,l}_2 = H^k(B_G) \otimes H^l(X) \), for the action on \( H^*(X) \) is trivial. Let \( r \geq 2 \) be the integer such that \( d_r \neq 0 \). By the multiplicative properties of the spectral sequence, we have \( d_r(1 \otimes v_1) \neq 0 \) or \( d_r(1 \otimes v_2) \neq 0 \).

First, suppose that \( d_r(1 \otimes v_1) \neq 0 \). Then \( r = m + 1 \) where \( m \) is odd. We can write \( d_r(1 \otimes v_1) = at^{(m+1)/2} \otimes 1, 0 \neq a \in Q \). If \( n = 2m \), \( d_{m+1}(1 \otimes v_2) = bt^{(m+1)/2} \otimes v_1 \), then

\[
0 = d_{m+1}(1 \otimes v_2^2) = 2bt^{(m+1)/2} \otimes v_1 v_2 \neq 0,
\]
a contradiction. Therefore, either \( d_{m+1}(1 \otimes v_2) = 0 \) or \( m = n, d_{m+1}(1 \otimes v_2) = bt^{m+1} \otimes 1, a \neq b \in Q \). In the first case, \( d_{m+1}(1 \otimes v_1 v_2) = ad^{(m+1)/2} \otimes v_2 \); so the differentials

\[
d_{m+1} : E^{k,m}_{m+1} \to E^{k+1,m+1}_{m+1}, \quad \text{and}
\]

\[
d_{m+1} : E^{k,m+n}_{m+1} \to E^{k,m+n+1}_{m+1}
\]
are isomorphisms and we have $E_\infty = E_{m+2}$. Thus, the only nonzero vector spaces in the $E_\infty$-term are $E^{k,n}_\infty = Q = E^{k,0}_\infty$, $k$ even, $0 \leq k \leq m - 1$. We have

$$H^k(X_G) = \begin{cases} Q, & 0 \leq k \leq m - 1, \ k \text{ even}; \ n \leq k \leq n + m - 1, \ k - n \text{ even}; \\ 0, & \text{otherwise}. \end{cases}$$

In this case, $d_{m+1}(1 \otimes v_2) = bq^{(m+1)/2} \otimes 1$, $b \neq 0$, $d_{m+1}(1 \otimes v_1v_2) = \ell^{(m+1)/2} \otimes (av_2 - bv_1)$. So the differential

$$d_{m+1} : E^{k,m}_{m+1} \to E^{k+m+1,0}_{m+1}$$

is surjective, with $\ker(d_{m+1})$ generated by $\xi_k \otimes (av_2 - bv_1)$, $\xi_k$ is the generator of $H^k(B_G)$ and the differential

$$d_{m+1} : E^{k,2m}_{m+1} \to E^{k+m+1,m}_{m+1}$$

is injective with $im(d_{m+1})$ generated by $\xi_{k+m+1} \otimes (av_2 - bv_1)$. Consequently, $E_\infty = E_{m+2}$ and the nonzero $E_\infty$-terms are $E^k,n_\infty = Q = E^{k,0}_\infty$, $0 \leq k \leq m - 1$ and $k$ even. We have

$$H^k(X_G) = \begin{cases} Q, & 0 \leq k \leq m - 1, \ k \text{ even}; \ m \leq k \leq 2m - 1, \ k \text{ odd}; \\ 0, & 2m \leq k. \end{cases}$$

If $m = 1$, then $H^*(X_G) = Q[z]/(z^2)$. So, we can assume that $m > 1$. Then multiplication by $t \in H^2(B_G)$,

$$t \cup (\cdot) : E^{2,l}_\infty \to E^{2+l,0}_\infty,$$

regarded as a spectral sequence endomorphism, is an isomorphism, for $0 \leq k \leq m - 2$ and $l = 0, n$. If $m \neq n$ (resp. $m = n$) the element $1 \otimes v_2$ (resp. $1 \otimes (av_2 - bv_1)$) of $E^{0,n}_2$ is a permanent cocycle and gives a nonzero element $w \in E^{0,n}_2$. Let $y = \pi^*(t) \in H^2(X_G)$. Then the total complex

