

ON C*-EXTREME POINTS

BOJAN MAGAJNA

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ABSTRACT. Each weak* compact C*-convex set in a hyperfinite factor (in particular in $B(\mathcal{H})$) is the weak* closure of the C*-convex hull of its C*-extreme points.

1. INTRODUCTION AND MAIN RESULTS

A subset K of a unital C*-algebra R is called R -convex (or C^* -convex if R is clear from the context) if $\sum_{j=1}^n a_j^* x_j a_j \in K$ whenever $x_j \in K$ and $a_j \in R$ for all j and $\sum_{j=1}^n a_j^* a_j = 1$. As defined in [10], a point $x \in K$ is C^* -extreme for K if the condition

$$(1.1) \quad x = \sum_{j=1}^n a_j^* x_j a_j, \quad \sum_{j=1}^n a_j^* a_j = 1, \quad x_j \in K, \quad a_j \text{ invertible in } R, \quad n \in \mathbb{N},$$

implies that all x_j are unitarily equivalent to x in R . C*-extreme points for subsets K of $R = \mathbb{M}_n := \mathbb{M}_n(\mathbb{C})$ are extreme in the usual sense by [10], but the converse is not true (see [8] or [6]). It was conjectured already in [10] that a variant of the Krein-Milman theorem should hold for compact C*-convex sets, and indeed, much later for subsets of \mathbb{M}_n such a theorem was established by Morenz [12] using some previous work of Farenick and Morenz (see [4], [5] and [6]). Another Krein-Milman type theorem for the so-called matricially convex sets in locally convex spaces has been proved recently by Webster and Winkler [15]. Methods in [12] and [15] (although different) both used in an essential way the finite dimensionality of \mathbb{M}_n . In this note we shall prove a variant of the Krein-Milman theorem for C*-convex subsets of $B(\mathcal{H})$ where \mathcal{H} is a separable Hilbert space. But since the same proof is actually valid for subsets in hyperfinite factors, we shall formulate the result for such factors. For this we need a special kind of C*-extreme points.

A point $x \in K$, where K is a C*-convex subset in a C*-algebra R , is R -extreme for K if the condition

$$(1.2) \quad x = \sum_{j=1}^n a_j x_j a_j, \quad \sum_{j=1}^n a_j^2 = 1, \quad x_j \in K, \quad a_j \text{ invertible and positive in } R,$$

implies that $x_j = x$ and $a_j x = x a_j$ for all j . Using the polar decomposition of the coefficients a_j in (1.1) it is easy to show that each R -extreme point is C*-extreme.

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It suffices to check the condition (1.2) for an R -extreme point in the case $n = 2$. (Indeed, suppose that x satisfies the condition for $n = 2$. For a general n put $y = \bigoplus_{j=2}^n x_j$, $a = (a_2, \dots, a_n)^T$ and let $a = u|a|$ be the polar decomposition of the column a . Then, writing (1.2) as $x = a_1 x_1 a_1 + |a|(u^* y u)|a|$, it follows that $x_1 = x$ and a_1 commutes with x . The same argument can be applied to $j = 2, \dots, n$.) If R is commutative, then it is not difficult to show that the R -extreme points (and the C^* -extreme points) are just the usual extreme points. Also, for a general R , all R -extreme points are extreme in the usual sense.

Now we can state the main result of this note.

Theorem 1.1. *Each weak* compact C^* -convex set K in a hyperfinite factor R is the weak* closure of the C^* -convex hull of the set $\text{ext}_R(K)$ of all R -extreme points of K .*

To prove Theorem 1.1 we shall first consider for each $x \in R$ the weak* closure $\overline{\text{co}}_R(x)$ of the C^* -convex hull $\text{co}_R(x)$ of x , where

$$\text{co}_R(x) = \left\{ \sum_{j=1}^n a_j^* x a_j : a_j \in R, \sum_{j=1}^n a_j^* a_j = 1, n \in \mathbb{N} \right\}.$$

Recall also that for each $n \in \mathbb{N}$ the matricial range $W_n(x)$ of an element $x \in R$ is the set of all $\phi(x)$, where ϕ is any unital completely positive map from R to \mathbb{M}_n (see [1] or [14]).

The following two results will be used in the proof of Theorem 1.1.

Theorem 1.2. *Let R be any factor and $A \subseteq R$ a subfactor (containing the unit of R) isomorphic to \mathbb{M}_n for some $n \in \mathbb{N}$. Then*

$$\overline{\text{co}}_R(x) \cap A = W_n(x)$$

for each $x \in R$, where $W_n(x)$ is regarded as a subset of A by identifying A with \mathbb{M}_n (using any C^* -isomorphism). Moreover, $\phi(K) \subseteq K$ for each unital completely positive map $\phi : R \rightarrow A$ and each weak* compact C^* -convex subset K of R .

