ANALYTIC NORMS IN ORLICZ SPACES

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(Communicated by Dale Alspach)

Abstract. It is shown that an Orlicz sequence space $h_M$ admits an equivalent analytic renorming if and only if it is either isomorphic to $l_2n$, or isomorphically polyhedral. As a consequence, we show that there exists a separable Banach space admitting an equivalent $C^\infty$-Fréchet norm, but no equivalent analytic norm.

In this note, we denote by $h_M$ as usual the subspace of an Orlicz sequence space $l_M$ generated by the unit vector basis.

More terminology and notation concerning Orlicz spaces can be found in [LT].

Let us also point out that by $C^k$-smoothness (or analyticity) of a norm we always mean away from the origin (as is usual in renorming theory).

The characterization of the best order of $C^k$-Fréchet smoothness of some renorming, $k \in \mathbb{N} \cup \{+\infty\}$, for $h_M$ was obtained in [M], [MT1], [MT2]. In our present note, we complete the characterization also for analytic renormings. We show that an Orlicz sequence space $h_M$ has an analytic renorming if and only if $h_M \cong l_2n$, $n \in \mathbb{N}$, or $h_M$ is isomorphically polyhedral. Let us recall that a separable Banach space $X$ is isomorphically polyhedral if it has an equivalent polyhedral norm. By a theorem of Fonf [F], this is the case if and only if $X$ admits an equivalent norm with a countable boundary. More precisely, there exists a sequence $\{f_i\}_{i \in \mathbb{N}}$ in $X^*$ such that

$$\|x\| = \max\{|f_i(x)|, \ i \in \mathbb{N}\}.$$  

According to one of the results from [DFH], we have the following:

Theorem 1. Every separable isomorphically polyhedral Banach space $X$ admits an equivalent analytic form.

We prove that the converse is also true if we impose additional conditions on the space $X$. In connection with our result it should be noted that by recent work of Gonzalo and Jaramillo ([GJ]) every separable Banach space with a symmetric basis and $C^\infty$-Fréchet smooth norm is isomorphic to $l_2n$, provided it does not contain a copy of $c_0$.

Our approach is entirely different from that in [MT1] and relies on methods from [DFH] and [H1]–[H3]. As a corollary, relying on an example of Leung [L], we show that there exists a separable Banach space with $C^\infty$-Fréchet smooth norm which...
admits no analytic norm. A search of such an example was in fact a motivation of our work, since the previously known examples of such spaces (e.g. $c_0(\Gamma)$, $\Gamma$ uncountable; see [P] and [BF] for a result of Kuiper) were nonseparable.

Let us recall that a Banach space $X$ with an unconditional basis is said to satisfy an upper $p$-estimate, $p \geq 1$, if for some $C > 0$:

$$\left\| \sum_{i=1}^{n} u_i \right\| \leq C \left( \sum_{i=1}^{n} \| u_i \|^p \right)^{\frac{1}{p}}$$

whenever $u_i$ are disjointly supported in $X$.

An important notion in our consideration is that of weak sequential continuity.

**Definition 2.** Let $U \subseteq X$ be an open, convex and bounded subset of a Banach space $X$, $f$ be a real function on $U$. We say that $f$ is weakly sequentially continuous (wsc-for short) if it maps weakly Cauchy sequences from $U$ into convergent ones. A function $f$ defined on an open subset $O \subseteq X$ is said to be locally wsc if there exists a covering of $O$ by a family of open sets $U$ as above such that $f$ is wsc on $U$ for all $U$.

In order to verify wsc-property for polynomials, it is sufficient to check the convergence only for weakly convergent sequences in $U$ ([AHV]).

Using this fact, the following lemma follows from results in [G].

**Lemma 3.** Let $X$ be a Banach space with an unconditional basis satisfying an upper $p$-estimate. Then all polynomials of degree $n < p$ on $X$ are wsc (on $B_X$).

The importance of the notion of wsc stems from the following lemma, which comes from [H3], and which was shown for polynomials in [AHV].

**Lemma 4.** Let $X$ be a Banach space $l_1 \nRightarrow X$, $f$ be a $C^2$-Fréchet differentiable real function defined on some open set $O \subseteq X$. TFAE:

1. $f$ is locally wsc,
2. $f'$ is locally norm compact.

By $f'$ being locally norm compact we mean that there exists a covering by a family of open sets $U$ of $O$ such that $f'(U)$ is relatively norm compact in $X^*$ for all $U$.

The following is a generalization of the main result in [H1].

**Theorem 5.** Let $(X, \| \cdot \|)$ be a Banach space, where $\| \cdot \|$ is analytic. If all polynomials on $X$ are wsc, then $X$ is separable and isomorphically polyhedral.

**Proof.** By $\partial$ we denote the duality map corresponding to $\| \cdot \|$, i.e.

$$\partial : X \setminus \{0\} \to S_X^* \quad \text{and} \quad \partial x(x) = \| x \| \quad \text{for all } x \in X \setminus \{0\}. \quad \text{(5.1)}$$

Since $\| \cdot \|$ is differentiable, $\partial x$ is the derivative of $\| x \|$ at $x \in X \setminus \{0\}$.

Let us first show that $\| \cdot \|$ is locally wsc on $X \setminus \{0\}$.

