SPECTRUM OF INTERPOLATED OPERATORS

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Abstract. Let \((X_0, X_1)\) be a compatible pair of Banach spaces and let \(T\) be an operator that acts boundedly on both \(X_0\) and \(X_1\). Let \(T_\theta\) (0 ≤ \(\theta\) ≤ 1) be the corresponding operator on the complex interpolation space \((X_0, X_1)_\theta\).

The aim of this paper is to study the spectral properties of \(T_\theta\). We show that in general the set-valued function \(\theta \mapsto \sigma(T_\theta)\) is discontinuous even in inner points \(\theta \in (0, 1)\) and show that each operator satisfies the local uniqueness-of-resolvent condition of Ransford. Further we study connections with the real interpolation method.

I. Complex interpolation

Let \(\tilde{X} = (X_0, X_1)\) be a compatible pair of Banach spaces, i.e., \((X_0, \| \cdot \|_0)\) and \((X_1, \| \cdot \|_1)\) are Banach spaces continuously embedded in a Hausdorff topological vector space. Then \(\tilde{X}_\Delta = X_0 \cap X_1\) and \(\tilde{X}_\Sigma = X_0 + X_1\) endowed with norms \(\|x\|_\Delta = \max\{\|x\|_0, \|x\|_1\}\) and \(\|x\|_\Sigma = \inf\{\|a\|_0 + \|b\|_1 : a \in X_0, b \in X_1, a + b = x\}\) are Banach spaces.

Recall the construction of complex interpolation spaces \(\tilde{X}_\theta\) (0 ≤ \(\theta\) ≤ 1) (see e.g. [C], [BL], [T]). Let \(G = \{z \in \mathbb{C} : 0 < \text{Re} z < 1\}\). Denote by \(\mathcal{F}\) the set of all continuous functions \(f : G \to \tilde{X}_\Sigma\) that are analytic on \(G\) such that \(f(j + it) \in X_j\) (\(j = 0, 1, t \in \mathbb{R}\)) and \(\lim_{|t| \to \infty} \|f(j + it)\|_j = 0\) (\(j = 0, 1\)). Then \(\mathcal{F}\) with the norm

\[
\|f\|_\mathcal{F} = \max_{t \in \mathbb{R}} \left\{ \max_{t \in \mathbb{R}} \|f(it)\|_0, \max_{t \in \mathbb{R}} \|f(1 + it)\|_1 \right\}
\]

becomes a Banach space. The intermediate spaces \(\tilde{X}_\theta\) are defined by \(\tilde{X}_\theta = \{f(\theta) : f \in \mathcal{F}\}\) with the norm \(\|x\|_\theta = \inf\{\|f\|_\mathcal{F} : f(\theta) = x\}\). Then \(\tilde{X}_\Delta \subset \tilde{X}_\theta \subset \tilde{X}_\Sigma\) and \(\tilde{X}_\Delta\) is dense in \(\tilde{X}_\theta\) (0 ≤ \(\theta\) ≤ 1). Further \(\tilde{X}_\theta\) and \(\tilde{X}_1\) are closed subspaces of \(X_0\) and \(X_1\), respectively.

Clearly \(\tilde{X}_\theta\) can be identified with the quotient space \(\mathcal{F}/N_\theta\) where \(N_\theta = \{f \in \mathcal{F} : f(\theta) = 0\}\).

Let \(T : \tilde{X}_\Sigma \to \tilde{X}_\Sigma\) be a linear mapping such that \(TX_0 \subset X_0\) and \(TX_1 \subset X_1\) and the restrictions \(T|X_j\) are bounded with respect to the norms on \(X_j\) (\(j = 0, 1\)) (we denote this situation by \(T : X \to \tilde{X}\)). Then \(TX_\theta \subset \tilde{X}_\theta\) for all \(\theta\); the restriction \(T|\tilde{X}_\theta\) is denoted by \(T_\theta\).
It is well known that $T_{[\theta]}$ is bounded with respect to the norm on $\tilde{X}_{[\theta]}$ and $\|T_{[\theta]}\| \leq \|T_0\|^{1-\theta} \cdot \|T_1\|^\theta$. By the spectral radius formula we also have

$$r(T_{[\theta]}) \leq r(T_0)^{1-\theta} \cdot r(T_1)^\theta.$$ 

The continuity properties of the set-valued function $\theta \mapsto \sigma(T_{[\theta]})$ were studied by a number of authors. By [S] (see also [A]), the function $\theta \mapsto \sigma(T_{[\theta]})$ is upper semi-continuous on $(0,1)$; at the points 0, 1 it is in general neither upper nor lower semi-continuous (see [H]).