Lemma 1.3. *Let R be a unital C^* -algebra and A a C^* -subalgebra containing the unit of R such that for each non-zero $x \in R$ there exists a conditional expectation $E : R \rightarrow A$ satisfying $E(x) \neq 0$. If K is a C^* -convex subset of R such that $\phi(K) \subseteq K$ for each unital completely positive map $\phi : R \rightarrow A$, then $\text{ext}_A(K \cap A) \subseteq \text{ext}_R(K)$.*

Theorem 1.2 and Lemma 1.3 will be proved in Section 2. In the case $R = \mathbb{M}_n$ Theorem 1.1 will be proved in Section 3 by using some results of [5] and [12] (or [15]). Assuming this, we can now deduce Theorem 1.1 for a general hyperfinite factor R .

Proof of Theorem 1.1. Since R is hyperfinite there exists a sequence of finite dimensional C^* -subalgebras $B_0 \subseteq B_1 \subseteq B_2 \dots$ of R and (normal) conditional expectations $E_n : R \rightarrow B_n$ such that the sequence $(E_n x)$ converges to x in the strong operator topology for each $x \in R$. Moreover, except in the case R is of type III_0 , B_n can be chosen to be factors since by the classification of injective factors ([3] and [7]) all such factors are infinite tensor products of finite dimensional factors (see also [9, 13.1.15]). In case R is of type III_0 the algebras B_n cannot always be chosen to be factors, but their existence together with the expectations E_n follows by [2]. In any case (since all non-zero projections in a countably decomposable type III

factor are equivalent) there exists for each n a subfactor $A_n \cong \mathbb{M}_{m_n}$ ($m_n \in \mathbb{N}$) in R such that $A_n \supseteq B_n$ for all n .

Let K be as in the statement of the theorem and $x \in K$. Regarding E_n as a unital completely positive map from R into A_n , we have $E_n(x) \in K \cap A_n$ by Theorem 1.2. Hence, assuming the finite dimensional version of Theorem 1.1 (proved in Section 3) it follows that

$$(1.3) \quad E_n(x) \in \overline{\text{co}}_{A_n}(\text{ext}_{A_n}(K \cap A_n)).$$

Since $R \cong A_n \otimes R_n$ for some subfactor $R_n \subseteq R$, the (normal) conditional expectations from R to A_n (namely the slice maps) separate points of R , hence by Lemma 1.3 $\text{ext}_{A_n}(K \cap A_n) \subseteq \text{ext}_R(K)$. Now (1.3) implies that $E_n(x) \in \overline{\text{co}}_R(\text{ext}_R(K))$, hence $x \in \overline{\text{co}}_R(\text{ext}_R(K))$ since $E_n(x)$ converges to x . \square

We remark that the existence of C*-extreme points in a weak* compact C*-convex subset K of a general von Neumann algebra is proved in [11], but the extreme points obtained in [11] are in general not sufficient to generate K . So for general von Neumann algebras the problem if each weak* compact C*-convex subset is generated by its C*-extreme points remains open.

2. PROOFS OF THEOREM 1.2 AND LEMMA 1.3

To prove Theorem 1.2 we need the following result from [11].

Lemma 2.1. *Let A be a unital C*-algebra, a_1, \dots, a_m elements of A and ρ a state on A in the weak* closure of the pure states. Then for each $\varepsilon > 0$ there exists an element $h \in A$ such that $\|h\| = 1$ and $\|h^*(a_i - \rho(a_i))h\| < \varepsilon$ for $i = 1, \dots, m$.*

Proof of Theorem 1.2. Let $y \in \overline{\text{co}}_R(x) \cap A$ and choose a net $(y_\nu) \subseteq \text{co}_R(x)$ converging to y in the strong operator topology. Since the map $w \mapsto \sum_{j=1}^n a_j^* w a_j$ on R is completely positive and unital for any a_1, \dots, a_n in R satisfying $\sum_{j=1}^n a_j^* a_j = 1$, it follows that $W_n(y_\nu) \subseteq W_n(x)$ for all ν and consequently $W_n(y) \subseteq W_n(x)$ (since unital completely positive maps into \mathbb{M}_n can be approximated by normal such maps in the point-norm topology). But $y \in W_n(y)$ since $y \in A \cong \mathbb{M}_n$, hence $y \in W_n(x)$. This proves the inclusion $\overline{\text{co}}_R(x) \cap A \subseteq W_n(x)$.

