SPECTRUM OF INTERPOLATED OPERATORS

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Abstract. Let \((X_0, X_1)\) be a compatible pair of Banach spaces and let \(T\) be an operator that acts boundedly on both \(X_0\) and \(X_1\). Let \(T_{[\theta]}\) \((0 \leq \theta \leq 1)\) be the corresponding operator on the complex interpolation space \((X_0, X_1)_{[\theta]}\).

The aim of this paper is to study the spectral properties of \(T_{[\theta]}\). We show that in general the set-valued function \(\theta \mapsto \sigma(T_{[\theta]})\) is discontinuous even in inner points \(\theta \in (0, 1)\) and show that each operator satisfies the local uniqueness-of-resolvent condition of Ransford. Further we study connections with the real interpolation method.

I. Complex interpolation

Let \(\tilde{X} = (X_0, X_1)\) be a compatible pair of Banach spaces, i.e., \((X_0, \| \cdot \|_0)\) and \((X_1, \| \cdot \|_1)\) are Banach spaces continuously embedded in a Hausdorff topological vector space. Then \(\tilde{X}_\Delta = X_0 \cap X_1\) and \(\tilde{X}_\Sigma = X_0 + X_1\) endowed with norms \(\|x\|_\Delta = \max\{\|x\|_0, \|x\|_1\}\) and \(\|x\|_\Sigma = \inf\{\|a\|_0 + \|b\|_1 : a \in X_0, b \in X_1, a + b = x\}\) are Banach spaces.

Recall the construction of complex interpolation spaces \(\tilde{X}_{[\theta]}\) \((0 \leq \theta \leq 1)\) (see e.g. [C], [BL], [T]). Let \(G = \{z \in \mathbb{C} : 0 < \Re z < 1\}\). Denote by \(\mathcal{F}\) the set of all continuous functions \(f : G \to \tilde{X}\) that are analytic on \(G\) such that \(f(j + it) \in X_j\) \((j = 0, 1, t \in \mathbb{R})\) and \(\lim_{|t| \to \infty} \|f(j + it)\|_j = 0\) \((j = 0, 1)\). Then \(\mathcal{F}\) with the norm 

\[
\|f\|_{\mathcal{F}} = \max_{t \in \mathbb{R}} \{\max_{\theta} \|f(it)\|_0, \max_{\theta} \|f(1 + it)\|_1\}
\]

becomes a Banach space. The intermediate spaces \(\tilde{X}_{[\theta]}\) are defined by \(\tilde{X}_{[\theta]} = \{f(\theta) : f \in \mathcal{F}\}\) with the norm \(\|x\|_{[\theta]} = \inf\{\|f\|_{\mathcal{F}} : f(\theta) = x\}\). Then \(\tilde{X}_\Delta \subset \tilde{X}_{[\theta]} \subset \tilde{X}_\Sigma\) and \(\tilde{X}_\Delta\) is dense in \(\tilde{X}_{[\theta]}\) \((0 \leq \theta \leq 1)\). Further \(\tilde{X}_{[0]}\) and \(\tilde{X}_{[1]}\) are closed subspaces of \(\tilde{X}_0\) and \(\tilde{X}_1\), respectively.

Clearly \(\tilde{X}_{[\theta]}\) can be identified with the quotient space \(\mathcal{F}/N_{\theta}\) where \(N_{\theta} = \{f \in \mathcal{F} : f(\theta) = 0\}\).

Let \(T : \tilde{X}_\Sigma \to \tilde{X}_\Sigma\) be a linear mapping such that \(TX_0 \subset X_0, TX_1 \subset X_1\) and the restrictions \(T|X_j\) are bounded with respect to the norms on \(X_j\) \((j = 0, 1)\) (we denote this situation by \(T : \tilde{X} \to \tilde{X}\)). Then \(TX_{[\theta]} \subset X_{[\theta]}\) for all \(\theta\); the restriction \(T|X_{[\theta]}\) is denoted by \(T_{[\theta]}\).

