

COMPACT RANGE PROPERTY AND OPERATORS ON C^* -ALGEBRAS

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ABSTRACT. We prove that a Banach space E has the compact range property (CRP) if and only if, for any given C^* -algebra \mathcal{A} , every absolutely summing operator from \mathcal{A} into E is compact. Related results for p -summing operators ($0 < p < 1$) are also discussed as well as operators on non-commutative L^1 -spaces and C^* -summing operators.

1. INTRODUCTION

A Banach space E is said to have the compact range property (CRP) if every E -valued countably additive measure of bounded variation has compact range. It is well known that every Banach space with the Radon-Nikodym property (RNP) has the (CRP) and for dual Banach spaces, the (CRP) were completely characterized as those whose predual do not contain any copies ℓ^1 . For more in depth discussions on Banach spaces with the (CRP), we refer to [9].

The following characterization can be found in [9]: A Banach space E has the (CRP) if and only if every 1-summing operator from $C[0, 1]$ into E is compact. Since $C[0, 1]$ is a (commutative) C^* -algebra, it is a natural question whether $C[0, 1]$ can be replaced by any C^* -algebras. Let us recall that in [7] it was shown that if X is a Banach space that does not contain any copies of ℓ^1 , then any 1-summing operator from any given C^* -algebra into X^* is compact, hinting that, as in commutative case, the (CRP) is the right condition to provide compactness. The present note is an improvement of [7]. Our main result confirms that, if \mathcal{A} is a C^* -algebra and E is a Banach space that has the (CRP), then every 1-summing from \mathcal{A} into E is compact. Our proof relies on factorizations of summing operators used in [7] and properties of integral operators.

There is another well known characterization of spaces with the (CRP) in terms of operators defined on $L^1[0, 1]$: a Banach space E has the (CRP) if and only every operator T from $L^1[0, 1]$ into E is Dunford-Pettis (completely continuous). Thus the (CRP) is also referred to as the *complete continuity property* (CCP). Unlike the 1-summing operators on C^* -algebras, operators defined on non-commutative L^1 -spaces do not behave the same way as those defined on $L^1[0, 1]$ do. In the

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last section of this note, we will discuss these operators along with C^* -summing operators studied by Pisier in [6].

Our terminology and notation are standard as may be found in [2] and [4] for Banach spaces, [5] and [8] for C^* -algebras and operator algebras.

2. PRELIMINARIES

In this section, we recall some definitions.

Definition 1. Let X and Y be Banach spaces and $0 < p < \infty$. An operator $T : X \rightarrow Y$ is said to be p -summing if there is a constant C such that for any finite sequence (x_1, x_2, \dots, x_n) of X , one has

$$\left(\sum_{i=1}^n \|Tx_i\|^p \right)^{\frac{1}{p}} \leq C \sup \left\{ \left(\sum_{i=1}^n |\langle x_i, x^* \rangle|^p \right)^{\frac{1}{p}} ; x^* \in X^*, \|x^*\| \leq 1 \right\}.$$

The smallest constant C for which the above inequality holds is denoted by $\pi_p(T)$ and is called the p -summing norm of T .

Definition 2. We say that an operator $T : X \rightarrow Y$ is an integral operator if it admits a factorization:

$$\begin{array}{ccc} X & \xrightarrow{i \circ T} & Y^{**} \\ \downarrow \alpha & & \uparrow \beta \\ L^\infty(\mu) & \xrightarrow{J} & L^1(\mu) \end{array}$$

where i is the natural inclusion from Y into Y^{**} , μ is a probability measure on a compact space K , J is the natural inclusion and α and β are bounded linear operators.

We define the integral norm $i(T) := \inf \{ \|\alpha\| \cdot \|\beta\| \}$ where the infimum is taken over all such factorizations.

Similarly, we shall say that T is *strictly integral* if T is integral and on the factorization above β takes its values in Y .

It is well known that integral operators are 1-summing but the converse is not true.

If $X = C(K)$ where K is a compact Hausdorff space, then it is well known that every 1-summing operator from X into Y is integral.

For more details on the different properties of the classes of operators involved, we refer to [3].

The following simple fact will be needed in the sequel.