To prove the reverse inclusion, let $y \in W_n(x)$ and let $\phi : R \rightarrow A = \mathbb{M}_n$ be a unital completely positive map such that $\phi(x) = y$. By the Stinespring representation theorem (see [13]) $\phi(w) = V^* \sigma(w) V$ ($w \in R$), where σ is a representation of R on a Hilbert space \mathcal{K} and $V : \mathbb{C}^n \rightarrow \mathcal{K}$ is an isometry. Since $R \cong \mathbb{M}_n(B)$, where $B = A' \cap R$, there is a representation π of B on a Hilbert space \mathcal{H} such that $\mathcal{K} = \mathcal{H}^n$ and σ is (unitarily equivalent to) $\pi_n : \mathbb{M}_n(B) \rightarrow \mathbb{M}_n(\mathcal{B}(\mathcal{H}))$, where $\pi_n([b_{ij}]) = [\pi(b_{ij})]$ for each $[b_{ij}] \in \mathbb{M}_n(B)$. Let ε_j ($j = 1, \dots, n$) be the standard basic vectors in \mathbb{C}^n and put $\xi_j = V \varepsilon_j \in \mathcal{H}^n$. Let $y_{ij} \in \mathbb{C}$ be the entries of y . Then

$$(2.1) \quad y_{ij} = \langle y \varepsilon_j, \varepsilon_i \rangle = \langle \phi(x) \varepsilon_j, \varepsilon_i \rangle = \langle V^* \pi_n(x) V \varepsilon_j, \varepsilon_i \rangle = \langle \pi_n(x) \xi_j, \xi_i \rangle.$$

Suppose first that σ (and hence π) is irreducible and let $\xi \in \mathcal{H}$ be any unit vector. Then by the Kadison transitivity theorem there exists a unitary $u \in R$ such that $\pi_n(u)(\varepsilon_j \otimes \xi) = \xi_j$ for $j = 1, \dots, n$, where $\varepsilon_j \otimes \xi \in \mathcal{H}^n$ denotes the vector with the j -th component equal to ξ and the remaining components equal to 0. Now (2.1) can be rewritten as

$$y_{ij} = \langle \pi_n(u^* x u)(\varepsilon_j \otimes \xi), \varepsilon_i \otimes \xi \rangle = \langle \pi((u^* x u)_{ij}) \xi, \xi \rangle = \omega((u^* x u)_{ij}),$$

where $(u^*xu)_{ij}$ is the entry of $u^*xu \in \mathbb{M}_n(B)$ on the position (i, j) and ω is the state $b \mapsto \langle \pi(b)\xi, \xi \rangle$ on B . Thus, denoting by $\omega_n : \mathbb{M}_n(B) \rightarrow \mathbb{M}_n$ the map induced by ω , we have $y = \omega_n(z)$, where $z = u^*xu$. Since $\overline{\text{co}}_R(z) = \overline{\text{co}}_R(x)$, it suffices now to prove that $y \in \overline{\text{co}}_R(z)$ (for then we will have $y \in \overline{\text{co}}_R(x) \cap A$). Let $\epsilon > 0$. Since ω is pure and $\omega(z_{ij}) = y_{ij}$ (where z_{ij} are the entries of z), by Lemma 2.1 there exists a positive element $e (= hh^*)$ in B such that $\|e\| = 1$ and

$$(2.2) \quad \|e(z_{ij} - y_{ij})e\| < \epsilon \quad (i, j = 1, \dots, n).$$

Moreover, replacing e by a suitable spectral projection of e in R , we may assume that e itself is a projection. Since B is a factor, there is a family of partial isometries $u_k \in B$ such that $u_k u_k^* \leq e$ and $\sum_k u_k^* u_k = 1$. Now (2.2) implies that

$$\left\| \sum_k u_k^* z_{ij} u_k - y_{ij} \right\| = \left\| \sum_k u_k^* (z_{ij} - y_{ij}) u_k \right\| \leq \epsilon \quad (i, j = 1, \dots, n),$$

hence $\left\| \sum_k u_k^* z_{ij} u_k - y_{ij} \right\| \leq n\epsilon$, where $u_k^{(n)} \in \mathbb{M}_n(B)$ is the diagonal matrix with u_k on the main diagonal. This implies that the distance of y to $\overline{\text{co}}_R(z) = \overline{\text{co}}_R(x)$ is dominated by $n\epsilon$ for all $\epsilon > 0$, hence $y \in \overline{\text{co}}_R(x)$.