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It is well known that \( T_{[\theta]} \) is bounded with respect to the norm on \( X_{[\theta]} \) and \( \|T_{[\theta]}\| \leq \|T_0\|^{1-\theta} \cdot \|T_1\|^\theta \). By the spectral radius formula we also have
\[
r(T_{[\theta]}) \leq r(T_0)^{1-\theta} \cdot r(T_1)^\theta.
\]

The continuity properties of the set-valued function \( \theta \mapsto \sigma(T_{[\theta]}) \) were studied by a number of authors. By [Sv] (see also [A]), the function \( \theta \mapsto \sigma(T_{[\theta]}) \) is upper semi-continuous on \((0,1)\); at the points \(0,1\) it is in general neither upper nor lower semi-continuous (see [R]).

The spectral radius \( r(T_{[\theta]}) \) is continuous on \((0,1)\) and upper semi-continuous on \([0,1)\).

By [S], the polynomial convex hull of the spectrum \( \hat{\sigma}(T_{[\theta]}) \) (= the union of \( \sigma(T_{[\theta]}) \) with the bounded components of \( C \setminus \sigma(T_{[\theta]}) \)) is upper semi-continuous on \((0,1)\) and the outer boundary \( \partial \hat{\sigma}(T_{[\theta]}) \) is lower semi-continuous on \((0,1)\).

Recall that the capacity of a compact subset \( K \) of \( C \) is defined by
\[
\text{cap} K = \inf \|p\|^1_{K_{deg p}},
\]
where the infimum is taken over all polynomials with leading coefficient equal to 1 and \( \|p\|_{K} = \sup\{|p(z)| : z \in K\} \). Clearly \( \text{cap} K = \text{cap} K = \text{cap} \partial K \). Consequently the capacity \( \text{cap} \sigma(T_{[\theta]}) \) is continuous on \((0,1)\) and upper semi-continuous on \([0,1)\).

Our first example shows that the spectrum \( \sigma(T_{[\theta]}) \) is in general discontinuous even in inner points \( \theta \in (0,1) \). This gives a negative answer to a question posed in [S].

**Example 1.** Let \( (X_0, \| \cdot \|_0) \) and \( (X_1, \| \cdot \|_1) \) be Hilbert spaces, each with orthogonal basis \( \{ e_j : j \in \mathbb{Z} \} \) such that \( \|e_j\|_0 = 2^{-j} \) and \( \|e_j\|_1 = 2^j \) \( (j \in \mathbb{Z}) \). For \( 0 \leq \theta \leq 1 \) set \( r_0 = (\frac{1}{2})^{1-\theta} \cdot 2^\theta = 2^{2\theta-1} \). Then (cf. [H], 1.18.5) the intermediate spaces \( X_{[\theta]} = (X_0, X_1)_{[\theta]} \) are Hilbert spaces of all sums \( \sum_{j \in \mathbb{Z}} \alpha_j e_j \) with complex coefficients \( \alpha_j \) such that
\[
\left\| \sum_{j \in \mathbb{Z}} \alpha_j e_j \right\|_{[\theta]} = \left( \sum_{j \in \mathbb{Z}} |\alpha_j|^2 r_j^{2\theta} \right)^{1/2} < \infty.
\]

Let \( S : \hat{X} \to \hat{X} \) be defined by \( S e_j = e_{j+1} \) \( (j \in \mathbb{Z}) \). Clearly \( S_{[\theta]} \) is the \( r_\theta \)-multiple of the unitary bilateral shift of multiplicity 1.

