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ALMOST PERIODIC HYPERFUNCTIONS

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Abstract. We characterize the almost periodic hyperfunctions by showing that the following statements are equivalent for any bounded hyperfunction T. (i) T is almost periodic. (ii) $T * \varphi \in C_{ap}$ for every $\varphi \in \mathcal{F}$. (iii) There are two functions $f, g \in C_{ap}$ and an infinite order differential operator P such that $T = P(D^2)f + g$. (iv) The Gauss transform $u(x,t) = T * E(x,t)$ of T is almost periodic for every $t > 0$. Here C_{ap} is the space of almost periodic continuous functions, $\mathcal{F}$ is the Sato space of test functions for the Fourier hyperfunctions, and $E(x,t)$ is the heat kernel. This generalizes the result of Schwartz on almost periodic distributions and that of Cioranescu on almost periodic (non-quasianalytic) ultradistributions to the case of hyperfunctions.

1. Introduction

Let $f(x)$ be a complex valued continuous function defined on $\mathbb{R}$. A number τ is called an ϵ-*almost period* of $f(x)$ if $\sup_{-\infty<x<\infty} |\, f(x+\tau) - f(x)\,| \le \epsilon$. If for any $\epsilon > 0$ there exists a number $l(\epsilon)$ such that every interval of length $l(\epsilon)$ contains an ϵ-almost period of f, then $f(x)$ is said to be *almost periodic*.

It is well known that the following three statements are equivalent:

(i) f is an almost periodic function.
(ii) The set of translations f_h for $h \in \mathbb{R}$ forms a relatively compact set with respect to the uniform topology.
(iii) $f(x)$ is the uniform limit of a sequence of (generalized) trigonometric polynomials

$$P_m(x) = \sum_{n=1}^{l} \alpha_n \exp i\lambda_n x, \qquad \lambda_n \in \mathbb{R}.$$

The definition of almost periodicity for the continuous functions cannot carry over to generalized functions. Instead, the equivalent statements (ii) or (iii) can be used to define almost periodic generalized functions.

Schwartz [S] used (ii) to define almost periodic distributions in the sense of Stepanoff and showed that the following statements are equivalent for any bounded

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distribution T:

(1) T is almost periodic.
(2) T is the finite sum of derivatives of functions in C_{ap}.
(3) $T * \varphi \in C_{ap}$ for all $\varphi \in \mathcal{D}$.

Here, C_{ap} is the space of almost periodic functions.

Cioranescu [C2] used (iii) instead to define almost periodic non-quasianalytic ultradistributions of Beurling type and showed that the following statements are equivalent for any bounded ultradistribution T:

(1) T is almost periodic.
(2) $T * \varphi \in C_{ap}$ for every $\varphi \in \mathcal{D}^{(M_p)}$ which is the space of ultradifferentiable functions of class (M_p) in [K].
(3) There are two functions f, $g \in C_{ap}$ and an ultradifferential operator P of class (M_p) such that $T = P(D^2)f + g$.

In this paper we generalize the above results of Schwartz and Cioranescu to the case of hyperfunctions and show that the following statements are equivalent for any bounded hyperfunction T:

(1) T is almost periodic.
(2) $T * \varphi \in C_{ap}$ for every $\varphi \in \mathcal{F}$ which is the Sato space defined in Section 2.
(3) There exist two functions f and g belonging to C_{ap} and an ultradifferential operator P of class $\{p!^2\}$ such that $T = P(D^2)f + g$.
(4) The Gauss transform $u(x,t) = (T * E)(x,t)$ of T is almost periodic for each $t > 0$, where $E(x,t)$ is the heat kernel.

We also obtain the similar result for quasianalytic ultradistributions T, which, in fact, includes all the above results of Schwartz, Cioranescu and ourselves. Although Cioranescu imposed the condition (M.3) in addition to (M.1) and (M.2) we impose a much weaker condition (C) to prove our characterization theorem for quasi-analytic ultradistributions.

For the proof we apply the characterization of bounded hyperfunctions in Chung-Kim-Lee [CKL] and make use of the heat kernel method which represents various generalized functions as the initial values of solutions of the heat equation with suitable growth conditions.

2. Preliminaries

Let M_p, $p = 0, 1, 2, \cdots$, be a sequence of positive numbers. We impose the following conditions on M_p:

(M.1) $M_p^2 \le M_{p-1}M_{p+1}, \quad p \in \mathbb{N}$;

(M.2) There exist positive constants A and H such that

$$M_p \le AH^p \min_{0 \le q \le p} M_q M_{p-q}, \quad p \in \mathbb{N}_0.$$

The following *strong non-quasianalyticity* condition (M.3) has been assumed in the main result of Cioranescu [C2], but we do not impose the condition (M.3) to study hyperfunctions and a class of quasianalytic ultradistributions.