Consider now a more general situation, when σ is a direct sum of a family of irreducible representations $\sigma_k : R \rightarrow B(\mathcal{K}_k)$. Then, with respect to the decomposition $\mathcal{K} = \bigoplus_k \mathcal{K}_k$, V can be represented as a (possibly infinite) column $V = (v_k)$, where each $v_k \in B(\mathbb{C}^n, \mathcal{K}_k)$ has the polar decomposition $v_k = u_k |v_k|$ with $|v_k| = \sqrt{u_k^* v_k}$ and $u_k \in B(\mathbb{C}^n, \mathcal{K}_k)$ an isometry. (This is possible since $\dim \mathcal{K}_k \geq n$ even if R is finite dimensional, since $R \supseteq A \cong \mathbb{M}_n$ and σ_k is injective if the factor R is finite dimensional.) Now $y = V^* \sigma(x) V = \sum_k |v_k| (u_k^* \sigma_k(x) u_k) |v_k|$. Since each σ_k is irreducible and u_k an isometry, $u_k^* \sigma_k(x) u_k \in \overline{\text{co}}_R(x)$ by the argument from the previous paragraph, hence it follows easily that $y \in \overline{\text{co}}_R(x)$ since $\sum_k |v_k|^2 = V^* V = 1$.

Finally, in general we can approximate the map $\phi(\cdot) = V^* \sigma(\cdot) V$ in the point-norm topology by the maps of the form $V_\nu^* \sigma_\nu(\cdot) V_\nu$, where each σ_ν is a direct sum of irreducible representations. To get such an approximation one can use the fact that each state on $\mathbb{M}_n(R)$ can be approximated in the weak* topology by convex combinations of pure states and the well known connection between completely positive maps from R to \mathbb{M}_n and positive functionals on $\mathbb{M}_n(R)$ (see [13, Chapter 5], we leave the details to the reader). Thus, in any case we have $y \in \overline{\text{co}}_R(x)$, hence $W_n(x) \subseteq \overline{\text{co}}_R(x) \cap A$ (since by definition $W_n(x) \subseteq A \cong \mathbb{M}_n$).

Let now K be any weak* compact C*-convex subset of R and $\phi : R \rightarrow A$ a unital completely positive map. Then $\phi(x) \in W_n(x) \subseteq \overline{\text{co}}_R(x) \subseteq K$ for each $x \in K$. Thus $\phi(K) \subseteq K$. □

Proof of Lemma 1.3. Let $x \in \text{ext}_A(K \cap A)$ and suppose that

$$x = \sum_{j=1}^n a_j x_j a_j,$$

where $x_j \in K$, $a_j \in R$ is positive and invertible for each j and $\sum_{j=1}^n a_j^2 = 1$. Then for each conditional expectation $E : R \rightarrow A$ we have

$$(2.3) \quad x = \sum_{j=1}^n E(a_j x_j a_j) = \sum_{j=1}^n E(a_j^2)^{1/2} \phi_j(x_j) E(a_j^2)^{1/2},$$

where $\phi_j : R \rightarrow A$, $\phi_j(w) := E(a_j^2)^{-1/2} E(a_j w a_j) E(a_j^2)^{1/2}$ are unital completely positive mappings. Since $\phi_j(x_j) \in K \cap A$ by the hypothesis and $x \in \text{ext}_A(K \cap A)$,

(2.3) implies that $\phi_j(x_j) = x$ and $E(a_j^2)x = xE(a_j^2)$ for all j . By the definition of ϕ_j it follows now that $E(a_jx_ja_j) = E(a_j^2)^{1/2}xE(a_j^2)^{1/2} = E(a_j^2)x = xE(a_j^2)$, hence

$$(2.4) \quad E(a_jx_ja_j - a_j^2x) = 0 \quad \text{and} \quad E(a_j^2x - xa_j^2) = 0$$

since a conditional expectation from R to A is A -linear. By hypothesis the conditional expectations from R to A separate points of R , hence (2.4) implies that $a_jx_ja_j = a_j^2x$ and $a_j^2x = xa_j^2$, but since a_j is invertible and positive it follows that $a_jx = xa_j$ and $x_j = x$ for all $j = 1, \dots, n$. Thus $x \in \text{ext}_R(K)$. \square

3. THE FINITE DIMENSIONAL CASE

In this section we shall prove Theorem 1.1 in the case $R = \mathbb{M}_n$ ($n \in \mathbb{N}$). For convenience we recall now some results from [12] in a form suitable for our application.

Observe first that if $x \in \text{ext}_R(K)$ for some C*-convex set K in a unital C*-algebra R and if

$$x = \sum_{j=1}^n a_jx_ja_j, \text{ where } x_j \in K, a_j \geq 0, \sum_{j=1}^n a_j^2 = 1,$$

then $x_1 = x$ and a_1 commutes with x if a_1 is invertible. (To see this, note that $x = \frac{1}{2}a_1x_1a_1 + a^*(\bigoplus_{j=1}^n x_j)a$, where a is the column $(\frac{1}{\sqrt{2}}a_1, a_2, \dots, a_n)^T$, and use the polar decomposition of a to write x in the form $x = \frac{1}{2}a_1x_1a_1 + |a|y|a|$, where $y \in K$, a_1 and $|a|$ are invertible and $\frac{1}{2}a_1^2 + |a|^2 = 1$.)

