A TWO-DIMENSIONAL HAHN-BANACH THEOREM

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ABSTRACT. Let $\tilde{T} = \sum_{i=1}^{n} \tilde{u}_i \otimes v_i : V \to V = [v_1, \ldots, v_n] \subset X$, where $\tilde{u}_i \in V^*$ and $X$ is a Banach space. Let $T = \sum_{i=1}^{n} u_i \otimes v_i : X \to V$ be an extension of $\tilde{T}$ to all of $X$ (i.e., $u_i \in X^*$) such that $T$ has minimal (operator) norm. In this paper we show in particular that, in the case $n = 2$ and the field is $\mathbb{R}$, there exists a rank-$n$ $\tilde{T}$ such that $kTk = k\tilde{T}k$ for all $X$ if and only if the unit ball of $V$ is either not smooth or not strictly convex. In this case we show, furthermore, that, for some $kTk = k\tilde{T}k$, there exists a choice of basis $v = v_1, v_2$ such that $k u_i k = k\tilde{u}_i k$, $i = 1, 2$; i.e., each $u_i$ is a Hahn-Banach extension of $\tilde{u}_i$.

1. Introduction

Let $X$ be a Banach space and $V$ be a subspace of $X$. If $T$ is an operator from $X$ into $V$, then $T|_V$ is the restriction of $T$ onto $V$. Let $\tilde{T} : V \to V$.

Define

$$e(\tilde{T}; X) = \inf \{ \|T\| : T|_V = \tilde{T} \}$$

and

$$e(\tilde{T}) = \sup_X \{ e(\tilde{T} ; X) : X \supset V \}.$$ 

The classical Hahn-Banach Theorem states that, for every rank-one operator $\tilde{T}$,

$$e(\tilde{T}) = \|\tilde{T}\|.$$ 

The Nachbin Theorem shows that, if $V$ is $n$-dimensional and, for every $\tilde{T} : V \to V$,

$$e(\tilde{T}) = \|\tilde{T}\|,$$

then $V \cong \ell_1^n$.

Definition 1.1. An operator $\tilde{T} : V \to V$ is called a Hahn-Banach operator if $e(\tilde{T}) = \|\tilde{T}\|$.

In this paper we are concerned with the existence of non-trivial Hahn-Banach operators on a Banach space $V$. In particular we show (Theorem 2.2) that the two-dimensional real Banach space $V$ possesses a rank-two Hahn-Banach operator if and only if $V$ is either non-smooth or not strictly convex.

For the proof of the theorem we will need a number of definitions.

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Let $X$ be a Banach space and $V$ be a finite-dimensional subspace. Let $T$ be a bounded linear operator from $X$ into $V$ and let $\tilde{T}$ be the restriction of $T$ onto $V$. ($\tilde{T} = T_{|_V}$.)

**Definition 1.2.** $T$ is a minimal norm extension of $\tilde{T}$ means

$$||T|| = \inf \{ ||S|| : S : X \to V : S|_V = \tilde{T} \}.$$ 

**Definition 1.3.** The extremal set of $T : X \to V$ is

$$\mathcal{E}(T) = \{(x^*, x^{**}) \in B(X^*) \times B(X^{**}) : x^*(T^{**}x^{**}) = ||T||\},$$

where $B$ denotes the unit ball and $T^{**}$ denotes the second adjoint extension of $T$ ($T^{**}x^{**} = \sum_{i=1}^n \langle u_i, x^{**} \rangle v_i$ and $||T^{**}|| = ||T||$).

**Note 1.1.** In the following we will designate by $x^* \otimes x^{**} : X \to X^{**}$ the usual dyad operator given by $x^* \otimes x^{**}(x) = \langle x, x^* \rangle x^{**}$.

**Theorem 1.1** ([2]). $T$ is a minimal norm extension of $\tilde{T}$ if and only if there exists a probability measure $\mu$ on $\mathcal{E}(T)$ such that the operator

$$E_T = \int x^* \otimes x^{**} \, d\mu$$

maps $V$ into $V$.

**Definition 1.4.** An $n$-dimensional subspace $V \subset L_1[\mathcal{T}, \nu]$ is said to be smooth if no non-zero element $v$ of $V$ vanishes on a set of positive $\nu$-measure.

**Note 1.2.** It is well known that the $n$-dimensional real space $V$ is smooth if and only if the unit sphere $S(V)$ is differentiable (has a unique $(n - 1)$-dimensional tangent plane) at every point. This fact follows immediately from $||\tilde{a}|| := ||\tilde{a} \cdot \tilde{v}||$ and the formula (see e.g. [1])

$$\frac{\partial}{\partial a_i} ||\tilde{a}|| = \int_T v_i(s) \operatorname{sgn}[\tilde{a} \cdot \tilde{v}(s)] \, d\nu(s)$$

for $\tilde{a} \neq 0$ and the Implicit Function Theorem. A point where $S(V)$ is non-differentiable is called a point of non-smoothness.