The spectral radius $r(T_{[\theta]})$ is continuous on $(0,1)$ and upper semi-continuous on $(0,1)$. By [S], the polynomial convex hull of the spectrum $\hat{\sigma}(T_{[\theta]})$ (the union of $\sigma(T_{[\theta]})$ with the bounded components of $C \setminus \sigma(T_{[\theta]})$) is upper semi-continuous on $(0,1)$ and the outer boundary $\partial \hat{\sigma}(T_{[\theta]})$ is lower semi-continuous on $(0,1)$.

Recall that the capacity of a compact subset $K$ of $C$ is defined by

$$\text{cap } K = \inf \|p\|_{K}^{1/d_p},$$

where the infimum is taken over all polynomials with leading coefficient equal to 1 and $\|p\|_K = \sup \{|p(z)| : z \in K\}$. Clearly $\text{cap } K = \text{cap } \hat{K} = \text{cap } \partial K$. Consequently the capacity $\text{cap } \sigma(T_{[\theta]})$ is continuous on $(0,1)$ and upper semi-continuous on $(0,1)$.

Our first example shows that the spectrum $\sigma(T_{[\theta]})$ is in general discontinuous even in inner points $\theta \in (0,1)$. This gives a negative answer to a question posed in [S].

**Example 1.** Let $(X_0, \| \cdot \|_0)$ and $(X_1, \| \cdot \|_1)$ be Hilbert spaces, each with orthogonal basis $\{e_j : j \in \mathbb{Z}\}$ such that $\|e_j\|_0 = 2^{-j}$ and $\|e_j\|_1 = 2^j$ ($j \in \mathbb{Z}$). For $0 \leq \theta \leq 1$ set $r_\theta = (\frac{1}{2})^{1-\theta} \cdot 2^\theta = 2^{2\theta-1}$. Then (cf. [H], 1.18.5) the intermediate spaces $X_{[\theta]} = (X_0, X_1)_{[\theta]}$ are Hilbert spaces of all sums $\sum_{j \in \mathbb{Z}} \alpha_je_j$ with complex coefficients $\alpha_j$ such that

$$\left\| \sum_{j \in \mathbb{Z}} \alpha_je_j \right\|_{[\theta]} = \left( \sum_{j \in \mathbb{Z}} |\alpha_j|^2 r_\theta^{2j} \right)^{1/2} < \infty.$$

Let $S : \tilde{X} \to \tilde{X}$ be defined by $Se_j = e_{j+1}$ ($j \in \mathbb{Z}$). Clearly $S_{[\theta]}$ is the $r_\theta$-multiple of the unitary bilateral shift of multiplicity 1.

Let $H_0 = \bigoplus_{j \in \mathbb{Z}} X_0$, $H_1 = \bigoplus_{j \in \mathbb{Z}} X_1$ and $\tilde{H} = (H_0, H_1)$. Then the intermediate spaces $H_{[\theta]}$ are orthogonal sums of infinitely many copies of $X_{[\theta]}$. Define $T : H \to \tilde{H}$ by

$$T(\ldots, x_{-1}, x_0, x_1, \ldots) = (\ldots, Sx_{-2}, \boxed{Sx_{-1}}, (S-I)x_0, Sx_1, \ldots)$$

where the box denotes the zero position and $I$ is the identity operator; the same formula also defines $T_{[\theta]}$ for all $\theta$. For each $n \geq 1$ we have

$$T^n(\ldots, x_{-1}, x_0, x_1, \ldots) = (\ldots, S^n x_{-n-1}, \boxed{S^n x_{-n}}, (S^n - S^{n-1})x_{-n+1}, \ldots, (S^n - S^{n-1})x_0, S^n x_1, \ldots).$$

