THE COHOMOLOGY RINGS OF THE ORBIT SPACES
OF FREE TRANSFORMATION GROUPS
OF THE PRODUCT OF TWO SPHERES

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Abstract. Let $G = \mathbb{Z}_p$, $p$ a prime (resp. $S^1$), act freely on a finitistic space $X$ with mod $p$ (resp. rational) cohomology ring isomorphic to that of $S^m \times S^n$. In this paper we determine the possible cohomology algebra of the orbit space $X/G$.

1. Introduction

Let $G = \mathbb{Z}_p$, $p$ a prime (resp. $S^1$, the circle group), act on a space $X$ with mod $p$ (resp. rational) cohomology ring isomorphic to that of $S^m \times S^n$; we abbreviate this as $X \sim_p S^m \times S^n$ (resp. $X \sim Q S^m \times S^n$). There are two spaces associated with the transformation group $(G, X)$; viz. the fixed point set $X^G$ and the orbit space $X/G$. The homological nature of $X^G$ has been studied in detail by Adem [1], Bredon [3], Hsiang [4], Su [6] and Tomter [7]. However, to our knowledge, no one has investigated the homological structure of the space $X/G$. We find here the possibilities for the cohomology algebra $H^*(X/G)$ when the action is free. Throughout this paper, we use Čech cohomology with coefficients in the field $F_p$ of $p$ elements or $Q$ of rational numbers, unless otherwise indicated. The mod $p$ Bockstein cohomology operation associated with the coefficient sequence $0 \to \mathbb{Z}_p \to \mathbb{Z}_p^2 \to \mathbb{Z}_p \to 0$ will be denoted by $\beta$. We prove the following:

Theorem 1. Let $G = \mathbb{Z}_p$, $p$ an odd prime, act freely on a finitistic space $X \sim_p S^m \times S^n$, $0 < m \leq n$, and assume that $H^*(X; Z)$ is of finite type. Then $H^*(X/G; \mathbb{Z}_p)$ is isomorphic to $\mathbb{Z}_p[x, y, z]/\phi(x, y, z)$ as a graded commutative algebra, where $\phi(x, y, z)$ is one of the following graded ideals:

(i) $(x^2, y^{(m+1)/2}, z^2)$, $m$ odd, $\deg x = 1, \deg y = \beta(x), \deg z = n$;

(ii) $(x^2, y^{(m+n+1)/2}, y^{(n-m+1)/2} - ay^{(n+1)/2}, z^2 - by^m)$, $m$ even, $\deg x = 1, y = \beta(x), \deg z = m, a, b \in \mathbb{Z}_p$ and $a = 0$ necessarily when $n < 2m$;

(iii) $(x^2, y^{(n+1)/2}, z^2 - by^m)$, $n$ odd, $\deg x = 1, y = \beta(x), \deg z = m, b \in \mathbb{Z}_p, b \neq 0$ only when $m$ is even and $2m < n$.

Theorem 2. Let $G = \mathbb{Z}_2$ act freely on a finitistic space $X \sim_2 S^m \times S^n$, $0 < m \leq n$. Then $H^*(X/G, \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2[y, z]/\psi(y, z)$ as a graded algebra, where $\psi(y, z)$ is one of the following graded ideals:

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Hence the Leray-Serre spectral sequence of the map $E_{2}^{p,q}$ converges to $H^{p+q}(X_{G})$. Proposition 1. If $G$ acts trivially on $H^{*}(X)$ and the spectral sequence of the map $G X_{G} \rightarrow B G$, which has $E_{2}^{p,q} = H^{p}(B G; H^{q}(X))$ as its $E_{2}$-term and converges to $H^{p+q}(X_{G})$, in the sense of Bredon [2], where the coefficients $H^{*}(X)$ are twisted by the action of $\pi_{1}(B G)$ and $X_{G} = (E G \times X)/G$ is the Borel construction of $X$ associated to a universal $G$-bundle $E G \rightarrow B G$. It can easily be seen that $X_{G}$ is paracompact, when $X$ is so.

2. SOME KNOWN RESULTS

Suppose $G = Z_{p}$, $p$ a prime, acts on a finitistic space $X \sim_{p} S^{m} \times S^{n}$. The following facts can be easily deduced.

Proposition 1. If $G$ acts trivially on $H^{*}(X)$ and the spectral sequence of the map $\pi : X_{G} \rightarrow B G$, degenerates, then $\sum_{k} \text{rk} H^{k}(X_{G}) = 4$ [2 VII, 1.6].

Proposition 2. If $m$ and $n$ are even and $p > 2$, then $X_{G} \neq \Phi$ [2 III, 7.10].

Proposition 3. If $H^{*}(X; Z)$ is of finite type, $p > 2$ and $G$ acts nontrivially on $H^{*}(X)$, then $p = 3$ and $X_{G} \neq \Phi$ [6].

Proposition 4. If $p = 2$ and $G$ acts nontrivially on $H^{*}(X)$, then $X_{G} \neq \Phi$ and $m = n$ [2 VII, 7.5].