Let \( H_0 = \bigoplus_{j \in \mathbb{Z}} X_0 \), \( H_1 = \bigoplus_{j \in \mathbb{Z}} X_1 \) and \( \hat{H} = (H_0, H_1) \). Then the intermediate spaces \( H_{[\theta]} \) are orthogonal sums of infinitely many copies of \( \hat{X}_{[\theta]} \). Define \( T : H \to \hat{H} \) by
\[
T(\ldots, x_{-1}, \underbrace{0}_{x_0}, x_1, \ldots) = (\ldots, S_{x_2}, \ldots, x_{-1}, (S - I)x_0, Sx_1, \ldots)
\]
where the box denotes the zero position and \( I \) is the identity operator; the same formula also defines \( T_{[\theta]} \) for all \( \theta \). For each \( n \geq 1 \) we have
\[
T^n(\ldots, x_{-1}, \underbrace{0}_{x_0}, x_1, \ldots) = (\ldots, S^n_{x_{-n-1}}, \ldots, S^n_{x_{-1}}, (S^n - S^{n-1})x_{-n+1}, \ldots, (S^n - S^{n-1})x_0, S^n x_1, \ldots).
\]
Thus \( \|T_{[\theta]}\| \leq r_\theta^n + r^{n-1}_\theta \) and \( r(T_{[\theta]}) = \lim_{n \to \infty} \|T^n_{[\theta]}\|^{1/n} \leq r_\theta \). Hence \( \sigma(T_{[\theta]}) \subset \{ z \in C : |z| \leq r_\theta \} \).
Let $\theta \neq 1/2$. Then $r_\theta \neq 1$ so that $S_\theta - I_\theta$ is invertible. Let $m$ denote the injectivity modulus, $m(T_\theta) = \inf\{||T_\theta x|| : x \in \mathcal{H}_\theta, ||x|| = 1\}$. Clearly

$$m(T_\theta) \geq \min\{r_\theta^n, |r_\theta^n - r_\theta^{n-1}|\} = |r_\theta^n - r_\theta^{n-1}|$$

(since $1/2 \leq r_\theta \leq 2$). Hence

$$r(T_\theta^{-1}) = \lim_{n \to \infty} ||T_\theta^{-n}||^{1/n} = \lim_{n \to \infty} m(T_\theta)^{-1/n} \geq r_\theta^{-1} \quad (\theta \neq 1/2)$$

and thus $\sigma(T_\theta) \subset \{z \in \mathbb{C} : |z| = r_\theta\}$.

On the other hand, for $\theta = 1/2$ we have $m(S_{1/2} - I_{1/2}) = 0$ so that $m(T_{1/2}) = 0$ and $0 \in \sigma(T_{1/2})$. Thus the function $\theta \mapsto \sigma(T_\theta)$ is discontinuous at $\theta = 1/2$.

In fact it is easy to check that $\sigma(T_\theta) = \{z : |z| = r_\theta\}$ for $\theta \neq 1/2$ and $\sigma(T_{1/2}) = \{z : |z| \leq r_{1/2} = 1\}$.

It is not difficult to construct an example of an operator $T : \tilde{X} \to \tilde{X}$ with a much greater set of discontinuity points of the function $\theta \mapsto \sigma(T_\theta)$ in $(0, 1)$.

Since this function is upper semi-continuous, the set of all continuity points of $\sigma(T_\theta)$ is a dense $G_\delta$ set (see e.g. [1]). In fact this is the only condition.

**Theorem 2.** Let $M$ be a dense $G_\delta$ subset of $(0, 1)$. Then there exists a compatible pair $\tilde{Z} = (Z_0, Z_1)$ of separable Hilbert spaces and an operator $W : \tilde{Z} \to \tilde{Z}$ such that $M$ is the set of all continuity points of the function $\theta \mapsto \sigma(W_\theta)$ in $(0, 1)$.

**Proof.** As in Example 1 set $r_\theta = 2^{2\theta-1} \quad (0 \leq \theta \leq 1)$.

We give an outline of the proof in three steps:

(a) For each $\alpha \in (0, 1)$ there is a compatible pair of separable Hilbert spaces $\tilde{H} = (H_0, H_1)$ and $T : \tilde{H} \to \tilde{H}$ such that

$$\sigma(T_\theta) = \{z : |z| = r_\theta\} \quad (0 \leq \theta \leq 1, \theta \neq \alpha),$$

$$\sigma(T_\alpha) = \{z : |z| \leq r_\alpha\}.$$

Indeed, define $\tilde{H}$ as in Example 1 and let

$$T(\ldots, x_{-1}, x_0, x_1, \ldots) = (\ldots, Sx_{-2}, Sx_{-1}, (S - r_\alpha I)x_0, Sx_1, \ldots).$$

It is easy to check that $T$ satisfies the conditions of (a).