(M.3) There exists a positive constant A such that

$$\sum_{q=p+1}^{\infty} M_{q-1}/M_q \le ApM_p/M_{p+1}, \quad p \in \mathbb{N}.$$

We briefly give definitions of Fourier hyperfunctions and bounded hyperfunctions. We refer to [KCK], [CKL], [C2] for more details. We denote by $\mathcal{F}$ the Sato space of all infinitely differentiable functions φ in $\mathbb{R}$ satisfying

$$\|\varphi\|_{h,k} = \sup_{\substack{x\in\mathbb{R}\\ p\in\mathbb{N}_0}} \frac{|\varphi^{(p)}(x)| \exp k|x|}{h^{|p|}p!} < \infty$$

for some h, $k > 0$. Also, we denote by $\mathcal{F}'$ the strong dual of $\mathcal{F}$ and call its elements *Fourier hyperfunctions*.

We denote by $\mathcal{D}_{L^1}^{\{M_p\}}$ the space of functions $\varphi \in C^\infty(\mathbb{R})$ satisfying

$$\|\varphi\|_{L^1,h} = \sup_p \frac{\|\varphi^{(p)}\|_{L^1}}{h^p M_p} < \infty$$

for some constant $h > 0$. We say that $\varphi_j \to 0$ as $j \to \infty$ in $\mathcal{D}_{L^1}^{\{M_p\}}$ if $\|\varphi_j\|_{L^1,h} \to 0$ as $j \to \infty$ for some $h > 0$. Also, we denote by ${\mathcal{D}'}_{L^\infty}^{\{M_p\}}$ the strong dual space of $\mathcal{D}_{L^1}^{\{M_p\}}$ and call its elements *bounded ultradistributions* of Roumieu type. In particular, we denote the space $\mathcal{D}_{L^1}^{\{p!\}}$ by $\mathcal{A}_{L^1}$ and denote by $\mathcal{B}_{L^\infty}$ the dual space of $\mathcal{A}_{L^1}$ and call its elements *bounded hyperfunctions*. It is easy to see the following topological inclusions: $\mathcal{F} \hookrightarrow \mathcal{A}_{L^1}$, $\mathcal{B}_{L^\infty} \hookrightarrow \mathcal{F}'$.

Let $D = -id/dt$. Then an operator of the form $P(D) = \sum a_n D^n$ is called an *ultradifferential* operator (of class $\{M_p\}$) if for every L there exists C such that

$$|a_n| \le CL^n/M_n,\ n \in \mathbb{N}_0. \tag{2.1}$$

The following result establishes the existence of a *parametrix* of an ultradifferential operator, which will be very useful later.

Lemma 2.1 ([M])**.** *For any $h > 0$ and $\epsilon > 0$ there exist functions $v(t) \in C_c^\infty([0,\epsilon])$, $w(t) \in C_c^\infty([\epsilon/2,\epsilon])$ and an ultradifferential operator $P(d/dt)$ such that*

$$|v^{(k)}(t)| \le Ch^{-k}k!^2, \quad k = 0, 1, \cdots,$$
$$|v(t)| \le C\exp(-h/t), \quad 0 < t < \infty,$$
$$P(d/dt) = \sum_{k=0}^\infty a_k (d/dt)^k,\ |a_k| \le C_1 h_1^k/k!^2, \quad 0 < h_1 < h,$$
$$P(d/dt)v(t) + w(t) = \delta.$$

3. Almost periodic hyperfunctions

We first recall the characterization of the bounded hyperfunctions as in [CKL]. Let $E(x,t)$ be the heat kernel

$$E(x,t) = \begin{cases} \frac{1}{\sqrt{4\pi t}} \exp(-x^2/4t), & t > 0, \\ 0, & t \le 0. \end{cases}$$

Note that $E(x,t)$ belongs to the Sato space $\mathcal{F}$ for each $t > 0$ and $E(x-\cdot,t)$ belongs to $\mathcal{A}_{L^1}$ for each $(x,t) \in \mathbb{R}\times\mathbb{R}^+$. So, for each $T \in \mathcal{B}_{L^\infty}$ its *Gauss transform* $u(x,t) = T_y(E(x-y,t)$ is a C^∞ function in $\mathbb{R}\times\mathbb{R}^+$.