We recall that for $G = Z_{p}$,

$$H^{*}(B G; Z_{p}) = \begin{cases} Z_{p}[t], & \text{deg } t = 1, p = 2, \\ \wedge(s) \otimes Z_{p}[t], & \text{deg } s = 1, t = \beta(s), p > 2, \end{cases}$$

and for $G = S^{1}$,

$$H^{*}(B G; Q) = Q[t], \quad \text{deg } t = 2.$$

3. PROOFS

**Proof of Theorem** 7. Since there are no fixed points, it follows from Propositions 1 and 3 that $m$ and $n$ cannot both be even and that $Z_{p}$ acts trivially on $H^{*}(X)$. Hence the Leray-Serre spectral sequence of the map $\pi : X_{G} \rightarrow B G$ does not collapse at the $E_{2}$-term and $E_{2}^{p,q} = H^{p}(B G) \otimes H^{q}(X)$. Let $r \geq 2$ be the smallest integer such that $d_{r} \neq 0$. By the multiplicative properties of the spectral sequence, we have $d_{r}(1 \otimes v_{1}) \neq 0$ or $d_{r}(1 \otimes v_{2}) \neq 0$. Suppose, first, that $d_{r}(1 \otimes v_{1}) \neq 0$. Then $r = m + 1$
and \( m \) must be odd. So we can write \( d_{m+1}(1 \otimes v_1) = t^{(m+1)/2} \otimes 1 \). Now, we either have \( d_{m+1}(1 \otimes v_2) = 0 \) or \( n = m \) and \( d_{m+1}(1 \otimes v_2) = at^{m+1} \otimes 1, \ 0 \neq a \in \mathbb{Z}_p \). For \( n \neq m \), obviously, \( d_{m+1}(1 \otimes v_2) = 0 \) and \( d_{m+1}(1 \otimes v_3) = t^{(m+1)/2} \otimes v_2 \). Thus the differentials

\[
d_{m+1} : E^{k,m}_{m+1} \to E^{k+m+1,0}_{m+1}, \text{ and}
\]

\[
d_{m+1} : E^{k,m+n}_{m+1} \to E^{k+m+1,n}_{m+1}
\]

are isomorphisms and we have \( E^\infty = E_{m+2} \). Consequently, the only nonzero vector spaces in the \( E^\infty \)-term are \( E^{k,n}_{\infty} = \mathbb{Z}_p = E^{k,0}_{\infty}, \ 0 \leq k \leq m \). Thus,

\[
H^k(X_G) = \begin{cases} 
\mathbb{Z}_p, & 0 \leq k \leq m \text{ and } n \leq k \leq m+n; \\
0, & \text{otherwise.}
\end{cases}
\]

If \( n = m \) and \( d_{m+1}(1 \otimes v_2) = at^{(m+1)/2} \otimes 1, \ a \in \mathbb{Z}_p \), then \( d_{m+1}(1 \otimes v_3) = t^{(m+1)/2} \otimes (v_2 - av_1) \). So the differential

\[
d_{m+1} : E^{k,m}_{m+1} \to E^{k+m+1,0}_{m+1}
\]

is surjective with \( \ker(d_{m+1}) \) generated by \( \zeta_k \otimes (v_2 - av_1) \), \( \zeta_k \) is the generator of \( H^k(B_G) \) and the differential

\[
d_{m+1} : E^{k,2m}_{m+1} \to E^{k+m+1,m}_{m+1}
\]

is injective with \( \text{im}(d_{m+1}) \) generated by \( \zeta_{k+m+1} \otimes (v_2 - av_1) \). Consequently, \( E^\infty = E_{m+2} \) and the only nonzero entries in the \( E^\infty \)-term are \( E^{k,m}_{\infty} = \mathbb{Z}_p = E^{k,0}_{\infty}, \ 0 \leq k \leq m \). We obtain

\[
H^k(X_G) = \begin{cases} 
\mathbb{Z}_p, & 0 \leq k \leq 2m \text{ and } k \neq m; \\
\mathbb{Z}_p \oplus \mathbb{Z}_p, & k = m; \\
0, & 2m < k.
\end{cases}
\]

The multiplication by \( t \in H^2(B_G) \):

\[
t \cup (\cdot) : E^{k,l}_{\infty} \to E^{k+2,l}_{\infty},
\]

regarded as a spectral sequence endomorphism, is an isomorphism for \( 0 \leq k \leq m-2, \ m > 1 \) and \( l = 0 \) or \( n \). If \( m \neq n \) (resp. \( m = n \)) the element \( 1 \otimes v_2 \) (respectively \( 1 \otimes (v_2 - av_1) \)) of \( E^{0,n}_2 \) is a permanent cocycle and gives a nonzero element \( w \in E^{0,n}_2 \). There are elements \( x \in H^1(X_G) \) and \( y \in H^2(X_G) \) with \( x = \pi^*(s) \) and \( y = \pi^*(t) \). It is easily checked that the total complex