The following lemma is a variation on [6, Theorem 4.1], but we do not require finite dimensionality of R .

Lemma 3.1. *Let K be a norm closed C*-convex subset in any von Neumann algebra R . Suppose that*

$$x = \sum_{j=1}^n a_jx_ja_j,$$

where $x_j \in K$, $a_j \in R$, $a_j \geq 0$ and $\sum_{j=1}^n a_j^2 = 1$. If $x \in \text{ext}_R(K)$, then the range projection p of a_1 reduces x and $xp = px_1p$. In particular, if x is irreducible in R (that is, the commutant of $\{x, x^*\}$ in R consists of scalars only) and $a_1 \neq 0$, then $x_1 = x$.

Proof. By the Dixmier approximation theorem ([9, 8.3.5]) K contains a central element, so translating by a central element, we may assume that $0 \in K$. Then $a^*ya \in K$ for all $y \in K$ and $a \in R$ with $\|a\| \leq 1$. Using the polar decomposition of the column $(a_2, \dots, a_n)^T$ we may write x in the form $x = a_1x_1a_1 + aya$, where $y \in K$ and $a = \sqrt{1 - a_1^2}$. For each $\alpha \in [0, 1]$ we have

$$(3.1) \quad x = (1 - \alpha)aya + (\alpha aya + a_1x_1a_1) = (1 - \alpha)aya + |t(\alpha)|u(\alpha)^*(y \oplus x_1)u(\alpha)|t(\alpha)|,$$

where $t(\alpha) = (\alpha^{1/2}a)$ and $t(\alpha) = u(\alpha)|t(\alpha)|$ is the polar decomposition of $t(\alpha)$. Since $|t(\alpha)|^2 = \alpha a^2 + a_1^2$ is invertible if $\alpha > 0$, (3.1) implies that $u(\alpha)^*(y \oplus x_1)u(\alpha) = x$ if $\alpha \in (0, 1]$ by the observation preceding the lemma. But from

$$u(\alpha) = t(\alpha)|t(\alpha)|^{-1} = \begin{pmatrix} \alpha^{1/2}a(\alpha a^2 + a_1^2)^{-1/2} \\ a_1(\alpha a^2 + a_1^2)^{-1/2} \end{pmatrix} \quad (\alpha \in (0, 1])$$

we see that $\lim_{\alpha \rightarrow 0} u(\alpha) = \begin{pmatrix} p^\perp \\ p \end{pmatrix}$ in the strong operator topology, hence by continuity $x = \lim_{\alpha \rightarrow 0} u(\alpha)^*(y \oplus x_1)u(\alpha) = px_1p + p^\perp y p^\perp$ (where the convergence is in the weak operator topology). This implies that $px = xp = px_1p$. If x is irreducible and $a_1 \neq 0$, then $p = 1$, hence $x = x_1$. \square

It can be proved [11] that each weak* compact C*-convex subset K of a factor R contains a scalar multiple of 1 as an R -extreme point (in the case $R = \mathbb{M}_n$, which is the only case that will be needed in this section, this also follows from [4]). Thus, translating, we may assume without loss of generality that $0 \in \text{ext}_R(K)$.

From now on let K be a C*-convex compact subset of \mathbb{M}_n with $0 \in \text{ext}_{\mathbb{M}_n}(K)$.

Notation. For each $k \leq n$ let $P_k \in \mathbb{M}_n$ be the orthogonal projection onto the first k coordinates. Put

$$K_k = P_k K P_k$$

and regard K_k as a subset of \mathbb{M}_k by identifying \mathbb{M}_k with $P_k \mathbb{M}_n P_k$. It is easy to see that K_k is C*-convex. For each $x \in \mathbb{M}_k$ we shall denote by \tilde{x} the element $x \oplus 0 \in \mathbb{M}_n$.

Remark 3.2. If x is an irreducible C*-extreme point of K , then $x \in \text{ext}_{\mathbb{M}_n}(K)$ by [12, Corollary 1.8]. In our present situation this can also be seen in a more straightforward way as follows. Suppose that $x = \sum_{j=1}^2 a_j x_j a_j$, where $x_j \in K$ and all $a_j \neq 0$ are positive with $\sum_{j=1}^2 a_j^2 = 1$. Then by Lemma 3.1 $x_j = x$ for all j , hence

$$(3.2) \quad x = \sum_{j=1}^2 a_j x a_j.$$

Writing a_1 (and hence also $a_2 = \sqrt{1 - a_1^2}$) as a diagonal matrix, $a_1 = \bigoplus_{i=1}^m \alpha_i 1_{n_i}$, where $\alpha_i \neq \alpha_k$ if $i \neq k$ and writing x correspondingly as a block matrix, $x = [x_{ik}]$ ($x_{ik} \in \mathbb{M}_{n_i, n_k}$), it follows from (3.2) by a straightforward computation that $x_{ik} = 0$ if $i \neq k$. But, since x is irreducible, this implies that $m = 1$ and a_1, a_2 are scalar multiples of 1.