Theorem 3.1 ([CKL]). *The following statements are equivalent*:

(i) $T \in \mathcal{B}_{L^\infty}$.

(ii) $T * \varphi \in L^\infty$ *for every* $\varphi \in \mathcal{F}$.

(iii) *There exist two functions* f *and* g *belonging to* C_b *and an ultradifferential operator* P *of class* $\{p!^2\}$ *such that* $T = P(D^2)f + g$.

(iv) *The Gauss transform* $u(x,t)$ *of* T *belongs to* $C^\infty(\mathbb{R}^2_+)$ *and satisfies the following:*

$$(\partial_t - \Delta)u(x,t) = 0 \quad \text{in} \ \ \mathbb{R}^2_+;$$

for every $\epsilon > 0$ *there exists a constant* $C > 0$ *such that*

$$\|u(x,t)\|_{L^\infty(\mathbb{R})} \le C\, e^{\epsilon/t} \quad \text{in} \ \ \mathbb{R}^2_+;$$

and $u(\,\cdot\,,t) \to T$ *as* $t \to 0^+$ *in the sense that*

$$T(\varphi) = \lim_{t\to 0^+} \int_{\mathbb{R}} u(x,t)\varphi(x)dx, \quad \varphi \in \mathcal{A}_{L^1}.$$

Here, C_b *is the space of bounded continuous functions on* $\mathbb{R}$.

Using the equivalent condition (iii) in Section 1 we now define almost periodic hyperfunctions.

Definition 3.2. A hyperfunction $T \in \mathcal{B}_{L^\infty}$ is called *almost periodic* if T is the limit of a sequence of trigonometric polynomials $P_m(x) = \sum_{n=1}^{k(m)} \alpha_n \exp(i\lambda_n x)$ in the space $\mathcal{B}_{L^\infty}$ with respect to the strong topology where $\lambda_n \in \mathbb{R}$ and $\alpha_n \in \mathbb{C}$ depend on m.

For the proof of our main result we need the following lemmas.

Lemma 3.3 ([CKL]). *For any* $\varphi \in \mathcal{A}_{L^1}$, *let*

$$\varphi_t(x) = \int E(x-y,t)\varphi(y)dy, \quad t > 0.$$

Then $\varphi_t \in \mathcal{A}_{L^1}$ *for every* $t > 0$ *and* $\varphi_t \to \varphi$ *in* $\mathcal{A}_{L^1}$ *as* $t \to 0^+$.

We now prove the continuity of the ultradifferential operator.

Lemma 3.4. *Let* $P(d/dt)$ *be an ultradifferential operator of class* $\{p!^2\}$. *Then the operator* $P(D^2) : \mathcal{A}_{L^1} \longrightarrow \mathcal{A}_{L^1}$ *is a continuous linear mapping.*

Proof. Let $\varphi \in \mathcal{A}_{L^1}$. Then there exists $s > 0$ such that $\|\varphi^{(p)}\|_{L^1} \le s^p p! \|\varphi\|_{L^1,s}$ for all p. Using the inequalities (2.1) we obtain that for some constant C,

$$\begin{aligned}\|(a_k(D^2)^k\varphi(x))^{(p)}\|_{L^1} &\le |a_k|\|\varphi^{(2k+p)}\|_{L^1} \\ &\le |a_k| s^{(2k+p)}(2k+p)!\|\varphi\|_{L^1,s} \\ &\le C(L3^2s^2)^k(3s)^p p!\|\varphi\|_{L^1,s},\end{aligned}$$

since $(2k+p)! \le 3^{2k+p}(k!)^2 p!$. Therefore we obtain

$$\|a_k(D^2)^k\varphi\|_{L^1,3s} \le C(L3^2s^2)^k\|\varphi\|_{L^1,s}.$$

If L is so small that $L3^2s^2 \le 1/2$, then $\|a_k(D^2)^k\varphi\|_{L^1,3s} \le C(1/2)^k\|\varphi\|_{L^1,s}$. It follows easily that $\|P(D^2)\varphi\|_{L^1,3s} \le 2C\|\varphi\|_{L^1,s}$. Since the constant C is independent of φ we complete the proof.

We are now in a position to state and prove the main result.