\[
\text{Tot} E^{*,*}_{\infty} \cong \mathbb{Z}_p[x,y,w]/(x^2, y^{(m+1)/2}, w^2)
\]

as a graded commutative algebra. We choose an element \( z \in H^m(X_G) \) such that \( t^*(z) = v_2 \) (resp. \( v_2 - av_1 \)) if \( m \neq n \) (resp. \( m = n \)). Then \( xz \neq 0 \neq yz \) and \( z^2 = 0 \). The multiplication

\[
y \cup (\cdot) : H^k(X_G) \to H^{k+2}(X_G)
\]

is an isomorphism in degrees \( k \) such that \( 0 \leq k \leq m-2 \) and \( n < k \leq m+n-2 \). So

\[
H^*(X_G) \cong \mathbb{Z}_p[x,y,z]/(x^2, y^{(m+1)/2}, z^2).
\]

For \( m = 1 \), we have

\[
H^*(X_G) \cong \mathbb{Z}_p[x,z]/(x^2, z^2).
\]
Since $G$ acts freely on $X$, $H^*(X_G)$ is isomorphic to $H^*(X/G)$ as a ring and we are in case (1).

Suppose, now, that $d_r(1 \otimes v_1) = 0$ and $d_r(1 \otimes v_2) \neq 0$. We then have either $r = n - m + 1$ and $d_r(1 \otimes v_2) = A \otimes v_1$ or $r = n + 1$ and $d_r(1 \otimes v_2) = A \otimes 1$, $0 \neq A \in H^*(B_G)$. In the former case if $n$ were even, then we would have

$$0 = d_r(1 \otimes v_2^2) = A \otimes v_1 v_2 + (-1)^{n(1 + r)} A \otimes v_2 v_1 = 2A \otimes v_3 \neq 0.$$  

Hence $n$ is odd. We then observe that $m$ must be even. Assume the contrary and consider the spectral sequence of the map $\pi$ with coefficients in $\mathbb{Z}$, the ring of integers. Since $H^*(X; \mathbb{Z})$ is finitely generated, it has no $p$-torsion elements; consequently, we have $\tilde{E}_2^{k,l} = H^k(B_G; H^l(X; \mathbb{Z})) = 0$, for all $k$ odd. Thus $\tilde{E}_2^{k,l} = 0$, for all $k$ odd and $r \geq 2$. The coefficients homomorphism $q : \mathbb{Z} \to \mathbb{Z}_p$ gives the commutative diagram:

$$
\begin{array}{ccc}
E^{0,n}_{n-m+1} & \xrightarrow{d_{n-m+1}} & E^{r,n}_{n-m+1} \\
\downarrow{q^*} & & \downarrow{q^*} \\
E^{0,n}_{n-m+1} & \xrightarrow{d_{n-m+1}} & E^{r,n}_{n-m+1}
\end{array}
$$

The composition $d_{n-m+1} \circ q^*$ is the trivial homomorphism, for $n - m + 1$ is odd. Since $q^*$ in the left is surjective, the bottom $d_{n-m+1}$ is trivial. But this is not the case; hence our assertion. Thus we must have $m$ even and $n$ odd so that we can write $d_{n-m+1}(1 \otimes v_2) = t^{(n-m+1)/2} \otimes v_1$. It follows that the differential

$$d_{n-m+1} : E^{s,m}_{n-m+1} \to E^{s,m+1}_{n-m+1}$$

is an isomorphism and $d_{n-m+1}(E^{s,m}_{n-m+1}) = 0 = d_{n-m+1}(E^{s,m+1}_{n-m+1})$. So we have $E^{k,n}_{r} = 0 = E^{k+m-n-1,n}_{r}$. $E^{k,n}_{n-m+1} = E^{k,n}_{2,m+n}$ and $E^{k,0}_{r} = E^{k,0}_{2}$ for all $k \geq 0$ and $r = n - m + 2$. It is easily seen that the differential

$$d_{m+1} : E^{k,m}_{m+1} \to E^{k,m+1,0}_{m+1}$$

is trivial for $0 \leq k \leq n - m$, since $E^{k,m}_{m+1} = E^{k,m}_2$. Because there are no fixed points, the differential

$$d_{n+m+1} : E^{0,m+n}_{n+m+1} \to E^{0,m+n+1,0}_{n+m+1}$$

must be nontrivial so that we can assume $d_{n+m+1}(1 \otimes v_3) = t^{(n+m+1)/2} \otimes 1$. Then, the differential

$$d_{n+m+1} : E^{s,m+n}_{n+m+1} \to E^{s,0}_{n+m+1}$$

is an isomorphism. Consequently, $E_\infty = E_{m+n+2}$ and the only nonzero vector spaces in the $E_\infty$-term are $E^{k,m}_{\infty} = \mathbb{Z}_p$ for $0 \leq k \leq n - m$ and $E^{0,0}_{\infty} = \mathbb{Z}_p$ for $0 \leq k \leq m + n$. It follows that

$$H^k(X_G) = \begin{cases} 
\mathbb{Z}_p, & 0 \leq k < m \text{ and } n < k \leq m + n; \\
\mathbb{Z}_p \oplus \mathbb{Z}_p, & m \leq k \leq n; \\
0, & m + n < k.
\end{cases}$$