In general, K may have no irreducible C*-extreme points and one has to consider also the possible irreducible C*-extreme points of the compressions K_k . Note that if x is an irreducible C*-extreme point of K_k ($k < n$) such that x is a compression of an irreducible C*-extreme point y of K_j for some $k < j \leq n$, say $\tilde{x} = p\tilde{y}p$ for a projection $p \in \mathbb{M}_n$, then \tilde{x} can be expressed as the C*-convex combination $\tilde{x} = p\tilde{y}p + p^\perp 0 p^\perp$.

Notation. Denote by S_n the set of all irreducible C*-extreme points of K and (for $k < n$) by S_k the set of all irreducible C*-extreme points of K_k which are not compressions of irreducible C*-extreme points of K_j for $k < j \leq n$. (The elements of K_k are called structural elements in [12].)

From [12, Theorem 4.5] or [15, Theorem 4.3] and [5] we have the following.

Theorem 3.3 ([12]). *Each $x \in K$ can be expressed as a finite sum*

$$x = \sum_{i=1}^l t_i^* x_i t_i,$$

where $x_i \in S_{n_i}$ ($n_i \in \mathbb{N}$, $n_i \leq n$), $t_i \in \mathbb{M}_{n_i, n}$ and $\sum_{i=1}^l t_i^* t_i = 1$. Thus K is equal to the C^* -convex hull in \mathbb{M}_n of the set $\tilde{S} := \bigcup_{k=1}^n \tilde{S}_k$, where $\tilde{S}_k = \{\tilde{y} : y \in S_k\}$.

We will show (Proposition 3.5) that elements of the form $\bigoplus_{i=1}^r x_i$, where $x_i \in S_{n_i}$ ($\sum_{i=1}^r n_i = n$) are \mathbb{M}_n -extreme in K . Since each \tilde{x}_i can obviously be expressed as a C^* -convex combination of $\bigoplus_{i=1}^r x_i$ and 0, it follows then from Theorem 3.3 (since $0 \in \text{ext}_{\mathbb{M}_n}(K)$) that $K = \text{co}_{\mathbb{M}_n}(\text{ext}_{\mathbb{M}_n}(K))$. This will prove Theorem 1.1 in the case $R = \mathbb{M}_n$.

We need the following slight variation of [12, Lemma 4.6]. For convenience of the reader we shall present a simple proof.

Lemma 3.4 ([12]). *Let $x \in K_k$ ($k \leq n$) be of the form $x = \bigoplus_{i=1}^r x_i$, where $x_i \in S_{m_i}$ ($m_i \in \mathbb{N}$). Suppose that $x = v^* y v$ for some $y \in K$ and some isometry $v \in \mathbb{M}_{n, k}$. Then there exist a unitary $U \in \mathbb{M}_n$ and a matrix $z \in K_{n-k}$ such that $y = U(x \oplus z)U^*$ and $v = U \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, where $1 \in \mathbb{M}_k$ is the identity matrix.*

Proof. Assume first that $r = 1$ (that is, $x \in S_k$) and $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (that is, v is the inclusion of C^k into C^n). By Theorem 3.3

$$(3.3) \quad y = \sum_{j=1}^l t_j^* y_j t_j,$$

where $y_j \in S_{n_j}$, $t_j \in \mathbb{M}_{n_j, n}$ and $\sum_{j=1}^l t_j^* t_j = 1$. Let $t_j v = u_j |t_j v|$ be the polar decomposition of $t_j v$ with $u_j \in \mathbb{M}_{n_j, k}$ an isometry or a coisometry. Then