Thus $\|T_{[\theta]}\| \leq r_\theta^n + r_\theta^{n-1}$ and $r(T_{[\theta]}) = \lim_{n \to \infty} \|T_{[\theta]}\|^{1/n} \leq r_\theta$. Hence $\sigma(T_{[\theta]}) \subset \{z \in C : |z| \leq r_\theta\}$. 
As in (b) we can construct operators $V$. It is easy to check that
\[
\sigma(V_{\theta}) \supset \{ z : |z - r_{\theta}| \leq r^n_{\theta} \}
\]
(since $1/2 \leq r_{\theta} \leq 2$). Hence
\[
r(T_{\theta})^{-1} = \lim_{n \to \infty} \| T_{\theta}^{-n} \|^{1/n} = \lim_{n \to \infty} m(T_{\theta}^{-n})^{-1/n} \geq r_{\theta}^{-1}
\]
and thus $\sigma(T_{\theta}) \subset \{ z \in \mathbb{C} : |z| = r_{\theta} \}$.

On the other hand, for $\theta = 1/2$ we have $m(S_{1/2} - I_{1/2}) = 0$ so that $m(T_{1/2}) = 0$ and $0 \in \sigma(T_{1/2})$. Thus the function $\theta \mapsto \sigma(T_{\theta})$ is discontinuous at $\theta = 1/2$.

In fact it is easy to check that $\sigma(T_{\theta}) = \{ z : |z| = r_{\theta} \}$ for $\theta \neq 1/2$ and $\sigma(T_{1/2}) = \{ z : |z| \leq r_{1/2} = 1 \}$.

It is not difficult to construct an example of an operator $T : X \to X$ with a much greater set of discontinuity points of the function $\theta \mapsto \sigma(T_{\theta})$ in $(0, 1)$.

Since this function is upper semi-continuous, the set of all continuity points of $\sigma(T_{\theta})$ is a dense $G_\delta$ set (see e.g. [F]). In fact this is the only condition.

**Theorem 2.** Let $M$ be a dense $G_\delta$ subset of $(0, 1)$. Then there exists a compatible pair $\hat{Z} = (\hat{Z}_0, \hat{Z}_1)$ of separable Hilbert spaces and an operator $W : \hat{Z} \to \hat{Z}$ such that $M$ is the set of all continuity points of the function $\theta \mapsto \sigma(W_{\theta})$ in $(0, 1)$.

**Proof.** As in Example 1 set $r_{\theta} = 2^{2\theta - 1}$ ($0 \leq \theta \leq 1$).

We give an outline of the proof in three steps:

(a) For each $\alpha \in (0, 1)$ there is a compatible pair of separable Hilbert spaces $\hat{H} = (H_0, H_1)$ and $T : \hat{H} \to \hat{H}$ such that
\[
\sigma(T_{\theta}) = \{ z : |z| = r_{\theta} \} \quad (0 \leq \theta \leq 1, \theta \neq \alpha),
\]
\[
\sigma(T_{\alpha}) = \{ z : |z| \leq r_{\alpha} \}.
\]

Indeed, define $\hat{H}$ as in Example 1 and let
\[
T(x_0, x_1, \ldots) = (S - r_{\alpha} I)x_0, Sx_1, \ldots).
\]

It is easy to check that $T$ satisfies the conditions of (a).

(b) Let $F$ be a closed rare subset of $(0, 1)$ (i.e., the interior of $F$ is empty). Then there exists a compatible pair $Y = (Y_0, Y_1)$ of separable Hilbert spaces and $V : Y \to Y$ such that
\[
\sigma(V_{\theta}) = \{ z : |z| = r_{\theta} \} \quad (\theta \notin F),
\]
\[
\sigma(V_{\theta}) = \{ z : |z| \leq r_{\theta} \} \quad (\theta \in F).
\]

Indeed, let $F_0$ be a countable dense subset of $F$. For each $\alpha \in F_0$ let $T^{(\alpha)}$ be as in (a). Set $V = \bigoplus_{\alpha \in F_0} T^{(\alpha)}$. As in Example 1 it is easy to check that $V$ satisfies the conditions of (b).