We note that $1 \otimes v_1 \in E^{0,m}_{2}$ is a permanent cocycle and determines an element $w \in E^{0,m}_{\infty}$. We have

$$\text{Tot } E^{*,*}_{\infty} \cong \mathbb{Z}_p[x, y, w]/(x^2, w^2, y^{(m+n+1)/2}, y^{(n-m+1)/2}w)$$
as graded commutative algebras, where $x$ and $y$ satisfy $\pi^*(s) = x$ and $\pi^*(t) = y$. The multiplication

$$y \cup (\cdot) : H^k(X_G) \to H^{k+2}(X_G)$$

is an isomorphism in degrees $k$ for $0 \leq k \leq n - 2$ and $n < k \leq m + n - 2$. We choose an element $z \in H^m(X/G)$ such that $\iota^*(z) = v_1$. Then $y^k z$ and $y^{(m+2r)/2}$ are linearly independent over $Z_p$ for $r \leq (n - m - 1)/2$. It is possible to change $z$ suitably so that $y^{(n-m+1)/2} z = 0$ and $z^2 = by^m$, $b \in Z_p$, when $n < 2m$ and $y^{(n-m+1)/2} z = ay^{(n+1)/2}$ and $z^2 = by^m$, $a, b \in Z_p$, when $2m < n$. Therefore,

$$H^*(X_G) \cong Z_p[x, y, z]/(x^2, y^{(m+1)/2}, y^{(n-m+1)/2} z - ay^{(n+1)/2}, z^2 - by^m)$$

and we are in case (ii).

Finally, consider the possibility $r = n + 1$, $d_r(1 \otimes v_1) = 0$ and $d_r(1 \otimes v_2) = A \otimes 1$, $0 \neq A \in H^*(B_G)$. Then $n$ must be odd and we can set $d_{n+1}(1 \otimes v_2) = t^{(n+1)/2} \otimes 1$. So $d_{n+1}(1 \otimes v_3) = \pm t^{(n+1)/2} \otimes v_1$; consequently the differentials

$$d_{n+1} : E^{k,n}_{n+1} \to E^{k+n+1,0}_{n+1},$$

$$d_{n+1} : E^{k,n+m}_{n+1} \to E^{k+n+1,m}_{n+1}$$

are isomorphisms. We obtain $E_{\infty} = E_{n+2}$ and the only nonzero vector spaces in the $E_{\infty}$-term are $E^{k,0}_{\infty} = Z_p = E^{k,0}_0$ for $0 \leq k \leq n$. Thus

$$H^k(X_G) = \begin{cases} Z_p, & 0 \leq k < m \text{ and } n < k \leq m + n; \\ Z_p \oplus Z_p, & m \leq k \leq n; \\ 0, & m + n < k. \end{cases}$$

We note that $1 \otimes v_1 \in E^{0,m}_{0}$ is, again, a permanent cocycle and gives an element $w \in E^{0,m}_{0}$. Choosing $x \in H^1(X_G)$ and $y \in H^2(X_G)$ such that $\pi^*(s) = x$ and $\pi^*(t) = y$, we obtain

$$\text{Tot} E^{*,*}_{\infty} \cong Z_p[x, y, w]/(x^2, y^{(n+1)/2}, w^2)$$

as graded commutative algebras. Now, we choose $z \in H^m(X_G)$ such that $\iota^*(z) = v_1$. Then $z$ represents $w$ and the multiplication

$$y \cup (\cdot) : H^k(X_G) \to H^{k+2}(X_G)$$

is an isomorphism for $0 \leq k \leq n - 2$ and $n < k \leq m + n - 2$, so that $y^2 z \neq 0$ for $r \leq (n - 1)/2$. If $m$ is even and $n < 2m$, then $z^2 = by^{m/2} z, b \in Z_p$. We replace $z$ by $z - (b/2)y^{m/2}$ and have $z^2 = 0$. If $m$ is even and $2m < n$, then $z^2 = by^{m/2} z + cy^m$, $b, c \in Z_p$. Replacing $z$ by $z - (b/2)y^{m/2}$ we obtain the relation $z^2 = b' y^m$. Thus

$$H^*(X_G) \cong Z_0[x, y, z]/(x^2, y^{(n+1)/2}, z^2 - by^m)$$

as a graded commutative algebra and we obtain case (iii). This completes the proof of the theorem.