(b) Let $F$ be a closed rare subset of $(0, 1)$ (i.e., the interior of $F$ is empty). Then there exists a compatible pair $\tilde{Y} = (Y_0, Y_1)$ of separable Hilbert spaces and $V : \tilde{Y} \to \tilde{Y}$ such that

$$\sigma(V_\theta) = \{z : |z| = r_\theta\} \quad (\theta \notin F),$$

$$\sigma(V_\theta) = \{z : |z| \leq r_\theta\} \quad (\theta \in F).$$

Indeed, let $F_0$ be a countable dense subset of $F$. For each $\alpha \in F_0$ let $T^{(\alpha)}$ be as in (a). Set $V = \bigoplus_{\alpha \in F_0} T^{(\alpha)}$. As in Example 1 it is easy to check that $V$ satisfies the conditions of (b).

(c) Let $M$ be a dense $G_\delta$ subset of $(0, 1)$, i.e., $\{0, 1\} \setminus M = \bigcup_{n=1}^\infty F_n$ where $F^{(n)}$ are closed rare sets. Set $s_n = 1 - 2^{-n}$ ($n = 1, 2, \ldots$) and $r_\theta^{(n)} = 2^{2\theta-1} \cdot 2^{-n+2}$.

As in (b) we can construct operators $V^{(n)}$ such that

$$\sigma(V^{(n)}_\theta) = \{z : |z - s_n| = r_\theta^{(n)}\} \quad (\theta \notin F^{(n)}),$$

$$\sigma(V^{(n)}_\theta) = \{z : |z - s_n| \leq r_\theta^{(n)}\} \quad (\theta \in F^{(n)}).$$
Set \( W = \bigoplus_{n=1}^{\infty} V^{(n)} \). It is easy to check that

\[
\sigma(W_{[\theta]}) = \bigcup_{\{n: \theta \not\in F^{(n)}\}} \{z: |z - s_n| = r_{\theta}^{(n)}(\theta)\} \cup \bigcup_{\{n: \theta \in F^{(n)}\}} \{z: |z - s_n| \leq r_{\theta}^{(n)}(\theta)\} \cup \{1\}.
\]

Clearly, \( M \) is the set of all continuity points of the function \( \theta \mapsto \sigma(W_{[\theta]}) \) in \((0,1)\).

Let \( \tilde{X} = (X_0, X_1) \) be a compatible pair of Banach spaces and \( T : \tilde{X} \to \tilde{X} \). Let \( 0 \leq \alpha < \beta \leq 1 \). In general it is possible that both \( T_{[\alpha]} \) and \( T_{[\beta]} \) are invertible but the inverses \( T_{[\alpha]}^{-1}, T_{[\beta]}^{-1} \) do not coincide on \( X_\Delta \). An operator \( T : \tilde{X} \to \tilde{X} \) is said to have the uniqueness-of-resolvent (U.R.) property if the restrictions \( (T_{[\alpha]} - z)^{-1}|X_\Delta \) and \( (T_{[\beta]} - z)^{-1}|X_\Delta \) coincide for all \( \alpha, \beta \in (0,1) \) and \( z \notin \sigma(T_{[\alpha]}) \cup \sigma(T_{[\beta]}) \) (see \( \mathbb{Z} \)).

In \( \mathbb{R} \) a weaker property was suggested:

**Definition 3.** Let \( \tilde{X} = (X_0, X_1) \) be a compatible pair of Banach spaces. An operator \( T : \tilde{X} \to \tilde{X} \) satisfies the local uniqueness-of-resolvent (local U.R.) condition if, for all \( \alpha \in (0,1) \) and \( w \notin \sigma(T_{[\alpha]}) \), there exists a neighbourhood \( U \) of \( \alpha \) such that \( (T_{[\theta]} - w)^{-1} \) exists and \( (T_{[\theta]} - w)^{-1}|X_\Delta \) coincides with \( (T_{[\alpha]} - w)^{-1}|X_\Delta \).

We show that in fact each operator \( T : \tilde{X} \to \tilde{X} \) satisfies the local U.R. condition.

The proof uses the argument outlined in \( \mathbb{S}^{12} \); for the convenience of the reader we give a detailed proof.