**Theorem 1.2** ([1]). Let $V = [\tilde{v}]$ be a smooth real subspace of $L_1[\mathcal{T}, \nu]$. Then $T = \sum_{i=1}^n u_i \otimes v_i$ is a minimal extension of $\tilde{T}$ (to $L_1[\mathcal{T}, \nu]$) implies that, for almost all $t$,

$$\tilde{u}(t) = ||T|| \tilde{z}(t)$$

with $\tilde{z}(t)$ being a point of intersection of $S(V)$ and its tangent plane perpendicular to $M\tilde{u}(t)$, where $M$ is the $n \times n$ matrix $M$ of $(E_T)_{|_V}$ with respect to the basis $\tilde{v}$. (See Figure 1.)

**Corollary 1.1.** Let $V = [\tilde{v}]$ be a smooth real subspace of $L_1[\mathcal{T}, \nu]$. Then without loss $T$ can be enlarged so that there exists $T = \sum_{i=1}^n u_i \otimes v_i$ which is a minimal extension of $\tilde{T}$ and the operator $T^*$ is an extension of $\tilde{T}^* : V^* \to V^*$.

**Proof.** Let $T = \sum_{i=1}^n u_i \otimes v_i$ be a minimal extension of $\tilde{T}$. Theorem 1.2 provides that $\tilde{u}/||T||$ lies on $S(V)$ a.e.$[\nu]$. Next replace $T$ by $T \cup \Delta T$, where $\Delta T = S(V) - \{ \tilde{u}/||T|| \}$, enlarge $\nu$ and $\tilde{v}$ to be zero on $\Delta T$, and enlarge $\tilde{u}/||T||$ to be the coordinate functions on $\Delta T$. Then let $T$ and $\tilde{T}$ be replaced by their corresponding enlargements, recall that $V^*$ is isometric to the space of coordinate functions of $S(V)$ regarded as a subspace of $L_\infty(S(V))$, and note that $T^*_{|_{V^*}} = \tilde{T}^*$. \qed
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Figure 1.

2. Main theorem

Lemma 2.1. Let $T$ be an extension of $\tilde{T}$. Then $\|T\| = \|\tilde{T}\|$ if and only if there exists a pair 

$$(x^*, v) \in \mathcal{E}(T), \ v \in V.$$ 

Hence the dyad operator $E_T = x^* \otimes v$ satisfies the conclusion of Theorem [1.2].

Proof. If $(x^*, v) \in \mathcal{E}(T)$, then 

$$\|T\| = x^*(Tv) = \|Tv\|.$$ 

Thus $T$ attains its norm on $V$ and 

$$\|T\| = \|Tv\| \leq \|	ilde{T}v\|.$$ 

Conversely, if $\|T\| = \|\tilde{T}\|$, let $v \in V$ be such that $\|	ilde{T}v\| = \|Tv\| = \|\tilde{T}\| = \|T\|$. Let $x^* \in X^*$ be such that $x^*(Tv) = \|Tv\|$. Then $(x^*, v) \in \mathcal{E}(T)$. 

The following theorem shows that, if $n > 1$ and the field is $\mathbb{R}$, a smooth, strictly convex, $n$-dimensional $L_1$-subspace $V$ possesses no rank-$n$ Hahn-Banach operator. In fact it shows more, namely, that $V$ possesses no rank-$k$ operator which extends to $L_1$ of the same norm, $k > 1$.

Theorem 2.1. Let $V$ be a smooth, strictly convex, $n$-dimensional real subspace of $L_1[-1, 1]$. Let $\tilde{T} : V \to V$ be such that there exists an extension $T : L_1[-1, 1] \to V$ with $\|T\| = \|\tilde{T}\|$. Then $\tilde{T}$ is a rank-one operator.

Proof. By Lemma 2.1 corresponding to $T = \sum_{i=1}^n u_i \otimes v_i$, there exists a rank-one operator $E_T$ satisfying the conclusion of the lemma. Hence the matrix $M$ defined as the matrix of $(E_T)_{uv}$, with respect to the basis $v$, has rank one.

Since $V$ is smooth there exists a minimal extension $T$ of $\tilde{T}$ satisfying the conditions of Theorem 1.2. But $M$ has rank one, whence the vectors $M\tilde{u}(t)$ lie on the same one-dimensional subspace of $\mathbb{R}^n$ for almost all $t \in [-1, 1]$.

Since $V$ is strictly convex, the vectors $\tilde{u}(t)$ also lie in the same one-dimensional subspace of $\mathbb{R}^n$ for almost all $t \in [-1, 1]$. Hence the operator $T$ given by Theorem 1.2 has rank one and thus so does $\tilde{T}$.