**Proof of Theorem** The proof is analogous to that of Theorem 11; we describe it rather briefly. The spectral sequence of the map $\pi$ is nontrivial. Let $r \geq 2$ be the least integer such that $d_r \neq 0$. Then, we must have $d_r(1 \otimes v_1) \neq 0$ or
\(d_r(1 \otimes v_2) \neq 0\). Suppose that \(r = m + 1, d_{m+1}(1 \otimes v_1) = t^{m+1} \otimes 1\) and \(m \neq n\). Then \(d_{m+1}(1 \otimes v_2) = 0, d_{m+1}(1 \otimes v_3) = t^{m+1} \otimes v_2\) so that \(E_\infty = E_{m+2}\) and we obtain

\[
H^k(X_G) = \begin{cases} 
Z_2, & 0 \leq k \leq m \text{ and } n \leq k \leq m + n; \\
0, & \text{otherwise}.
\end{cases}
\]

The element \(1 \otimes v_2 \in E_2^{0,n}\), being a permanent cocycle, yields an element \(w \in E_0^{0,n}\). So, the total complex \(\text{Tot } E_\infty^{*,*}\) is the graded algebra \(Z_2[y, w]/(y^1, w^2)\), where \(y = \pi^*(t), t \in H^1(B_G)\). Choose \(z \in H^n(X_G)\) such that \(t^*(z) = v_2\). Then \(z\) determines \(w\) and satisfies \(z^2 = 0\). Since the multiplication

\[
y \cup (\cdot) : H^k(X_G) \to H^{k+1}(X_G)
\]

is an isomorphism for \(0 \leq k < m\) and \(n \leq k < m + n\), we have \(y^r z \neq 0, 1 \leq r \leq m\). Therefore

\[
H^*(X_G) \cong Z_2[y, z]/(y^{m+1}, z^2).
\]

As the action of \(G\) is free, \(H^*(X_G) \cong H^*(X/G)\) and we have the case (i). If \(m = n, d_{m+1}(1 \otimes v_1) = t^{m+1} \otimes 1\) and \(d_{m+1}(1 \otimes v_2) = ct^{m+1} \otimes 1, c \in Z_2\), then \(d_{m+1}(1 \otimes v_3) = t^{m+1} \otimes (v_2 + cv_1)\). We obtain \(E_\infty = E_{m+2}\) and

\[
H^k(X_G) = \begin{cases} 
Z_2, & 0 \leq k \leq 2m \text{ and } k \neq m; \\
Z_2 \oplus Z_2, & k = m; \\
0, & 2m < k.
\end{cases}
\]

In this case, the element \(1 \otimes (v_2 + cv_1) \in E_2^{0,m}\) is a permanent cocycle and determines an element \(w \in E_0^{0,m}\). Let \(y \in H^1(X_G)\) and \(z \in H^m(X_G)\) be such that \(\pi^*(t) = y\) and \(t^*(z) = v_2 + cv_1\). Then \(z\) represents \(w\) in \(E_0^{0,m}\) and satisfies \(z^2 = ay^m z\) for some \(a \in Z_2\). The element \(yz\) represents \(0 \neq tw \in E_0^{1,m}\) and the multiplication

\[
y \cup (\cdot) : H^k(X_G) \to H^{k+1}(X_G)
\]

is an isomorphism for \(m < k < 2m\) so that \(y^r z \neq 0, 1 \leq r \leq m\). Thus

\[
H^*(X_G) \cong Z_2[y, z]/(y^{m+1}, z^2 - ay^m z)
\]

and we are in case (iii) with \(b = 0\).

Next, when \(d_r(1 \otimes v_1) = 0, r = n - m + 1\) and \(d_r(1 \otimes v_2) = t^r \otimes v_1\), then \(d_r(1 \otimes v) = Q\). So, we have \(E_{r+1}^{k,m+n} = E_2^{k,m+n}, E_{r+1}^{k,0} = E_2^{k,0}\), for \(k \geq 0\), and \(E_{r+1}^{k,m}\) is trivial for \(k > n - m\) and \(E_2^{k,m}\) for \(k \leq n - m\). It is easily seen that the differential

\[
d_{m+1} : E_{m+1}^{k,m} \to E_{m+1}^{k+1,0}
\]

is also trivial for \(0 \leq k \leq n - m\). Then the differential

\[
d_{n+m+1} : E_{n+m+1}^{k,m} \to E_{n+m+1}^{k+m+n+1,0}
\]

must be an isomorphism for all \(k\), because the action of \(G\) on \(X\) is free. Thus, the only nonzero vector spaces in the \(E_\infty\)-term are \(E_\infty^{k,m} = Z_2\) for \(0 \leq k \leq n - m\) and \(E_\infty^{k,0} = Z_2\), for \(0 \leq k \leq m + n\). Consequently

\[
H^k(X_G) = \begin{cases} 
Z_2, & 0 \leq k < m \text{ and } n < k \leq m + n; \\
Z_2 \oplus Z_2, & m \leq k \leq n; \\
0, & m + n < k.
\end{cases}
\]
The element $1 \otimes v_1 \in E_2^{0,m}$ is a permanent cocycle and gives an element $w \in E_\infty^{0,m}$. We obtain