Theorem 3.5. *For $T \in \mathcal{B}_{L^\infty}$ the following statements are equivalent*:

(i) *T is almost periodic.*
(ii) *$T * \varphi \in C_{ap}$ for every $\varphi \in \mathcal{F}$.*
(iii) *There exist two functions f and g belonging to C_{ap} and an ultradifferential operator P of class $\{p!^2\}$ such that $T = P(D^2)f + g$.*
(iv) *The Gauss transform $u(x,t)$ of T is almost periodic.*

Proof. (i) $\Rightarrow$ (ii): Let T be almost periodic. Then there exists a sequence (P_n) of trigonometric polynomials converging to T in $\mathcal{B}_{L^\infty}$. A simple computation yields that for every $\varphi \in \mathcal{F}$, $P_n * \varphi$ is also a trigonometric polynomial. It is easy to see that the set of translations $(\tau_s\varphi)_{s\in\mathbb{R}}$ is bounded in $\mathcal{A}_{L^1}$ where $(\tau_s\varphi)(t) = \varphi(t-s)$. Then it follows that

$$(P_n * \varphi)(s) = \langle P_n, \tau_s\check{\varphi}\rangle \longrightarrow \langle T, \tau_s\check{\varphi}\rangle = (T * \varphi)(s)$$

as $n \to \infty$ and the convergence is uniform on $\mathbb{R}$. Hence $T * \varphi \in C_{ap}$.

(ii) $\Rightarrow$ (iii): Let $v(t)$, $w(t)$ and P be as in Lemma 2.1. Define

$$V(x,t) = \int_0^\infty E(x,t+s)v(s)ds, \quad W(x,t) = \int_0^\infty E(x,t+s)w(s)ds.$$

Then $V(x,t)$ and $W(x,t)$ belong to $C^\infty(\mathbb{R}^2_+)$, and for fixed $t > 0$ they also belong to $\mathcal{F}$ as functions of x.

By the assumption we have that for fixed $t > 0$ the functions $T*_xV(x,t)$ and $T*_xW(x,t)$ belong to C_{ap}.

Also, we have

$$\begin{aligned} P(-d/dt)V(x,t) &= \int_0^\infty P(-d/dt)E(x,t+s)v(s)ds \\ &= \int_0^\infty P(-d/ds)E(x,t+s)v(s)ds \\ &= \int_0^\infty E(x,t+s)P(d/ds)v(s)ds \\ &= \int_0^\infty E(x,t+s)(-w(s)+\delta(s))ds \\ &= -W(x,t) + E(x,t). \end{aligned}$$

Therefore, we obtain the following:

$$\begin{aligned} P(-d/dt)V(x,t) + W(x,t) &= E(x,t), \\ P(-d/dt)T*_xV(x,t) + T*_xW(x,t) &= T*_xE(x,t), \\ P(D^2)T*_xV(x,t) + T*_xW(x,t) &= T*_xE(x,t). \end{aligned}$$

Then by the uniqueness of the Cauchy problem for the heat equation in [W] there exist functions $g(x)$ and $h(x)$ in L^∞ such that

$$T*_xV(x,t) = g(x) * E(x,t) \quad \text{and} \quad T*_xW(x,t) = h(x) * E(x,t).$$

In fact, since $T * V(x,0) = g(x)$ and $T * W(x,0) = h(x)$ it follows that $g(x)$, $h(x) \in C_{ap}$.

(iii) $\Rightarrow$ (i): Let $T = P(D^2)g(x) + h(x)$ where $g(x), h(x) \in C_{ap}$. Note that every element in C_{ap} is almost periodic in $\mathcal{B}_{L^\infty}$. So $P(D^2)g$ and h are almost periodic in $\mathcal{B}_{L^\infty}$ by Lemma 3.4. Therefore, T is almost periodic.

(ii) $\Rightarrow$ (iv): Since the heat kernel $E(x,t)$ belongs to the Sato space $\mathcal{F}$ for each $t > 0$, the Gauss transform $u(x,t) = T * E(x,t)$ belongs to C_{ap} for each $t > 0$.

(iv) $\Rightarrow$ (ii): Assume that the Gauss transform

$$u(x,t) = T_y(E(x-y,t)) = (T * E_t)(x)$$

of T is almost periodic for each $t > 0$. Then for every $\varphi \in \mathcal{F}$, $(T*E_t)*\varphi = (T*\varphi)*E_t$ is an almost periodic function for each $t > 0$, which converges uniformly to $T * \varphi$ as $t \to 0^+$, since $T * \varphi$ and $(T * \varphi)'$ are bounded. Thus $T * \varphi$ is almost periodic for every $\varphi \in \mathcal{F}$.

As an application to the Dirichlet problem for the half plane for the case of hyperfunctions we state the following theorem without proof, which generalizes the result in [C2].