$$x = v^* y v = \sum_{j=1}^l |t_j v| (u_j^* y_j u_j) |t_j v|,$$

where $u_j^* y_j u_j \in K$ (since K is C^* -convex and $0 \in K$). Put $\alpha_j = |t_j v|$. Since $x \in S_k$ (hence x is irreducible and C^* -extreme in K_k), $x \in \text{ext}_{\mathbb{M}_k}(K_k)$ by Remark 3.2, hence if $\alpha_j \neq 0$, then Lemma 3.1 and the irreducibility of x imply that $u_j^* y_j u_j = x$ and α_j is a scalar. Decompose the set $\mathbb{L} := \{1, \dots, l\}$ as $\mathbb{L} = \mathbb{L}_1 \cup \mathbb{L}_2$, where $\mathbb{L}_1 = \{j \in \mathbb{L} : \alpha_j \neq 0\}$ and $\mathbb{L}_2 = \mathbb{L} \setminus \mathbb{L}_1$. Since u_j is an isometry or a coisometry, the identity $u_j^* y_j u_j = x$ means that one of x, y_j is unitarily equivalent to the compression of the other; but since $y_j \in S_{n_j}$ and $x \in S_k$ this is possible only if $y_j \cong x$ (where \cong denotes unitary equivalence). Hence u_j must be unitary and $n_j = k$ for all $j \in \mathbb{L}_1$. The identity $t_j v = \alpha_j u_j$ then implies (since $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$) that $t_j = (\alpha_j u_j, b_j)$ for some $b_j \in \mathbb{M}_{k, n-k}$. If $j \in \mathbb{L}_2$, then $t_j v = u_j \alpha_j = 0$ implies that t_j is of the form $t_j = (0, b_j)$. Thus from (3.3) we have (since $\sum_{j \in \mathbb{L}_1} \alpha_j^2 = \sum_{j=1}^l |t_j v|^2 = v^* \sum_{j=1}^l t_j^* t_j v = 1$)

$$y = \sum_{j \in \mathbb{L}_1} \begin{pmatrix} \alpha_j u_j^* \\ b_j^* \end{pmatrix} u_j x u_j^* (\alpha_j u_j, b_j) + \sum_{j \in \mathbb{L}_2} \begin{pmatrix} 0 \\ b_j^* \end{pmatrix} y_j (0, b_j) = \begin{pmatrix} x & g \\ h & z \end{pmatrix},$$

where $z \in \mathbb{M}_{n-k}$,

$$(3.4) \quad g = \sum_{j \in \mathbb{L}_1} \alpha_j x u_j^* b_j \quad \text{and} \quad h = \sum_{j \in \mathbb{L}_1} \alpha_j b_j^* u_j x.$$

But, from

$$1 = \sum_{j=1}^l t_j^* t_j = \sum_{j \in \mathbb{L}_1} \begin{pmatrix} \alpha_j u_j^* \\ b_j^* \end{pmatrix} (\alpha_j u_j, b_j) + \sum_{j \in \mathbb{L}_2} \begin{pmatrix} 0 \\ b_j^* \end{pmatrix} (0, b_j)$$

we see (looking at the entries on the position (1,2)) that $\sum_{j \in \mathbb{L}_1} \alpha_j u_j^* b_j = 0$, hence (3.4) implies that $g = 0$. Similarly $h = 0$ and y is of the form $x \oplus z$.

Assume now that $x = \bigoplus_{i=1}^r x_i$, where $r > 1$, but v is still of the form $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so that y can be represented as a matrix of the form

$$y = \begin{pmatrix} x & g \\ h & z \end{pmatrix}.$$

Applying (to elements x_i) what we have just proved above, it follows inductively that the successive block rows and columns of the matrices g and h (respectively) corresponding to the summands x_i of x are all 0. Thus $g = 0$, $h = 0$ and $y = x \oplus z$.

Finally, a general isometry $v : \mathbb{C}^k \rightarrow \mathbb{C}^n$ can be extended to a unitary operator U on \mathbb{C}^n , so that $v = U \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the proof is completed by applying the above argument to U^*yU in place of y . \square

Proposition 3.5. *If $x \in K$ is of the form $x = \bigoplus_{i=1}^r x_i$, where $x_i \in S_{n_i}$ ($n_i \in \mathbb{N}$), then $x \in \text{ext}_{\mathbb{M}_n}(K)$.*

Proof. Suppose first that $x \cong y^{(r)}$ (= the direct sum of r copies of y) for some $y \in S_k$, where $rk = n$. Since y is an irreducible C^* -extreme point of K_k , y is \mathbb{M}_k -extreme in K_k (see Remark 3.2). Observe that the canonical isomorphism $\mathbb{M}_k \cong \mathbb{M}_k^{(r)} \subseteq \mathbb{M}_n$ identifies K_k with $K \cap \mathbb{M}_k^{(r)}$. (Note that $w^{(r)} = \sum_{j=1}^r e_{j1} w e_{1j} \in K$ for each $w \in K_k$ since K is C^* -convex and $0 \in K$, where $\{e_{ij}\}_{i,j=1}^r$ is the appropriate matrix unit in $\mathbb{M}_r(\mathbb{M}_k)$, namely the standard matrix unit in \mathbb{M}_r tensored with $1 \in \mathbb{M}_k$.) It follows then by Lemma 1.3 (and Theorem 1.2 which justifies the application of Lemma 1.3, as in Section 1) that $y^{(r)} \in \text{ext}_{\mathbb{M}_n}(K)$.