$$\text{Tot} E_\infty^{*,*} \cong Z_2[y,w]/(y^{m+n+1}, w^2, y^{n-m+1}w)$$

as graded algebras. The multiplication by $\pi^*(t) = y$

$$y \cup (\cdot) : H^k(X_G) \to H^{k+1}(X_G),$$

is an isomorphism for $0 \leq k \leq n-1$ and $n < k < m + n$. Since the composition $\iota^* \pi^*$ is trivial in positive degrees, it is possible to choose $z \in H^m(X_G)$ such that $\iota^*(z) = v_1$, $y^{n-m+1}z = 0$ and $z^2 = ay^mz + by^{2m}$, where $a = 0$ when $2m > n$. Thus we have

$$H^*(X_G) \cong Z_2[y,z]/(y^{m+n+1}, y^{n-m+1}z, z^2 - ay^mz - by^{2m})$$

as graded algebras and we are in case (ii).

Finally, consider the case $d_r(1 \otimes v_1) = 0$, $r = n + 1$ and $d_{n+1}(1 \otimes v_2) = t^{n+1}$. Then $d_{n+1}(1 \otimes v_3) = t^{n+1} \otimes v_1$. So the differentials

$$d_{n+1} : E_{n+1}^{k,n} \to E_{n+1}^{k,n+1,0},$$

and

$$d_{n+1} : E_{n+1}^{k,m+n} \to E_{n+1}^{k,m+n+1,m},$$

are isomorphisms. We have $E_\infty = E_{n+2}$ and the only nonzero vector spaces in the $E_\infty$-term are $E_\infty^{k,m} = Z_2 = E_\infty^{b,0}$, for $0 \leq k \leq n$. Therefore

$$H^k(X_G) = \begin{cases} 
Z_2, & 0 \leq k < m \text{ and } n < k \leq m + n; \\
Z_2 \oplus Z_2, & m \leq k \leq n; \\
0, & m + n < k.
\end{cases}$$

Taking $y \in H^1(X_G)$ and $z \in H^m(X_G)$ with $\pi^*(t) = y$ and $\iota^*(z) = v_1$, it can be easily seen that

$$H^*(X_G) \cong Z_2[y,z]/(y^{n+1}, z^2 - ay^mz - by^{2m}),$$

as graded algebras, where $b = 0$, if $n < 2m$. This completes the proof. \qed

\textbf{Proof of Theorem 3} Though arguments given herein are different, the technique of the proof remains the same. Since there are no fixed points, the spectral sequence of the map $\pi : X_G \to B_G$ does not collapse at the $E_2$-term and $E_2^{k,l} = H^k(B_G) \otimes H^l(X)$, for the action on $H^*(X)$ is trivial. Let $r \geq 2$ be the integer such that $d_r \neq 0$. By the multiplicative properties of the spectral sequence, we have $d_r(1 \otimes v_1) \neq 0$ or $d_r(1 \otimes v_2) \neq 0$.

First, suppose that $d_r(1 \otimes v_1) \neq 0$. Then $r = m + 1$ where $m$ is odd. We can write $d_r(1 \otimes v_1) = at^{(m+1)/2} \otimes 1, 0 \neq a \in Q$. If $n = 2m$, $d_{m+1}(1 \otimes v_2) = bt^{(m+1)/2} \otimes v_1$, then

$$0 = d_{m+1}(1 \otimes v_2^2) = 2bt^{(m+1)/2} \otimes v_1 v_2 \neq 0,$$ a contradiction. Therefore, either $d_{m+1}(1 \otimes v_2) = 0$ or $m = n$, $d_{m+1}(1 \otimes v_2) = bt^{m+1} \otimes 1, a \neq b \in Q$. In the first case, $d_{m+1}(1 \otimes v_1 v_2) = at^{(m+1)/2} \otimes v_2$; so the differentials

$$d_{m+1} : E_{m+1}^{k,m} \to E_{m+1}^{k,m+1,0},$$

and

$$d_{m+1} : E_{m+1}^{k,m+n} \to E_{m+1}^{k,m+n+1,n}.$$
are isomorphisms and we have $E^\infty = E_{m+2}$. Thus, the only nonzero vector spaces in the $E^\infty$-term are $E^{k,n}_\infty = Q = E^{k,0}_\infty$, $k$ even, $0 \leq k \leq m - 1$. We have

$$H^k(X_G) = \begin{cases} Q, & 0 \leq k \leq m - 1, \text{ } k \text{ even}; \quad n \leq k \leq n + m - 1, \text{ } k - n \text{ even}; \\
0, & \text{otherwise.}
\end{cases}$$