Theorem 3.6. *Let $T \in \mathcal{B}_{L^\infty}$ be almost periodic. Then there exists a harmonic function $u(x,y)$ in the right half-plane such that*

(i) *for every $x > 0$, the function $y \to u(x,y)$ is almost periodic*;
(ii) *$u(x,y) \to T$ in $\mathcal{B}_{L^\infty}$ as $x \to 0$.*

4. Almost periodic quasianalytic ultradistributions

As mentioned in Section 1, Cioranescu has already obtained the characterization of almost periodic ultradistributions of Beurling type in [C2]. We give a similar result for quasianalytic ultradistributions of Roumieu type provided that M_p satisfies the conditions (M.1), (M.2) and the following condition (C): There exists a positive integer k such that $\liminf_{p\to\infty}(m_{kp}/m_p)^2 > k$ where $m_p = M_p/M_{p-1}$ $p = 1, 2, \cdots$.

The condition (C) is much weaker than (M.3). In fact, the condition (C) is equivalent to the fact that M_p^2 satisfies (M.3). Hereafter, we always assume the conditions (M.1), (M.2) and (C). See [CK] for more details.

Here we use the following Lemma which is a modified version of Lemma 2.1.

Lemma 4.1 ([CK])**.** *For any $h > 0$ and $\epsilon > 0$ there exist functions $v(t) \in C_c^\infty([0,\epsilon])$, $w(t) \in C_c^\infty([\epsilon/2, \epsilon])$ and an ultradifferential operator $P(d/dt)$ such that*

$$|v^{(k)}(t)| \le Ch^{-k}M_k^2, \quad k = 0, 1, \cdots,$$
$$|v(t)| \le C\exp\left(-M^*(h/t)\right), \quad 0 < t < \infty,$$
$$P(d/dt) = \sum_{k=0}^{\infty} a_k (d/dt)^k, \ |a_k| \le C_1 h_1^k / M_k^2, \quad 0 < h_1 < h,$$
$$P(d/dt)v(t) + w(t) = \delta,$$

where $M^*(t) = \sup_p \log \dfrac{p!t^p}{M_p^2}$.

Employing Lemma 4.1 and adapting the proof of Theorem 3.1 we can characterize bounded quasianalytic ultradistributions as follows.

Lemma 4.2. *The following statements are equivalent:*

(i) $T \in \mathcal{D}'^{\{M_p\}}_{L^\infty}$.

(ii) $T * \varphi \in L^\infty$ *for every* $\varphi \in \mathcal{D}^{\{M_p\}}_{L^1}$.

(iii) *There exist two functions* f, $g \in C_b$ *and an ultradifferential operator of class* $\{M_p^2\}$ *such that* $T = P(D^2)f + g$.

(iv) *The Gauss transform* $u(x,t)$ *of* T *belongs to* $C^\infty(\mathbb{R}^2_+)$ *and satisfies the following:*

$$(\partial_t - \Delta)u(x,t) = 0 \quad in \ \ \mathbb{R}^2_+;$$

for every $\epsilon > 0$ *there exists a constant* $C > 0$ *such that*

$$\|u(x,t)\|_{L^\infty(\mathbb{R}_x)} \le C \exp\left(M^*(\epsilon/t)\right) \quad in \ \ \mathbb{R}^2_+,$$

where $M^*(t) = \sup_p \log \dfrac{p!t^p}{M_p^2}$; *and* $u(\,\cdot\,,t) \to T$ *as* $t \to 0^+$ *in the sense that*

$$T(\varphi) = \lim_{t\to 0^+} \int_{\mathbb{R}} u(x,t)\varphi(x)dx, \quad \varphi \in \mathcal{D}^{\{M_p\}}_{L^1}.$$

Making use of Lemma 4.1 and Lemma 4.2 and suitably modifying the proof of Theorem 3.5 we obtain the following characterization of almost periodic ultradistributions which unify previous results on characterizations of various almost periodic generalized functions.

Theorem 4.3. *For* $T \in \mathcal{D}'^{\{M_p\}}_{L^\infty}$ *the following are equivalent*:

(i) T *is almost periodic.*

(ii) $T * \varphi \in C_{ap}$ *for every* $\varphi \in \mathcal{D}^{\{M_p\}}_{L^1}$.

(iii) *There exist two functions* $f, g \in C_{ap}$ *and an ultradifferential operator* P *of class* $\{M_p^2\}$ *such that* $T = P(D^2)f + g$.

(iv) *The Gauss transform* $u(x,t)$ *of* T *is almost periodic.*

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