$$\text{Tot}^* E^{*,*}_\infty \cong Q[y, w]/(y^{(m+1)/2}, w^2)$$

as a graded commutative algebra. We choose an element $z \in H^n(X_G)$ such that $\iota^*(z) = v_2$ (resp. $av_2 - bv_1$) if $m \neq n$ (resp. $m = n$). Then $yz \neq 0$ and $z^2 = 0$. The multiplication

$$y \cup (\cdot) : H^k(X_G) \to H^{k+2}(X_G)$$

is an isomorphism in degrees $k$ such that $0 \leq k \leq m - 2$ and $n \leq k \leq m + n - 3$. So

$$H^*(X_G) \cong Q[y, z]/(y^{(m+1)/2}, z^2),$$

as mentioned in case (i).

Next, suppose that $d_r(1 \otimes v_1) = 0$ and $d_r(1 \otimes v_2) \neq 0$, where $r = n - m + 1$. If $n$ is even, then $m$ must be odd. Consequently, $0 = d_{n-m+1}(1 \otimes v^2_2) = 2at^q \otimes v_1v_2 \neq 0$, where $d_r(1 \otimes v_2) = at^q \otimes v_1$, $2q = n - m + 1$. Therefore $n$ is odd and $m$ is even. It follows that $d_r(1 \otimes v_1v_2) = 0$ and the differential

$$d_r : E^{*,n}_r \to E^{*,m}_r$$

is an isomorphism. Thus

$$E^{k,n}_{n-m+2} = 0 = E^{k+n-m+1,m}_{n-m+2}, \ E^{*,m+n}_{n-m+2} = E^{*,m+n}_{2}, \ E^{*,0}_{n-m+2} = E^{*,0}_{2}.$$  

Since $m$ is even,

$$d_{m+1} : E^{k,m}_{m+1} \to E^{k+m+1,0}_{m+1}$$
is trivial, for $0 \leq k \leq n - m$. Since there are no fixed points, the differential
\[ d_{n+m+1} : E_{n+m+1}^{0,m+n} \to E_{n+m+1}^{n+m+1,0} \]
must be nontrivial so that we can assume $d_{n+m+1}(1 \otimes v_1 v_2) = a t^{(n+m+1)/2} \otimes 1$, $a \neq 0$. Then the differential
\[ d_{n+m+1} : E_{n+m+1}^{*,m+n} \to E_{n+m+1}^{*,0} \]
is an isomorphism. So $E_\infty = E_{n+m+2}$ and the nonzero vector spaces in the $E_\infty$-term are $E_{k,m}^k = Q$, $0 \leq k \leq n - m$ and $E_{k,0}^k = Q$, $0 \leq k \leq m + n$; $k$ even. We have
\[ H^k(X_G) = \begin{cases} Q, & 0 \leq k \leq m - 1, n + 1 \leq k \leq n + m - 1, k \text{ even}; \\ Q \oplus Q, & m \leq k \leq n - 1, k \text{ even}; \\ 0, & \text{otherwise}. \end{cases} \]
The element $1 \otimes v_1 \in E_{2,m}^0$ is a permanent cocycle and determines an element $w \in E_\infty^{0,m}$. We see that
\[ \text{Tot } E_{\infty,*}^* \cong Q[y,w]/(w^2, y^{(n+m+1)/2}, wy^{(n-m+1)/2}), \]
as graded commutative algebras, where $y = \pi^*(t)$, as above. The multiplication
\[ y \cup (\cdot) : H^k(X_G) \to H^{k+2}(X_G) \]
is an isomorphism in degrees $k$, for $0 \leq k \leq n - 2$ and $n + 1 \leq k \leq n + m - 3$. We find an element $z \in H^m(X_G)$ such that $t^*(z) = v_1$. Then $y^r z$ and $y^{(m+2r)/2}$ are linearly independent over $Q$, for $r \leq (n - m - 1)/2$. We can change $z$ suitably so that $y^{(n-m+1)/2} = 0$ and $z^2 = b y^m$, $b \in Q$, when $n < 2m$ and $z^2 = a y^{(n+1)/2}$ and $z^2 = b y^m$, $a, b \in Q$, when $2m < n$. Therefore,
\[ H^*(X_G) \cong Q[y, z]/(y^{(n+m+1)/2}, z^2 = b y^m - a y^{(n+1)/2}), \]
as mentioned in case (ii).