In this case, $d_{m+1}(1 \otimes v_2) = b(d(m+1)/2 \otimes 1, b \neq 0, d_{m+1}(1 \otimes v_1 v_2) = \ell(m+1)/2 \otimes (av_2 - bv_1)$. So the differential

$$d_{m+1} : E^{k,m}_{m+1} \rightarrow E^{k+m+1,0}_{m+1}$$

is surjective, with $\ker(d_{m+1})$ generated by $\xi_k \otimes (av_2 - bv_1)$, $\xi_k$ is the generator of $H^k(B_G)$ and the differential

$$d_{m+1} : E^{k,2m}_{m+1} \rightarrow E^{k+m+1,m}_{m+1}$$

is injective with $im(d_{m+1})$ generated by $\xi_{k+m+1} \otimes (av_2 - bv_1)$. Consequently, $E^\infty = E_{m+2}$ and the nonzero $E^\infty$-terms are $E^{k,m}_\infty = Q = E^{k,0}_\infty$, $0 \leq k \leq m - 1$ and $k$ even. We have

$$H^k(X_G) = \begin{cases} Q, & 0 \leq k \leq m - 1, \text{ } k \text{ even}; \quad m \leq k \leq 2m - 1, \text{ } k \text{ odd}; \\
0, & 2m \leq k.
\end{cases}$$

If $m = 1$, then $H^*(X_G) = Q[z]/(z^2)$. So, we can assume that $m > 1$. Then multiplication by $t \in H^2(B_G)$,

$$t \cup (\cdot) : E^{k,l}_\infty \rightarrow E^{k+2,l}_\infty,$$

regarded as a spectral sequence endomorphism, is an isomorphism, for $0 \leq k \leq m - 2$ and $l = 0, n$. If $m \neq n$ (resp. $m = n$) the element $1 \otimes v_2$ (resp. $1 \otimes (av_2 - bv_1)$) of $E^{0,n}_2$ is a permanent cocycle and gives a nonzero element $w \in E^{0,n}_\infty$. Let $y = \pi^*(t) \in H^2(X_G)$. Then the total complex

$$\Tot E^{*,*}_\infty \cong Q[y, w]/(y^{(m+1)/2}, w^2)$$

as a graded commutative algebra. We choose an element $z \in H^n(X_G)$ such that $\iota^*(z) = v_2$ (resp. $av_2 - bv_1$) if $m \neq n$ (resp. $m = n$). Then $yz \neq 0$ and $z^2 = 0$. The multiplication

$$y \cup (\cdot) : H^k(X_G) \rightarrow H^{k+2}(X_G)$$

is an isomorphism in degrees $k$ such that $0 \leq k \leq m - 2$ and $n \leq k \leq m + n - 3$. So

$$H^*(X_G) \cong Q[y, z]/(y^{(m+1)/2}, z^2),$$

as mentioned in case (i).

Next, suppose that $d_r(1 \otimes v_1) = 0$ and $d_r(1 \otimes v_2) \neq 0$, where $r = n - m + 1$. If $n$ is even, then $m$ must be odd. Consequently, $0 = d_{n-m+1}(1 \otimes v_2^2) = 2atq \otimes v_1 v_2 \neq 0$, where $d_r(1 \otimes v_2) = at^q \otimes v_1, 2q = n - m + 1$. Therefore $n$ is odd and $m$ is even. It follows that $d_r(1 \otimes v_1 v_2) = 0$ and the differential

$$d_r : E^{r,n}_r \rightarrow E^{r,m}_r$$

is an isomorphism. Thus

$$E^{k,n}_{n-m+2} = 0 = E^{k+n-m+1,m}_{n-m+2}, \quad E^{r,m+n}_r = E^{r,m+n}_2, \quad E^{*,0}_{n-m+2} = E^{*,0}_2.$$
We have consequently the differentials $zy$ as mentioned in case (ii). $Q$ in independent over $z$ is an isomorphism in degrees $E$. The element $1 = 0$. Then the differential $d_{n+m+1} : E_n^{0,m+1} = E_{n+m+1}^{0,m+1,0}$ must be nontrivial so that we can assume $d_{n+m+1}(1 \otimes v_1v_2) = a(t(n+m+1)/2 \otimes 1$, $a \neq 0$. Then the differential $d_{n+m+1} : E_n^{*,m+1} \rightarrow E_{n+m+1}^{*,0}$ is an isomorphism. So $E_\infty = E_{n+m+2}$ and the nonzero vector spaces in the $E_\infty$-term are $E_{\infty}^{k,m} = Q$, $0 \leq k \leq n - m$ and $E_{\infty}^{k,0} = Q$, $0 \leq k \leq m + n$; $k$ even. We have

$$H^k(X_G) = \begin{cases} Q, & 0 \leq k \leq m - 1, n + 1 \leq k \leq n + m - 1, k \text{ even;} \\ Q \oplus Q, & m \leq k \leq n - 1, k \text{ even;} \\ 0, & \text{otherwise.} \end{cases}$$