Recall that the reduced minimum modulus \( \gamma(S) \) of an operator \( S : X \to Y \) is defined by \( \gamma(S) = \inf \{ \|Tx\|/\text{dist}\{x, \ker S\} : x \in X \setminus \ker S \} \). Clearly \( \gamma(S) > 0 \) if and only if \( S \) has closed range.

**Theorem 4.** Let \( \tilde{X} = (X_0, X_1) \) be a compatible pair of Banach spaces, \( T : \tilde{X} \to \tilde{X} \) and \( \alpha \in (0,1) \). Suppose that \( T_{[\alpha]} \) is invertible. Then there exists a neighbourhood \( U \) of \( \alpha \) such that \( T_{[\theta]} \) is invertible for all \( \theta \in U \) and \( T_{[\theta]}^{-1}x = T_{[\alpha]}^{-1}x \quad (\theta \in U, x \in X_\Delta) \).

**Proof.** Let \( \mathcal{F} \) be the space defined in the introduction. For \( w \in G \) define the operator \( V(w) : \mathcal{F} \to \mathcal{F} \) by

\[
(V(w)f)(z) = (w - z)f(z) \quad (f \in \mathcal{F}, z \in \mathcal{G}).
\]

Clearly \( \text{Im} \ V(w) = \{f \in \mathcal{F} : f(w) = 0\} \) so that \( \text{Im} \ V(w) \) is closed and \( \gamma(V(w)) > 0 \) for all \( w \in G \). Further \( w \mapsto V(w) \) is an analytic (in fact linear affine) function,

\[
V(w) = (w - \alpha)I_{\mathcal{F}} + V_0 \quad (w \in G),
\]

where \( I_{\mathcal{F}} \) is the identity operator on \( \mathcal{F} \) and \( V_0 = V(\alpha) : \mathcal{F} \to \mathcal{F} \) is defined by \( (V_0f)(z) = (\alpha - z)f(z) \quad (f \in \mathcal{F}, z \in \mathcal{G}) \).

For \( w \in G \) write \( N_w = \{f \in \mathcal{F} : f(w) = 0\} \) and define the operator \( \tilde{T} : \mathcal{F} \to \mathcal{F} \) by \( (\tilde{T}f)(z) = T_{[\alpha]}f(z) \quad (f \in \mathcal{F}, z \in \mathcal{G}) \). Clearly \( ||\tilde{T}|| \leq \max\{||T_0||, ||T_{[\alpha]}||\} \). The operator induced by \( \tilde{T} \) on the quotient space \( \mathcal{F}/N_w \) is isometrically equivalent to \( T_{[\text{Re} w]} \).

Let \( c_1, c \) be positive constants satisfying \( c_1 > ||T_{[\alpha]}^{-1}||, c > (1 + c_1||\tilde{T}||)\gamma(V_0)^{-1} \). Let \( U_0 = \{w \in G : |w - \alpha| < c^{-1} \} \) and \( T_{[\text{Re} w]} \) is invertible. Clearly \( U_0 \) is an open neighbourhood of \( \alpha \).

Let \( x \in X_\Delta \). Then the function \( k(z) = x \exp(z^2) \) belongs to \( \mathcal{F} \). We show that there exist analytic functions \( g, h : U_0 \to \mathcal{F} \) such that

\[
\tilde{T}g(w) + V(w)h(w) = k \quad (w \in U_0).
\]
If \( g(w) = \sum_{j=0}^{\infty} g_j (w-\alpha)^j \) and \( h(w) = \sum_{j=0}^{\infty} h_j (w-\alpha)^j \) are the Taylor expansions of \( g \) and \( h \) about \( \alpha \), (1) reduces to the construction of coefficients \( g_j, h_j \in \mathcal{F} \) satisfying
\[
\begin{align*}
\mathcal{T} g_0 + V_0 h_0 &= k, \\
\mathcal{T} g_1 + V_0 h_1 &= -h_0, \\
&\vdots \\
\mathcal{T} g_j + V_0 h_j &= -h_{j-1}, \\
&\vdots
\end{align*}
\]

such that the series defining \( g \) and \( h \) converge in \( U_0 \).