Proposition 3. *Let $T : X \rightarrow Y$ be a strictly integral operator. If Y has the (CRP), then T is compact.*

Proof. The operator T has a factorization $T = \beta \circ J \circ \alpha$ where $\alpha : X \rightarrow L^\infty(\mu)$, $J : L^\infty(\mu) \rightarrow L^1(\mu)$ and $\beta : L^1(\mu) \rightarrow Y$ are as in the above definition. Note that J is 1-summing so $\beta \circ J : L^\infty(\mu) \rightarrow Y$ is 1-summing and since $L^\infty(\mu)$ is a $C(K)$ -space and Y has the (CRP), $\beta \circ J$ (and hence T) is compact. \square

We recall that a von Neumann algebra \mathcal{M} is said to be σ -finite if the identity is countably decomposable, equivalently if there exists a faithful state $\varphi \in \mathcal{M}_*$. As is customary, for every functional $\varphi \in \mathcal{M}_*$ and $x \in \mathcal{M}$, $x\varphi$ (resp. φx) denotes the normal functional $y \rightarrow \varphi(yx)$ (resp. $y \rightarrow \varphi(xy)$).

3. MAIN RESULT

Theorem 4. *For a Banach space E , the following are equivalent:*

- (1) E has the CRP;
- (2) Every 1-summing operator $T : C[0, 1] \rightarrow E$ is compact;
- (3) For any given C^* -algebra \mathcal{A} , every 1-summing operator $T : \mathcal{A} \rightarrow E$ is compact.

The equivalence (1) \Leftrightarrow (2) is well known; we refer to [4], [9] for more details. Clearly (3) \Rightarrow (2) so what we need to show is (1) \Rightarrow (3). For this, it is enough to consider the following particular case (see [7] for this reduction).

Proposition 5. *Let E be a Banach space with the (CRP) and \mathcal{M} be a σ -finite von Neumann algebra. If $T : \mathcal{M} \rightarrow E$ is 1-summing and is weak* to weakly continuous, then T is compact.*

Proof. The proof is a refinement of the argument used in Proposition 3.2 of [7]. We will include most of the details for completeness. Without loss of generality, we can and do assume that E is separable.

Let $\delta > 0$. From Lemma 2.3 of [7],

$$\|Tx\| \leq 2(1 + \delta)\pi_1(T)\|xf + fx\|_{\mathcal{M}_*} \quad \text{for every } x \in \mathcal{M},$$

where f is a faithful normal state in \mathcal{M}_* . If $L^2(f)$ is completion of the prehilbertian space $(\mathcal{M}, \langle \cdot, \cdot \rangle)$ where $\langle x, y \rangle = f(\frac{xy^* + y^*x}{2})$, then we have the following factorization:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{T} & E \\ J \downarrow & & \swarrow L \\ L^2(f) & \xrightarrow{\theta} & L^2(f)^* \xrightarrow{J^*} \mathcal{M}_* \end{array}$$

where J is the inclusion map, $\theta(Jx) = \langle \cdot, J(x^*) \rangle$ for every $x \in \mathcal{M}$ and $L(\frac{xf+fx}{2}) = Tx$. We recall that L is a well defined bounded linear map since $\{xf + fx; x \in \mathcal{M}\}$ is dense in \mathcal{M}_* and $\|L(xf + fx)\| \leq 4(1 + \delta)\pi_1(T)\|xf + fx\|_{\mathcal{M}_*}$. Let $S := J^* \circ \theta \circ J$.

Claim: $J \circ L^* : E^* \rightarrow L^2(f)$ is compact.

For this, let us consider $L^* : E^* \rightarrow \mathcal{M}$. Since E is separable, it is isometric to a subspace of $C[0, 1]$. Let I_E be the isometric embedding of E in $C[0, 1]$ and i be the natural inclusion of $C[0, 1]$ into $C[0, 1]**$.

Define the following map \tilde{T} from \mathcal{M} into $C[0, 1]$ by setting $\tilde{T} = \overline{I_E \circ T(x^*)}$ for every $x \in \mathcal{M}$. (Here, \bar{f} is the map $t \rightarrow \overline{f(t)}$ for $f \in C[0, 1]$ with $\bar{f}(t)$ being the conjugate of the complex number $f(t)$.)