Thus we may assume now that not all the summands x_i of x are mutually unitarily equivalent, hence the set $\{1, \dots, r\}$ can be partitioned into two parts \mathbb{I} and \mathbb{J} such that $x_i \not\cong x_j$ if $i \in \mathbb{I}$ and $j \in \mathbb{J}$. Put

$$y = \bigoplus_{i \in \mathbb{I}} x_i \quad \text{and} \quad z = \bigoplus_{j \in \mathbb{J}} x_j$$

and let $k = \sum_{i \in \mathbb{I}} n_i$, $l = \sum_{j \in \mathbb{J}} n_j$. By induction on the dimension we may assume that $y \in \text{ext}_{\mathbb{M}_k}(K_k)$ and $z \in \text{ext}_{\mathbb{M}_l}(K_l)$. To prove that $x \in \text{ext}_{\mathbb{M}_n}(K)$, suppose that

$$(3.5) \quad x = \sum_{j=1}^m t_j^* w_j t_j,$$

where $w_j \in K$, $\sum_{j=1}^m t_j^2 = 1$ and each $t_j \in \mathbb{M}_n$ is positive and invertible. According to the decomposition $x = y \oplus z$, decompose each t_j as $t_j = (a_j, b_j)$, where $a_j \in \mathbb{M}_{n,k}$, $b_j \in \mathbb{M}_{n,l}$, and let $a_j = u_j |a_j|$, $b_j = v_j |b_j|$ be the polar decompositions. Then (3.5) implies (by considering the two diagonal blocks of sizes $k \times k$ and $l \times l$, respectively) that

$$y = \sum_{j=1}^m |a_j| (u_j^* w_j u_j) |a_j| \quad \text{and} \quad z = \sum_{j=1}^m |b_j| (v_j^* w_j v_j) |b_j|.$$

Since $y \in \text{ext}_{\mathbb{M}_k}(K_k)$ and $z \in \text{ext}_{\mathbb{M}_l}(K_l)$ it follows now that $|a_j|y = y|a_j|$, $|b_j|z = z|b_j|$ and

$$(3.6) \quad u_j^* w_j u_j = y, \quad v_j^* w_j v_j = z \quad (j = 1, \dots, m).$$

Since t_j is invertible, a_j is injective, hence $|a_j|$ is invertible and u_j is an isometry. Similarly v_j is an isometry. Thus (3.6) implies by Lemma 3.4 that there exist unitaries $U_j, V_j \in \mathbb{M}_n$ such that $u_j = U_j \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_j = V_j \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and

$$(3.7) \quad w_j = U_j(y \oplus h_j)U_j^*, \quad w_j = V_j(g_j \oplus z)V_j^*$$

for some matrices g, h . Putting $W_j = U_j^*V_j$, we have from (3.7) that $(y \oplus h_j)W_j = W_j(g_j \oplus z)$ and $(y \oplus h_j)^*W_j = W_j(g_j \oplus z)^*$. Since no irreducible direct summand x_i of y is unitarily equivalent to any irreducible direct summand x_j of z , the only operator intertwining the identity representations of $C^*(y)$ and $C^*(z)$ is 0, hence $W_j = c_j \oplus d_j$ for some unitary operators $c_j \in \mathbb{M}_k$ and $d_j \in \mathbb{M}_l$. Then $g_j = c_j^*y c_j$, $h_j = d_j z d_j^*$, $w_j = U_j(y \oplus d_j z d_j^*)U_j^*$ (from (3.7)), $u_j = U_j \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_j = V_j \begin{pmatrix} 0 \\ 1 \end{pmatrix} = U_j W_j \begin{pmatrix} 0 \\ 1 \end{pmatrix} = U_j(c_j \oplus d_j) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = U_j \begin{pmatrix} 0 \\ d_j \end{pmatrix}$. Thus $a_j = u_j |a_j| = U_j \begin{pmatrix} |a_j| \\ 0 \end{pmatrix}$, $b_j = v_j |b_j| = U_j \begin{pmatrix} 0 \\ |b_j| \end{pmatrix}$ and therefore

$$t_j = (a_j, b_j) = U_j(|a_j| \oplus |b_j|) = U_j(1 \oplus d_j)(|a_j| \oplus |b_j|).$$

Since t_j and $|a_j| \oplus |b_j|$ are positive, while $U_j(1 \oplus d_j)$ is unitary, it follows now by the uniqueness of the polar decomposition that $U_j(1 \oplus d_j) = 1$ and $t_j = |a_j| \oplus |b_j|$. This clearly implies that t_j commutes with $x = y \oplus z$ (since $|a_j|$ and $|b_j|$ commute with y and z , respectively) and $w_j = U_j(y \oplus d_j z d_j^*)U_j^* = U_j(1 \oplus d_j)(y \oplus z)(1 \oplus d_j^*)U_j^* = y \oplus z = x$. \square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LJUBLJANA, JADRANSKA 19, LJUBLJANA 1000,
SLOVENIA

E-mail address: `Bojan.Magajna@fmf.uni-lj.si`