(c) Let $M$ be a dense $G_\delta$ subset of $(0, 1)$, i.e., $(0, 1) \setminus M = \bigcup_{n=1}^{\infty} F^{(n)}$ where $F^{(n)}$ are closed rare sets. Set $s_n = 1 - 2^{-n}$ ($n = 1, 2, \ldots$) and $r_{\theta}^{(n)} = 2^{2\theta - 1} \cdot 2^{-(n+2)}$.

As in (b) we can construct operators $V^{(n)}$ such that
\[
\sigma(V_{\theta}^{(n)}) = \{ z : |z - s_n| = r_{\theta}^{(n)} \} \quad (\theta \notin F^{(n)}),
\]
\[
\sigma(V_{\theta}) = \{ z : |z - s_n| \leq r_{\theta}^{(n)} \} \quad (\theta \in F^{(n)}).
\]
Set \( W = \bigoplus_{n=1}^{\infty} V^{(n)} \). It is easy to check that

\[
\sigma(W[\theta]) = \bigcup_{\{n: \theta \notin F(n)\}} \{z : |z - s_n| = r^{(n)}_\theta\} \cup \bigcup_{\{n: \theta \in F(n)\}} \{z : |z - s_n| \leq r^{(n)}_\theta\} \cup \{1\}.
\]

Clearly, \( M \) is the set of all continuity points of the function \( \theta \mapsto \sigma(W[\theta]) \) in \( (0,1) \).

Let \( \bar{X} = (X_0, X_1) \) be a compatible pair of Banach spaces and \( T : \bar{X} \to \bar{X} \). Let \( 0 \leq \alpha < \beta \leq 1 \). In general it is possible that both \( T_{[\alpha]} \) and \( T_{[\beta]} \) are invertible but the inverses \( T_{[\alpha]}^{-1}, T_{[\beta]}^{-1} \) do not coincide on \( \bar{X}_\Delta \). An operator \( T : \bar{X} \to \bar{X} \) is said to have the uniqueness-of-resolvent (U.R.) property if the restrictions \( (T_{[\alpha]} - z)^{-1}\bar{X}_\Delta \) and \( (T_{[\beta]} - z)^{-1}\bar{X}_\Delta \) coincide for all \( \alpha, \beta \in (0,1) \) and \( z \notin \sigma(T_{[\alpha]}) \cup \sigma(T_{[\beta]}) \) (see [Z]).

In [R] a weaker property was suggested:

**Definition 3.** Let \( \bar{X} = (X_0, X_1) \) be a compatible pair of Banach spaces. An operator \( T : \bar{X} \to \bar{X} \) satisfies the local uniqueness-of-resolvent (local U.R.) condition if, for all \( \alpha \in (0,1) \) and \( w \notin \sigma(T_{[\alpha]}) \), there exists a neighbourhood \( U \) of \( \alpha \) such that \( (T_{[\alpha]} - w)^{-1} \exists X_\Delta \) exists and \( (T_{[\beta]} - w)^{-1}\bar{X}_\Delta \) coincides with \( (T_{[\alpha]} - w)^{-1}\bar{X}_\Delta \).

We show that in fact each operator \( T : \bar{X} \to \bar{X} \) satisfies the local U.R. condition. The proof uses the argument outlined in [ST2]; for the convenience of the reader we give a detailed proof.

Recall that the reduced minimum modulus \( \gamma(S) \) of an operator \( S : X \to Y \) is defined by \( \gamma(S) = \inf \{\|Tx\|/\text{dist}\{x, \ker S\} : x \in X \setminus \ker S\} \). Clearly \( \gamma(S) > 0 \) if and only if \( S \) has closed range.