Since \( T_{[\alpha]} \) is invertible, there exists a class \( y + N_\alpha \in \mathcal{F}/N_\alpha \) such that \( \mathcal{T} y + N_\alpha = k + N_\alpha \) and \( \|y + N_\alpha\|_{\mathcal{X}/N_\alpha} \leq \|T_{[\alpha]}^{-1}\| \cdot \|k\|_{\mathcal{X}} \). Thus there exists \( g_0 \in \mathcal{F} \) such that \( g_0 + N_\alpha = y + N_\alpha \) and \( \|g_0\|_{\mathcal{X}} \leq c_1 \|k\|_{\mathcal{X}} \). Hence \( \mathcal{T} g_0 - k \in N_\alpha \) and \( \|\mathcal{T} g_0 - k\|_{\mathcal{X}} \leq \|k\|_{\mathcal{X}} \cdot (1 + c_1 \|T_{[\alpha]}\|) \). Therefore there exists \( h_0 \in \mathcal{F} \) such that \( V_0 h_0 = \mathcal{T} g_0 - k \) and \( \|h_0\|_{\mathcal{X}} \leq c \cdot \|k\|_{\mathcal{X}} \).

In the same way we can find \( g_1 \) and \( h_1 \) in \( \mathcal{F} \) such that \( \mathcal{T} g_1 + V_0 h_1 = -h_0, \|g_1\|_{\mathcal{X}} \leq c_1 \|k\|_{\mathcal{X}} \) and \( \|h_1\|_{\mathcal{X}} \leq c^2 \|k\|_{\mathcal{X}} \).

If we continue in this way, we construct functions \( g_j, h_j \in \mathcal{F} \) (\( j = 0, 1, \ldots \)) satisfying (2) such that \( \|g_j\|_{\mathcal{X}} \leq c_1 c^{j-1} \|k\|_{\mathcal{X}} \) and \( \|h_j\|_{\mathcal{X}} \leq c^j \|k\|_{\mathcal{X}} \). Thus \( g(w) = \sum_{j=0}^{\infty} g_j (w-\alpha)^j \) and \( h(w) = \sum_{j=0}^{\infty} h_j (w-\alpha)^j \) are functions analytic in \( U_0 \) satisfying (1).

Define now a function \( \bar{g} : U_0 \to \overline{X}_\Sigma \) by
\[
\bar{g}(w) = (g(w)) (w) \cdot \exp(-w^2) \quad (w \in U_0).
\]
Clearly, \( \bar{g} \) is analytic in \( U_0 \) and \( T \bar{g}(w) = x \) (\( w \in U_0 \)). Further \( \bar{g}(w) \in \overline{X}_{[Re w]} \) so that \( \bar{g}(w) = T_{[Re w]}^{-1} x \) and the function \( \bar{g} \) is constant in the imaginary direction. Thus \( \bar{g} \) is constant in \( U_0 \). In particular, \( T_{[\theta]}^{-1} x = T_{[\alpha]}^{-1} x \) for \( \theta \in U_0 \cap R \).

By a standard argument it is possible to easily obtain the uniqueness of the inverse on any open interval:

**Corollary 5.** Let \( \bar{X} = (X_0, X_1) \) be a compatible pair of Banach spaces, \( T : \bar{X} \to \bar{X}, 0 \leq \alpha < \beta \leq 1 \). Suppose that \( T_{[\theta]} \) is invertible for all \( \theta \in (\alpha, \beta) \). Then the restriction \( T_{[\theta]}^{-1} |_{\bar{X}_\Delta} \) is constant on \( (\alpha, \beta) \).

**Corollary 6.** The set-valued function \( w \mapsto \sigma(T_{[Re w]}) \) is analytic on \( G \) (for the definition and properties of analytic set-valued functions (see [SII]).

Corollary 6 follows immediately from [R], Theorem 2.7; in fact it also follows from the general theory in [SII].

**Corollary 7** (cf. [A], 3.7 and 3.9). Let \( \bar{X} = (X_0, X_1) \) be a compatible pair of Banach spaces, \( T : \bar{X} \to \bar{X} \). Then
\[
\text{cap } \sigma(T_{[\theta]}) \leq \text{cap } \sigma(T_0)^{1-\theta} \cdot \text{cap } \sigma(T_1)^{\theta}
\]
for all \( \theta, 0 < \theta < 1 \).