Clearly, \tilde{T} is linear and bounded and it can be shown that \tilde{T} is 1-summing and is weak* to weakly continuous. In fact, if (x_1, x_2, \dots, x_n) is a finite sequence in \mathcal{M} , then

$$\begin{aligned} \sum_{i=1}^n \|\tilde{T}x_i\| &= \sum_{i=1}^n \|\overline{I_E \circ T(x_i^*)}\| \\ &= \sum_{i=1}^n \|I_E \circ T(x_i^*)\| \\ &= \sum_{i=1}^n \|T(x_i^*)\| \\ &\leq \pi_1(T) \sup \left\{ \sum_{i=1}^n |\langle x_i^*, \varphi \rangle|, \varphi \in \mathcal{M}^*, \|\varphi\| \leq 1 \right\} \\ &\leq \pi_1(T) \sup \left\{ \sum_{i=1}^n |\langle x_i, \varphi^* \rangle|, \varphi \in \mathcal{M}^*, \|\varphi\| \leq 1 \right\}, \end{aligned}$$

so \tilde{T} is 1-summing with $\pi_1(\tilde{T}) \leq \pi_1(T)$. Moreover if $(x_\alpha)_\alpha$ is a net that converges to zero weak* in \mathcal{M} , so does the net $(x_\alpha^*)_\alpha$ and since T is weak* to weakly continuous, $(T(x_\alpha^*))_\alpha$ converges to zero weakly in E and hence $(\tilde{T}(x_\alpha))_\alpha$ is weakly null which shows that \tilde{T} is weak* to weakly continuous.

To complete the proof, consider

$$E^* \xrightarrow{L^*} \mathcal{M} \xrightarrow{i \circ \tilde{T}} C[0, 1]**.$$

Since $C[0, 1]**$ has the Hahn-Banach extension property and $i \circ \tilde{T}$ is 1-summing, $i \circ \tilde{T}$ is an integral operator. Let $K : C[0, 1]^* \rightarrow \mathcal{M}_*$ such that $K^* = i \circ \tilde{T}$ (such an operator exists since $i \circ \tilde{T}$ is weak* to weakly continuous); K is integral ([3]) and since \mathcal{M}_* is a complemented subspace of its bidual \mathcal{M}^* (see for instance [8]), K is strictly integral and therefore $L \circ K : C[0, 1]^* \rightarrow E$ is strictly integral and by Proposition 3, $L \circ K$ (and hence $(L \circ K)^* = i \circ \tilde{T} \circ L^*$) is compact.

Let (U_n) be a bounded sequence in E^* . There exists a subsequence (U_{n_k}) so that $(i \circ \tilde{T} \circ L^*(U_{n_k}))_k$ is norm convergent in $C[0, 1]**$. Since i and I_E are isometries, we get that $(T \circ L^*(U_{n_k}))_k$ is norm convergent so

$$\lim_{k,m} \|T(L^*(U_{n_k})^*) - T(L^*(U_{n_m})^*)\| = 0.$$

As in [7], we get

$$\begin{aligned} &\lim_{k,m} \langle T(L^*(U_{n_k})^*) - T(L^*(U_{n_m})^*), U_{n_k} - U_{n_m} \rangle \\ &= \lim_{k,m} \|J \circ L^*(U_{n_k} - U_{n_m})\|_{L^2(f)}^2 = 0 \end{aligned}$$

which proves that $(J \circ L^*(U_{n_k}))_k$ is norm-convergent in $L^2(f)$. The proof is complete. \square

Theorem 6. *Let \mathcal{A} be a C^* -algebra, E be a Banach space and $0 < p < 1$. Every p -summing operator from \mathcal{A} into E is compact.*

Proof. Let $T : \mathcal{A} \rightarrow E$ be an operator with $\pi_p(T) < \infty$. One can choose, by the Pietsch Factorization Theorem, a probability space (Ω, Σ, μ) such that

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{T} & E \\
 \downarrow J & & \uparrow \tilde{T} \\
 S & \xrightarrow{i_p} & S_p \\
 \downarrow & & \downarrow \\
 L^\infty(\mu) & \xrightarrow{j_p} & L^p(\mu)
 \end{array}$$

where S is a subspace of $L^\infty(\mu)$, S_p is the closure of S in $L^p(\mu)$ and i_p is the restriction of the natural inclusion j_p .

Denote by S_1 the closure of S in $L^1(\mu)$, by i_1 the restriction of the natural inclusion and $i_{1,p}$ the natural inclusion of S_1 into S_p .

Claim: $\tilde{T} \circ i_{1,p} : S_1 \rightarrow E$ is weakly compact.