**Theorem 4.** Let \( \bar{X} = (X_0, X_1) \) be a compatible pair of Banach spaces, \( T : \bar{X} \to \bar{X} \) and \( \alpha \in (0,1) \). Suppose that \( T_{[\alpha]} \) is invertible. Then there exists a neighbourhood \( U \) of \( \alpha \) such that \( T_{[\alpha]} \) is invertible for all \( \theta \in U \) and \( T_{[\alpha]}^{-1}x = T_{[\alpha]}^{-1}x \quad (\theta \in U, x \in \bar{X}_\Delta) \).

**Proof.** Let \( \mathcal{F} \) be the space defined in the introduction. For \( w \in G \) define the operator \( V(w) : \mathcal{F} \to \mathcal{F} \) by

\[
(V(w)f)(z) = (w - z)f(z) \quad (f \in \mathcal{F}, z \in \bar{G}),
\]

Clearly \( \text{Im } V(w) = \{f \in \mathcal{F} : f(w) = 0\} \) so that \( \text{Im } V(w) \) is closed and \( \gamma(V(w)) > 0 \) for all \( w \in G \). Further \( w \mapsto V(w) \) is an analytic (in fact linear affine) function,

\[
V(w) = (w - \alpha)I_\mathcal{F} + V_0 \quad (w \in G),
\]

where \( I_\mathcal{F} \) is the identity operator on \( \mathcal{F} \) and \( V_0 = V(\alpha) : \mathcal{F} \to \mathcal{F} \) is defined by \( (V_0f)(z) = (\alpha - z)f(z) \quad (f \in \mathcal{F}, z \in \bar{G}) \).

For \( w \in G \) write \( N_w = \{f \in \mathcal{F} : f(w) = 0\} \) and define the operator \( \hat{T} : \mathcal{F} \to \mathcal{F} \) by \( (\hat{T}f)(z) = T_{[\alpha]}f(z) \quad (f \in \mathcal{F}, z \in \bar{G}) \). Clearly \( \|\hat{T}\| \leq \max\{|T_0|, \|T_1\|\} \). The operator induced by \( \hat{T} \) on the quotient space \( \mathcal{F}/N_w \) is isometrically equivalent to \( T_{[\text{Re}w]} \).

Let \( c_1, c \) be positive constants satisfying \( c_1 > \|T_{[\alpha]}^{-1}\| \) and \( c > (1 + c_1\|\hat{T}\|)\gamma(V_0)^{-1} \). Let \( U_0 = \{w \in G : |w - \alpha| < c^{-1} \} \) and \( T_{[\text{Re}w]} \) is invertible \( \}. \) Clearly \( U_0 \) is an open neighbourhood of \( \alpha \).

Let \( x \in \bar{X}_\Delta \). Then the function \( k(z) = x \exp(z^2) \) belongs to \( \mathcal{F} \). We show that there exist analytic functions \( g, h : U_0 \to \mathcal{F} \) such that

\[
(1) \quad \hat{T}g(w) + V(w)h(w) = k \quad (w \in U_0).
\]
If $g(w) = \sum_{j=0}^{\infty} g_j(w-\alpha)^j$ and $h(w) = \sum_{j=0}^{\infty} h_j(w-\alpha)^j$ are the Taylor expansions of $g$ and $h$ about $\alpha$, (1) reduces to the construction of coefficients $g_j, h_j \in \mathcal{F}$ satisfying

\begin{align*}
    \mathcal{T}g_0 + V_0h_0 &= k, \\
    \mathcal{T}g_1 + V_0h_1 &= -h_0, \\
    & \vdots \\
    \mathcal{T}g_j + V_0h_j &= -h_{j-1}, \\
    & \vdots
\end{align*}

such that the series defining $g$ and $h$ converge in $U_0$.