The element $1 \otimes v_1 \in E_2^{0,m}$ is a permanent cocycle and determines an element $w \in E_\infty^{0,m}$. We see that

$$\text{Tot } E_{\infty}^{*,*} \cong Q[y,w]/(w^2, y^{(n+m+1)/2}, wy^{(n-m+1)/2}),$$

as graded commutative algebras, where $y = \pi^*(t)$, as above. The multiplication

$$y \cup (\cdot) : H^k(X_G) \rightarrow H^{k+2}(X_G)$$

is an isomorphism in degrees $k$, for $0 \leq k \leq n - 2$ and $n+1 \leq k \leq n+m-3$. We find an element $z \in H^m(X_G)$ such that $v^*(z) = v_1$. Then $y^r z$ and $y^{(m+2r)/2}$ are linearly independent over $Q$, for $r \leq (n - m - 1)/2$. We can change $z$ suitably so that $z(y^{(n-m+1)/2} = 0$ and $z^2 = by^m$, $b \in Q$, when $n < 2m$ and $z(y^{(n-m+1)/2} = ay^{(n+1)/2}$ and $z^2 = by^m$, $a, b \in Q$, when $2m < n$. Therefore,

$$H^*(X_G) \cong Q[y,z]/(y^{(n+m+1)/2}, zy^{(n-m+1)/2} - ay^{(n+1)/2}, z^2 - by^m)$$

as mentioned in case (ii).

Finally, let us suppose that $d_r(1 \otimes v_1) = 0$, $r = n+1$ and $d_r(1 \otimes v_2) = a(t^{(n+1)/2} \otimes 1$, $0 \neq a \in Q$. Then, $n$ must be odd. We have $d_{n+1}(1 \otimes v_1v_2) = \pm a( t^{(n+1)/2} \otimes v_1;$ consequently the differentials

$$d_{n+1} : E_{n+1}^{k,n} \rightarrow E_{n+1}^{k+n+1,1}, \text{ and}$$

$$d_{n+1} : E_{n+1}^{k,m+n} \rightarrow E_{n+1}^{k+n+1,m}$$

are isomorphisms. We obtain $E_\infty = E_{n+2}$ and the only nonzero vector spaces in the $E_\infty$-term are $E_{\infty}^{k,m} = Q = e^{k,0}$, $0 \leq k \leq n$, $k$ even. Thus for $m$ even

$$H^k(X_G) = \begin{cases} Q, & 0 \leq k \leq m - 1, n + 1 \leq k \leq n + m - 1, k \text{ even;} \\ Q \oplus Q, & m \leq k \leq n - 1, k \text{ even;} \\ 0, & \text{otherwise,} \end{cases}$$

and for $m$ odd,

$$H^k(X_G) = \begin{cases} Q, & 0 \leq k \leq m - 1, k \text{ even; } m \leq k \leq m + n - 1, k \text{ odd;} \\ 0, & \text{otherwise.} \end{cases}$$

The element $1 \otimes v_1 \in E_2^{0,m}$ is a permanent cocycle and yields an element $w \in E_\infty^{0,m}$. We have

$$\text{Tot } E_{\infty}^{*,*} \cong Q[y,w]/(y^{(n+1)/2}, w^2)$$
as graded commutative algebras, where \( y = \pi^*(t) \in H^2(X_G) \). Choose \( z \in H^m(X_G) \) such that \( \iota^*(z) = v_1 \). Then \( z \) represents \( w \) and the multiplication

\[
y \cup (\cdot) : H^k(X_G) \rightarrow H^{k+2}(X_G)
\]

is an isomorphism, for \( 0 \leq k \leq n - 2 \) and \( n < k \leq m + n - 2 \). Accordingly, \( y^i z \neq 0 \) for \( i \leq (n - 1)/2 \). When \( m \) is even, \( z \) can be chosen so that \( z^2 = by^{m/2}z, b \in Q \), is zero for \( n < 2m \). So

\[
H^*(X_G) \cong Q[y, z]/(y^{(n+1)/2}, z^2 - by^m),
\]
as in case (iii).

Since \( G \) acts freely on \( X \), \( H^*(X_G) \cong H^*(X/G) \) and this completes the proof. \( \square \)

**Examples.** An example of case (i) in Theorem 1 is obtained by considering the diagonal action of \( G \) on \( S^m \times S^n \) where \( G \) acts freely on \( S^m \) and trivially on \( S^n \). In fact, this possibility can be put more succinctly as \( X/G \cong \mathbb{R}P^m \times S^n \). A similar consideration gives examples of case (3), except when \( 2m < n \) and \( m \) is even. By the same method one obtains an example of case (i) in Theorem 2, which can be described as \( X/G \cong \mathbb{R}P^m \times S^n \). The space \( \mathbb{C}P^{(m-1)/2} \times S^n \) has the cohomology of type (i) in Theorem 3 and is obtained by taking the diagonal action of \( S^1 \) on \( S^m \times S^n \), where \( S^1 \) acts freely on \( S^m \) and trivially on \( S^n \). Similarly, case (iii) can also be illustrated.

**References**

1. A. Adem, \( \mathbb{Z}/p\mathbb{Z} \) actions on \( (S^n)^k \), Trans. Amer. Math. Soc. **300** (1987), 791–809. \( \text{MR} \) 88b:57037

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