**Proof.** The previous corollary and [A], Theorem 7.1.3, imply that the function \( w \mapsto \log \text{cap } \sigma(T_{[Re w]}) \) is subharmonic on \( G \). Since it is constant in the imaginary
direction, the following lemma shows that the function \( \theta \mapsto \log \text{cap}(\sigma(T[\theta])) \) is convex in \((0, 1)\). In particular, for \( 0 < \theta_0 < \theta < \theta_1 < 1 \) we have
\[
\text{cap}(\sigma(T[\theta])) \leq \text{cap}(\sigma(T[\theta_0])) \frac{\theta_1 - \theta}{\theta_1 - \theta_0} \cdot \text{cap}(\sigma(T[\theta_1])) \frac{\theta_1 - \theta}{\theta_1 - \theta_0}.
\]
If \( \theta_0 \to 0 \) and \( \theta_1 \to 1 \), the upper semi-continuity of \( \text{cap}(\sigma(T[\theta])) \) gives the statement of Corollary 7.

**Lemma 8.** Let \( f : G \to \mathbb{R} \) be a subharmonic function such that, for all \( x \in (0, 1) \), the function \( y \mapsto f(x + iy) \) is constant on \( \mathbb{R} \). Then the function \( x \mapsto f(x) \) is convex on \((0, 1)\). In particular, \( f \) is continuous on \( G \).

**Proof.** For \( 0 < s < t < 1 \) the function
\[
z \mapsto g(z) = f(z) - \frac{\Re z - s}{t - s} \cdot f(t) - \frac{t - \Re z}{t - s} \cdot f(s)
\]
is subharmonic and hence attains a maximum on the compact set \((s,t) \times (-1,1)\). Because of \( g(s) = g(t) = 0 \) and the fact that \( y \mapsto g(x + iy) \) is constant on \( \mathbb{R} \) for all \( x \in (0,1) \), we conclude from the maximum principle for subharmonic functions that \( g(z) \leq 0 \) on \((s,t) \times (-1,1)\). In particular, for all \( \tau \in (s,t) \) we have
\[
f(\tau) - \frac{\tau - s}{t - s} \cdot f(t) - \frac{t - \tau}{t - s} \cdot f(s) \leq 0.
\]

\[\Box\]

### II. Applications to the Real Interpolation Method

Let \( \bar{X} = (X_0, X_1) \) be a compatible pair of Banach spaces. For \( 0 < \theta < 1 \) and \( 1 \leq p \leq \infty \) let \( \bar{X}_{\theta,p} \) be the real interpolation spaces; for definitions and basic properties see e.g. [BL], [T].

For an operator \( T : X \to \bar{X} \) denote by \( T_{\theta,p} \) the corresponding operator acting on \( \bar{X}_{\theta,p} \).

By [AS], \( \sigma(T_{\theta,p}) \) does not depend on \( p, 1 \leq p \leq \infty \), and it is upper semi-continuous as a function of \( \theta \) (see also [BKS]).

Clearly Example 1 shows that in general the function \( \theta \mapsto \sigma(T_{\theta,p}) \) is not continuous. Indeed, by [T], 1.18.5, in this case \( H_{\theta,2} = H_{[\theta]} \).

Using the connections between the real and complex interpolation spaces it is possible to obtain other results from Section I for the real interpolation spaces.

**Theorem 9** (local U.R. property for real interpolation spaces; cf. [K]). Let \( \bar{X} = (X_0, X_1) \) be a compatible pair of Banach spaces, let \( T : \bar{X} \to \bar{X} \) and \( 0 \leq \alpha < \beta \leq 1 \). Suppose that \( T_{\theta,1} \) is invertible for all \( \theta, \alpha < \theta < \beta \). Then \( T_{\theta,p}^{-1}X_\Delta \) is constant for all \( \theta, p \) with \( \alpha < \theta < \beta, 1 \leq p \leq \infty \).