Finally, let us suppose that $d_r(1 \otimes v_1) = 0$, $r = n+1$ and $d_r(1 \otimes v_2) = a t^{(n+1)/2} \otimes 1$, $0 \neq a \in Q$. Then, $n$ must be odd. We have $d_{n+1}(1 \otimes v_1 v_2) = \pm a t^{(n+1)/2} \otimes v_1$; consequently the differentials
\[ d_{n+1} : E_{n+1}^{k,n} \to E_{n+1}^{k+n+1,0}, \quad d_{n+1} : E_{n+1}^{k,m+n} \to E_{n+1}^{k+n+1,m} \]
are isomorphisms. We obtain $E_\infty = E_{n+2}$ and the only nonzero vector spaces in the $E_\infty$-term are $E_{k,0}^k = Q = e_\infty^0$, $0 \leq k \leq n$, $k$ even. Thus for $m$ even
\[ H^k(X_G) = \begin{cases} Q, & 0 \leq k \leq m - 1, n + 1 \leq k \leq n + m - 1, k \text{ even}; \\ Q \oplus Q, & m \leq k \leq n - 1, k \text{ even}; \\ 0, & \text{otherwise}, \end{cases} \]
and for $m$ odd,
\[ H^k(X_G) = \begin{cases} Q, & 0 \leq k \leq m = 1, k \text{ even}; m \leq k \leq m + n - 1, k \text{ odd}; \\ 0, & \text{otherwise}. \end{cases} \]
The element $1 \otimes v_1 \in E_{2,m}^0$ is a permanent cocycle and yields an element $w \in E_\infty^{0,m}$. We have
\[ \text{Tot } E_{\infty,*}^* \cong Q[y,w]/(y^{(n+1)/2}, w^2) \]
as graded commutative algebras, where \( y = \pi^*(t) \in H^2(X_G) \). Choose \( z \in H^m(X_G) \) such that \( \iota^*(z) = v_1 \). Then \( z \) represents \( w \) and the multiplication

\[
y \cup (\cdot) : H^k(X_G) \rightarrow H^{k+2}(X_G)
\]

is an isomorphism, for \( 0 \leq k \leq n - 2 \) and \( n < k \leq m + n - 2 \). Accordingly, \( y^iz \neq 0 \) for \( i \leq (n - 1)/2 \). When \( m \) is even, \( z \) can be chosen so that \( z^2 = by^{m/2}z, b \in Q \), is zero for \( n < 2m \). So

\[
H^*(X_G) \cong Q[y, z]/(y^{(n+1)/2}, z^2 - by^m),
\]

as in case (iii).

Since \( G \) acts freely on \( X \), \( H^*(X_G) \cong H^*(X/G) \) and this completes the proof. \( \square \)

**Examples.** An example of case (i) in Theorem 1 is obtained by considering the diagonal action of \( G \) on \( S^m \times S^n \) where \( G \) acts freely on \( S^m \) and trivially on \( S^n \). In fact, this possibility can be put more succinctly as \( X/G \cong L^m \times S^n \). A similar consideration gives examples of case (3), except when \( 2m < n \) and \( m \) is even. By the same method one obtains an example of case (i) in Theorem 2, which can be described as \( X/G \cong \mathbb{RP}^m \times S^n \). The space \( CP^{(m-1)/2} \times S^n \) has the cohomology of type (i) in Theorem 3 and is obtained by taking the diagonal action of \( S^1 \) on \( S^m \times S^n \), where \( S^1 \) acts freely on \( S^m \) and trivially on \( S^n \). Similarly, case (iii) can also be illustrated.

**References**


Department of Mathematics, University of Missouri, St. Louis, Missouri 63121

E-mail address: dotzel@umsl.edu

Department of Mathematics, University of Delhi, Delhi-110007, India

E-mail address: crl@delnet.ren.nic.in

Department of Mathematics, University of Delhi, Delhi-110007, India