To see this, let $(f_n)_n$ be a bounded sequence in $S_1 \subset L^1(\mu)$. By Komlòs's Theorem, there exist a subsequence $(f_{n_k})_k$ and a function $f \in L^1(\mu)$ such that $\lim_{m \rightarrow \infty} \frac{1}{m} \sum_{k=1}^m f_{n_k}(\omega) = f(\omega)$ for a.e. $\omega \in \Omega$. Since $0 < p < 1$,

$$\lim_{m \rightarrow \infty} \left\| \frac{1}{m} \sum_{k=1}^m f_{n_k} - f \right\|_p = 0.$$

This shows that $f \in S_p$ and $\left(\tilde{T} \circ i_{1,p} \left(\frac{1}{m} \sum_{k=1}^m f_{n_k} \right) \right)_m$ converges to $\tilde{T}(f)$ in E and the claim follows.

Using the factorization of a weakly compact operator [1], $i_{1,p} \circ \tilde{T}$ factors through a reflexive space and since $i_1 \circ J$ is 1-summing, the theorem follows from Theorem 4. □

4. CONCLUDING REMARKS

Let us recall some definitions

Definition 7. Let X and Y be Banach spaces. An operator $T : X \rightarrow Y$ is called Dunford-Pettis if T sends weakly compact sets into norm compact sets.

The following class of operators was introduced by Pisier in [6] as extension of the p -summing operators in the setting of C^* -algebras.

Definition 8. Let \mathcal{A} be a C^* -algebra and E be a Banach space, $0 < p < \infty$. An operator $T : \mathcal{A} \rightarrow E$ is said to be p - C^* -summing if there exists a constant C such that for any finite sequence (A_1, \dots, A_n) of Hermitian elements of \mathcal{A} , one has

$$\left(\sum_{i=1}^n \|T(A_i)\|^p \right)^{\frac{1}{p}} \leq C \left\| \left(\sum_{i=1}^n |A_i|^p \right)^{1/p} \right\|_{\mathcal{A}}.$$

Let \mathcal{M} be a finite von Neumann algebra with a faithful tracial state τ and let J be the canonical inclusion map from \mathcal{M} into $L^1(\mathcal{M}, \tau)$. As in the commutative case, we have the following:

Proposition 9. *Let E be a Banach space and $T : L^1(\mathcal{M}, \tau) \rightarrow E$ a bounded linear map. Then the following are equivalent:*

- (i) T is Dunford-Pettis;
- (ii) $T \circ J$ is compact.

Proof. (i) \implies (ii) is trivial. For the converse, let $(a_n)_n$ be a weakly null sequence in the unit ball of $L^1(\mathcal{M}, \tau)$. It is clear that $(a_n^*)_n$ is also weakly null so without loss of generality, we can assume that $(a_n)_n$ is a sequence of self-adjoint operators. For each $n \geq 1$, set $a_n = \int_{-\infty}^{\infty} t \, de_t^{(n)}$, the spectral decomposition of a_n , and for every $N \geq 1$, let

$$p_{n,N} = \int_{-N}^N 1 \, de_t^{(n)}.$$

It is clear that for every $n \geq 1$ and $N \geq 1$,

$$\tau(\mathbf{1} - p_{n,N}) = \tau\left(\int_{\{|t|>N\}} 1 \, de_t^{(n)}\right) \leq \frac{1}{N} \tau(|a_n|).$$

By Akeman's characterization of relatively weakly compact subset in $L^1(\mathcal{M}, \tau)$ (see for instance [8], Theorem 5.4, p.149), we conclude that for any given $\epsilon > 0$, there is $N_0 \geq 1$ such that for every $n \geq 1$, $\|a_n(\mathbf{1} - p_{n,N_0})\| \leq \epsilon$. Moreover $(a_n p_{n,N_0})_n$ is a bounded sequence in \mathcal{M} and since $T \circ J$ is compact, there is a compact subset K_ϵ of E such that $\{T(a_n); n \in \mathbb{N}\} \subset K_\epsilon + \epsilon B_E$. The proof is complete. \square

Fix a type II₁ von Neumann algebra \mathcal{M} such that \mathcal{M} contains a complemented copy of a Hilbert space H . The space H is reflexive (and therefore has (CRP)) but the projection map P from $L^1(\mathcal{M}, \tau)$ onto H cannot be Dunford-Pettis.

A very well known property of p -summing operators is that they are Dunford-Pettis. This is not the case for C^* -summing operators in general. By Proposition 9, $P \circ J$ is not compact. We remark that the argument used in [7] requires only that the operator is C^* -summing and Dunford-Pettis; hence since J is clearly C^* -summing and $P \circ J$ is not compact, $P \circ J$ should not be Dunford-Pettis.

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