**Proof.** By [AS], the existence of \( T_{\theta,p}^{-1} \) is independent of \( p, 1 \leq p \leq \infty \). Since \( X_\Delta \subset X_{\theta,1} \subset X_{\theta,p} \), the restriction \( T_{\theta,p}^{-1}X_\Delta \) is constant for \( 1 \leq p \leq \infty \). Therefore we may consider only the operators \( T_{\alpha,2}^{-1} \) \((\alpha < \theta < \beta)\). Clearly we can assume that \( 0 < \alpha < \beta < 1 \). Using the formula \( (X_{\alpha,2}, X_{\beta,2})_{[\eta]} = X_{\theta,2} \) where \( 0 < \eta < 1, \theta = (1 - \eta)\alpha + \eta\beta \) (see [BL], Theorem 4.7.2), Corollary 5 implies that \( T_{\theta,2}^{-1}X_\Delta \) is constant for \( \alpha < \theta < \beta \).

\[\Box\]

**Theorem 10.** Let \( \bar{X} = (X_0, X_1) \) be a compatible pair of Banach spaces, let \( T : \bar{X} \to \bar{X}, 0 < \theta < 1, 1 \leq p \leq \infty \). Then \( \sigma(T_{\theta,p}) \subset \sigma(T[\theta]) \).
Proof. Suppose that $T_{[\theta]}$ is invertible. By Theorem 4 there exist $\theta_0, \theta_1$, $0 < \theta_0 < \theta < \theta_1 < 1$ such that $T_{[\theta_0]}$ and $T_{[\theta_1]}$ are invertible and $T_{[\theta]}|X_{\Delta} = T_{[\theta_0]}|X_{\Delta}$. By [BL], Theorem 4.7.2, $X_{\theta_2} = (X_{[\theta_0]}, X_{[\theta_1]})_{\theta_2}$ where $\eta = \frac{\theta - \theta_0}{\theta_1 - \theta_0}$, so that $T_{\theta_2}$ is invertible. Since $\sigma(T_{\theta,p})$ is independent of $p$, we have $\sigma(T_{\theta,p}) \subset \sigma(T_{[\theta]}).$ 

Corollary 11.

$\text{cap} \sigma(T_{\theta,p}) \leq \text{cap} \sigma(T_0)^{1-\theta} \cdot \text{cap} \sigma(T_1)\theta$

for all $\theta, p$ with $0 < \theta < 1$, $1 \leq p \leq \infty$.

The next example shows that the real and complex methods yield in general different spectra.

Example 12. Choose $1 < p_0 < p_1 < \infty$, $0 < \theta < 1$. Let $p$ satisfy $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

For the definition and basic properties of the Lorentz spaces $L_{rs}$ (with respect to the Lebesgue measure on the real line), see [BL] or [T]. In particular, we have

$$
(L_{p_0}, L_{p_1})_{[\theta]} = L_p,
(L_{p_0}, L_{p_1})_{\theta,1} = L_{p,1},
(L_{p_0,1}, L_{p_1,1})_{[\theta]} = L_{p,1},
(L_{p_0,1}, L_{p_1,1})_{\theta,1} = L_{p,1}.
$$

Further $L_{r1} \subset L_r (= L_{rr})$ for all $r > 1$ and these two spaces are not isomorphic. Set

$$
X_0 = \cdots \oplus L_{p_0,1} \oplus L_{p_0,1} \oplus L_{p_0} \oplus L_{p_0} \oplus \cdots,
X_1 = \cdots \oplus L_{p_1,1} \oplus L_{p_1,1} \oplus L_{p_1} \oplus L_{p_1} \oplus \cdots
$$

($\ell_1$ direct sums). By [T], 1.18.1,

$$(X_0, X_1)_{[\theta]} = \cdots \oplus L_{p,1} \oplus L_p \oplus \cdots,$n$$(X_0, X_1)_{\theta,1} = \cdots \oplus L_{p,1} \oplus L_{p,1} \oplus \cdots.$$

Let $T$ be the right shift operator. Then $T_{\theta,1}$ is invertible but $T_{[\theta]}$ is not invertible.

Problem. In the previous example the polynomial convex hulls of $\sigma(T_{[\theta]})$ and $\sigma(T_{\theta,p})$ coincide (and consequently, $r(T_{[\theta]}) = r(T_{\theta,p})$). Is this always the case?

References


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