Since $T_{[\alpha]}$ is invertible, there exists a class $y + N_\alpha \in \mathcal{F}/N_\alpha$ such that $\mathcal{T}y + N_\alpha = k + N_\alpha$ and $\|y + N_\alpha\|_{\mathcal{F}/N_\alpha} \leq \|T_{[\alpha]}^{-1}\| \cdot \|k\|_{\mathcal{F}}$. Thus there exists $g_0 \in \mathcal{F}$ such that $g_0 + N_\alpha = y + N_\alpha$ and $\|g_0\|_{\mathcal{F}} \leq c_1\|k\|_{\mathcal{F}}$. Hence $\mathcal{T}g_0 - k \in N_\alpha$ and $\|\mathcal{T}g_0 - k\|_{\mathcal{F}} \leq \|k\|_{\mathcal{F}} \cdot (1 + c_1\|\mathcal{T}\|)$. Therefore there exists $h_0 \in \mathcal{F}$ such that $V_0h_0 = \mathcal{T}g_0 - k$ and $\|h_0\|_{\mathcal{F}} \leq c \cdot \|k\|_{\mathcal{F}}$.

In the same way we can find $g_1$ and $h_1 \in \mathcal{F}$ such that $\mathcal{T}g_1 + V_0h_1 = -h_0$, $\|g_1\|_{\mathcal{F}} \leq c_1c\|k\|_{\mathcal{F}}$ and $\|h_1\|_{\mathcal{F}} \leq c^2\|k\|_{\mathcal{F}}$.

If we continue in this way, we construct functions $g_j, h_j \in \mathcal{F}$ ($j = 0, 1, \ldots$) satisfying (2) such that $\|g_j\|_{\mathcal{F}} \leq c_1c^{j-1}\|k\|_{\mathcal{F}}$ and $\|h_j\|_{\mathcal{F}} \leq c^{j}\|k\|_{\mathcal{F}}$. Thus $g(w) = \sum_{j=0}^{\infty} g_j(w-\alpha)^j$ and $h(w) = \sum_{j=0}^{\infty} h_j(w-\alpha)^j$ are functions analytic in $U_0$ satisfying (1).

Define now a function $\tilde{g} : U_0 \to \tilde{X}_\Sigma$ by $\tilde{g}(w) = (g(w))(w) \cdot \exp(-w^2)$ ($w \in U_0$).

Clearly, $\tilde{g}$ is analytic in $U_0$ and $T\tilde{g}(w) = x$ ($w \in U_0$). Further $\tilde{g}(w) \in \tilde{X}_{[\Re w]}$ so that $\tilde{g}(w) = T_{[\Re w]}^{-1}x$ and the function $\tilde{g}$ is constant in the imaginary direction. Thus $\tilde{g}$ is constant in $U_0$. In particular, $T_{[\theta]}^{-1}x = T_{[\alpha]}^{-1}x$ for $\theta \in U_0 \cap \mathbb{R}$. \qed

By a standard argument it is possible to easily obtain the uniqueness of the inverse on any open interval:

**Corollary 5.** Let $\tilde{X} = (X_0, X_1)$ be a compatible pair of Banach spaces, $T : \tilde{X} \to \tilde{X}$, $0 \leq \alpha < \beta \leq 1$. Suppose that $T_{[\theta]}$ is invertible for all $\theta \in (\alpha, \beta)$. Then the restriction $T_{[\theta]}^{-1}|\tilde{X}_\Delta$ is constant on $(\alpha, \beta)$.

**Corollary 6.** The set-valued function $w \mapsto \sigma(T_{[\Re w]})$ is analytic on $G$ (for the definition and properties of analytic set-valued functions (see [SI2]).

Corollary 6 follows immediately from [R], Theorem 2.7; in fact it also follows from the general theory in [SI2].

**Corollary 7 (cf. [A], 3.7 and 3.9).** Let $\tilde{X} = (X_0, X_1)$ be a compatible pair of Banach spaces, $T : \tilde{X} \to \tilde{X}$. Then

$$\text{cap } \sigma(T_{[\theta]}) \leq \text{cap } \sigma(T_0)^{1-\theta} \cdot \text{cap } \sigma(T_1)^{\theta}$$

for all $\theta, 0 < \theta < 1$.