Fix $x \neq 0$. Since $\| \cdot \|$ is analytic at $x$ we can find $\delta > 0$ so that if $\| h \| < \delta$ we have

$$\| x + h \| = \sum_{n=1}^{\infty} p_n(h),$$

where $p_n(h)$ are the terms of the series decomposition.
Hahn-Banach theorem, there exists $x$ in this space.

On the other hand, by Theorem 6, no equivalent analytic norm exists on bounded subsets of $X$.

We proceed by showing that $X$ is separable. Since $X$ is an Asplund space, from Lemma 4 we get that there exist $0 < \eta < \delta$ so that the set $\{y : \|y - x\| < \eta\}$ is norm relatively compact. Thus the subspace $Y$ of $X^*$ generated by $\{\partial y : \|y - x\| < \eta\}$ is separable. If $Y = X^*$ the proof is finished. Otherwise, assume $Y \neq X^*$. By the Hahn-Banach theorem, there exists $x^{**} \in S_{X^{**}}$ such that

$$x^{**}(\partial y) = 0 \text{ whenever } \|y - x\| < \eta.$$  

Since $\|\cdot\|$ is analytic on $X \setminus \{0\}$ we get that $\partial$ is analytic as well on $X \setminus \{0\}$. Hence $f = x^{**} \circ \partial$ is a real analytic function on $X \setminus \{0\}$.

Since $f(y) = 0$ for $\|y - x\| < \eta$, clearly $f \equiv 0$ on $X \setminus \{0\}$. On the other hand, by the Bishop-Phelps theorem, $\partial S_X$ is dense in $S_{X^{**}}$, so there exists $y \in S_X$ such that $f(y) = x^{**}(\partial y) \neq 0$, a contradiction. So $X$ is separable.

Since $X$ is separable and $\|\cdot\|$ is locally wsc, by Lemma 3 and the Lindelöf property, $(S_X, \|\cdot\|)$ can be covered by a countable system $\{U_n\}_{n \in \mathbb{N}}$ of norm open convex bounded subsets of $X$ such that $\partial U_n$ is relatively compact. Thus the boundary of $(X, \|\cdot\|)$ can be covered by a countable system of compacts, and the result follows from [H2].

**Theorem 6.** Let $M$ be an Orlicz function. Then $h_M$ admits an equivalent analytic norm if and only if either $h_M \cong l_{2n}$, $n \in \mathbb{N}$, or $h_M$ is isomorphically polyhedral. In particular if

1. $\lim_{t \to 0} \frac{M(2t)}{M(t)} = +\infty$, then $h_M$ has an equivalent analytic norm.
2. $a_M = +\infty$ and there exists a sequence $t_i \downarrow 0$ such that $\sup_{t \in \mathbb{N}} \frac{M(at_i)}{M(t_i)} < +\infty$ for all $a > 1$, then $h_M$ does not admit an equivalent analytic norm.

**Proof.** The “if” part follows from the well-known result that the canonical norm on $l_{2n}$, $n \in \mathbb{N}$, is analytic and from Theorem 1.

The “only if” part: By classical results (LT1), the existence of an analytic norm on $X$ implies $\alpha_M = \beta_M \in \{2n\}_{n \in \mathbb{N}} \cup \{+\infty\}$.

The case $\alpha_M = 2n$ implies that $X \cong l_{2n}$ by [MT1].

If $\alpha_M = \infty$, then (LT1) $X$ has an upper $p$-estimate for every $p > 1$.

Combination of Lemma 3 and Theorem 1 finishes the proof of the “only if” part.

Leung [L] showed that if $M$ satisfies (1), then $h_M$ is isomorphically polyhedral and if $M$ satisfies (2) $h_M$ is not isomorphically polyhedral.

**Corollary 7.** There exists a $c_0$-saturated separable Banach space which admits an equivalent $C^\infty$-Fréchet norm but no equivalent analytic norm.

**Proof.** Leung [L] constructed an Orlicz function $M$ satisfying (2). By a result of [MT2] the corresponding space $h_M$ admits an equivalent $C^\infty$-Fréchet smooth norm. On the other hand, by Theorem 6 no equivalent analytic norm exists on this space.

Let us pass to some final remarks. A natural question is the following: Is there a separable $c_0$-saturated non-polyhedral Banach space with an equivalent analytic norm?
By a careful analysis of [DFH], we obtain that on every separable polyhedral space there exists a dense set of equivalent analytic norms whose boundaries can be covered by countably many compacts. Such norms in turn immediately imply the polyhedrality of the space (using [H2]).

However, there are examples of polyhedral spaces (e.g. [S], [PS]) with analytic norms failing this property.

More precisely, the space \( S \) of Schreier has an unconditional basis \( \{e_n\} \) such that the formal identity operator \( \text{id} \) from \( S \) into \( l_2 \) is bounded. It is easy to show that given an equivalent analytic norm \( \| \cdot \| \) on \( S \) whose boundary is covered by countably many compacts, the equivalent analytic norm

\[
\|x\| = (\|x\|^2 + \|\text{id}x\|^2)^{1/2}
\]

fails the covering property.

The problem is therefore how to recognize the polyhedrality of \( S \) based on its norm \( \| \cdot \| \).

ACKNOWLEDGEMENT

The authors would like to thank the Department of Analysis of Universidad Complutense in Madrid, for hospitality and excellent working conditions during the preparation of this note.

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