**Proof.** The previous corollary and [A], Theorem 7.1.3, imply that the function $w \mapsto \log \text{cap } \sigma(T_{[\Re w]})$ is subharmonic on $G$. Since it is constant in the imaginary
direction, the following lemma shows that the function \( \theta \mapsto \log \text{cap} \sigma(T_{[\theta]}) \) is convex in \((0, 1)\). In particular, for \(0 < \theta_0 < \theta < \theta_1 < 1\) we have

\[
\text{cap} \sigma(T_{[\theta]}) \leq \text{cap} \sigma(T_{[\theta_0]})^{\frac{\theta_1 - \theta}{\theta_1 - \theta_0}} \cdot \text{cap} \sigma(T_{[\theta_1]})^{\frac{\theta - \theta_0}{\theta_1 - \theta_0}}.
\]

If \( \theta_0 \to 0 \) and \( \theta_1 \to 1 \), the upper semi-continuity of \( \text{cap} \sigma(T_{[\theta]}) \) gives the statement of Corollary 7.

**Lemma 8.** Let \( f : G \to \mathbb{R} \) be a subharmonic function such that, for all \( x \in (0, 1) \), the function \( y \mapsto f(x + iy) \) is constant on \( \mathbb{R} \). Then the function \( x \mapsto f(x) \) is convex on \((0, 1)\). In particular, \( f \) is continuous on \( G \).

**Proof.** For \( 0 < s < t < 1 \) the function

\[
z \mapsto g(z) = f(z) - \frac{\text{Re} z - s}{t - s} \cdot f(t) - \frac{t - \text{Re} z}{t - s} \cdot f(s)
\]
is subharmonic and hence attains a maximum on the compact set \((s, t) \times (-1, 1)\). Because of \( g(s) = g(t) = 0 \) and the fact that \( y \mapsto g(x + iy) \) is constant on \( \mathbb{R} \) for all \( x \in (0, 1) \), we conclude from the maximum principle for subharmonic functions that \( g(z) \leq 0 \) on \((s, t) \times (-1, 1)\). In particular, for all \( \tau \in (s, t) \) we have

\[
f(\tau) - \frac{\tau - s}{t - s} \cdot f(t) - \frac{t - \tau}{t - s} \cdot f(s) \leq 0.
\]

\[\square\]

II. Applications to the real interpolation method

Let \( \bar{X} = (X_0, X_1) \) be a compatible pair of Banach spaces. For \( 0 < \theta < 1 \) and \( 1 \leq p \leq \infty \) let \( X_{\theta,p} \) be the real interpolation spaces; for definitions and basic properties see e.g. [BL], [T].

For an operator \( T : X \to \bar{X} \) denote by \( T_{\theta,p} \) the corresponding operator acting on \( X_{\theta,p} \).

By [AS], \( \sigma(T_{\theta,p}) \) does not depend on \( p, 1 \leq p \leq \infty \), and it is upper semi-continuous as a function of \( \theta \) (see also [HKS]).

Clearly Example 1 shows that in general the function \( \theta \mapsto \sigma(T_{\theta,p}) \) is not continuous. Indeed, by [T], 1.18.5, in this case \( H_{0,2} = H_{[\theta]} \).

Using the connections between the real and complex interpolation spaces it is possible to obtain other results from Section I for the real interpolation spaces.

**Theorem 9** (local U.R. property for real interpolation spaces; cf. [K]). Let \( \bar{X} = (X_0, X_1) \) be a compatible pair of Banach spaces, let \( T : X \to \bar{X} \) and \( 0 \leq \alpha < \beta \leq 1 \). Suppose that \( T_{\theta,1} \) is invertible for all \( \theta, \alpha < \theta < \beta \). Then \( T_{\theta,1}^{-1}|\bar{X}_\Delta \) is constant for all \( \theta, p \) with \( \alpha < \theta < \beta, 1 \leq p \leq \infty \).

**Proof.** By [AS], the existence of \( T_{\theta,1}^{-1} \) is independent of \( p, 1 \leq p \leq \infty \). Since \( \bar{X}_\Delta \subset \bar{X}_{\theta,1} \subset \bar{X}_{\theta,p} \), the restriction \( T_{\theta,1}^{-1}|\bar{X}_\Delta \) is constant for \( 1 \leq p \leq \infty \). Therefore we may consider only the operators \( T_{\theta,2} \) \( (\alpha < \theta < \beta) \). Clearly we can assume that \( 0 < \alpha < \beta < 1 \). Using the formula \((\bar{X}_{\alpha,2}, \bar{X}_{\beta,2})[\eta] = \bar{X}_{\theta,2}\) where \( 0 < \eta < 1 \), \( \theta = (1 - \eta)\alpha + \eta\beta \) (see [BL], Theorem 4.7.2), Corollary 5 implies that \( T_{\theta,2}^{-1}|\bar{X}_\Delta \) is constant for \( \alpha < \theta < \beta \).

\[\square\]

**Theorem 10.** Let \( \bar{X} = (X_0, X_1) \) be a compatible pair of Banach spaces, let \( T : \bar{X} \to X \), \( 0 < \theta < 1 \), \( 1 \leq p \leq \infty \). Then \( \sigma(T_{\theta,p}) \subset \sigma(T_{[\theta]}) \).
Proof. Suppose that $T_{[\theta]}$ is invertible. By Theorem 4 there exist $\theta_0, \theta_1$, $0 < \theta_0 < \theta < \theta_1 < 1$ such that $T_{[\theta_0]}$ and $T_{[\theta_1]}$ are invertible and $T_{[\theta_0]}^{-1} X_\Delta = T_{[\theta_1]}^{-1} X_\Delta$. By [BL], Theorem 4.7.2, $X_{\theta_2} = (X_{[\theta_0]}, X_{[\theta_1]} )_{\theta_2}$ where $\eta = \frac{\theta - \theta_0}{\theta_1 - \theta_0}$, so that $T_{\theta_2}$ is invertible. Since $\sigma(T_{\theta,p})$ is independent of $p$, we have $\sigma(T_{\theta,p}) \subset \sigma(T_{[\theta]} )$. 

Corollary 11. 

$$\cap \sigma(T_{\theta,p}) \leq \cap \sigma(T_0)^{1-\theta} \cdot \cap \sigma(T_1)^{\theta}$$

for all $\theta, p$ with $0 < \theta < 1$, $1 \leq p \leq \infty$.

The next example shows that the real and complex methods yield in general different spectra.

Example 12. Choose $1 < p_0 < p_1 < \infty$, $0 < \theta < 1$. Let $p$ satisfy $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

For the definition and basic properties of the Lorentz spaces $L_{rs}$ (with respect to the Lebesgue measure on the real line), see [BL] or [T]. In particular, we have

\[
(L_{p_0}, L_{p_1})_{[\theta]} = L_p, \\
(L_{p_0}, L_{p_1})_{\theta,1} = L_{p,1}, \\
(L_{p_0,1}, L_{p,1})_{[\theta]} = L_{p,1}, \\
(L_{p_0,1}, L_{p,1})_{\theta,1} = L_{p,1}.
\]

Further $L_{r,1} \subset L_r (= L_{rr})$ for all $r > 1$ and these two spaces are not isomorphic. Set

\[
X_0 = \cdots \oplus L_{p_0,1} \oplus L_{p_0,1} \oplus L_{p_0} \oplus L_{p_0} \oplus \cdots, \\
X_1 = \cdots \oplus L_{p_1,1} \oplus L_{p_1,1} \oplus L_{p_1} \oplus L_{p_1} \oplus \cdots
\]

($\ell_1$ direct sums). By [T], 1.18.1,

\[
(X_0, X_1)_{[\theta]} = \cdots \oplus L_{p,1} \oplus L_p \oplus \cdots, \\
(X_0, X_1)_{\theta,1} = \cdots \oplus L_{p,1} \oplus L_{p,1} \oplus \cdots.
\]

Let $T$ be the right shift operator. Then $T_{\theta,1}$ is invertible but $T_{[\theta]}$ is not invertible.

Problem. In the previous example the polynomial convex hulls of $\sigma(T_{[\theta]})$ and $\sigma(T_{\theta,p})$ coincide (and consequently, $r(T_{[\theta]}) = r(T_{\theta,p})$). Is this always the case?

References


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