Corollary 2.1. $n$-dimensional Hilbert space $(\ell_2^{(n)})$ possesses no Hahn-Banach operator with rank $> 1$. 

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Proof. It is well known (see [5]) that $\ell_2^2$ can be realized as a subspace of $L_1$. \qed

**Theorem 2.2.** Let $V$ be a two-dimensional real Banach space. Then the following are equivalent:

(1) $V$ is smooth and strictly convex.

(2) For every rank-two operator

$$\tilde{T} : V \to V, \quad e(\tilde{T}) > \|\tilde{T}\|.$$  

Proof. (1) implies (2) by Theorem 2.1 since every two-dimensional space $V$ can be isometrically imbedded into $L_1[-1,1]$ (see e.g. [6]).

We now show (2) implies (1).

Suppose that $V$ with unit sphere $S(V) \subset \mathbb{R}^2$ is not strictly convex. Then (via a linear transformation) we can assume that $S(V)$ lies inside the unit square $C$, touches $C$ on all four sides, and has a flat spot running from $(1,d)$ to $(1,-d)$ (and of course a corresponding flat spot running from $(-1,-d)$ to $(-1,d)$), where “flat spot” is defined in the usual obvious way. Let $P$ be the rectangle with corners $(\pm1,\pm d)$, whence $P$ lies inside $S(V)$. (See Figure A.)

![Figure A](image)

Now consider the linear transformation $\tilde{T}(a_1, a_2) := (a_1, da_2)$ taking $C$ into $P$. Thus $\tilde{T}(S(V))$ lies inside $P$, which lies inside $S(V)$, and, since $\tilde{T}(1,0) = (1,0) \in S(V) \cap \tilde{T}(S(V))$, we see that $\tilde{T} : V \to V$ has norm one. Note that we can also write $\tilde{T}$ in the form

$$\tilde{T} = \tilde{\phi}_1 \otimes (1,0) + d\tilde{\phi}_2 \otimes (0,1),$$

where $\tilde{\phi}_i$ are the norm-1 coordinate functionals on $V$ defined by

$$\tilde{\phi}_i(a_1, a_2) = a_i, \quad i = 1, 2.$$

Next let $\phi_i$ be a Hahn-Banach extension to $X$ of the functional $\tilde{\phi}_i$ on $V$, $i = 1, 2$. Then the extension operator $T : X \to V$ has the form

$$T = \phi_1 \otimes (1,0) + d\phi_2 \otimes (0,1).$$

Finally, since $T(S(X)) \subset \tilde{T}(\overline{\overline{C}}) = \overline{\overline{C}}P \subset \overline{\overline{S(V)}} = B(V)$, we see that $\|T\| \leq 1$ and hence, since $\|T\| \geq \|\tilde{T}\| = 1$, it follows that $\|T\| = \|\tilde{T}\|$.

Furthermore, suppose that $V$ is strictly convex but not smooth. Then, as is well known, $V^*$ is smooth but not strictly convex, and the above argument shows that $V^*$ possesses a rank-two Hahn-Banach operator. But then, by Corollary 1.1
A reasonable conjecture is that this phenomenon holds for every rank-2 Hahn-Banach operator. 

**Corollary 2.2.** If the unit ball of the 2-dimensional real space \( V \) is either not smooth or not strictly convex, then \( V \) possesses a basis \( v_1, v_2 \) and a rank-2 Hahn-Banach operator \( \tilde{T} = \sum_{i=1}^{2} u_i \otimes v_i \) such that, for the extension \( T = \sum_{i=1}^{2} u_i \otimes v_i \), each \( \| u_i \| = \| \tilde{u}_i \| \), \( i = 1, 2 \), i.e., each \( u_i \) is a Hahn-Banach extension of \( \tilde{u}_i \). A reasonable conjecture is that this phenomenon holds for every rank-2 Hahn-Banach operator on \( V \).

**Note 2.1.** In [3] it is shown in particular that, for each real non-singular \( 2 \times 2 \) real matrix \( A \) not equal to a scalar multiple of the identity \( I \), there exists a real space \( V = [\tilde{v}] \not\cong \ell_{2}^{2} \) which possesses a Hahn-Banach operator \( \tilde{T} \) such that \( \tilde{T}\tilde{v} = A\tilde{v} \). Theorem 2.2 explains why any such space must be either non-smooth or non-strictly convex. Indeed all the spaces provided in [3] are hexagonal spaces.

Note, furthermore, that the rectangle \( P \) in the proof of Theorem 2.2 can be replaced by any parallelogram (symmetric with respect to the origin) contained inside \( S(V) \) and intersecting the flat sides of \( S(V) \). In this way we see that \( V \) possesses an entire family of two-dimensional Hahn-Banach operators of norm-1 \( \tilde{T}p = p \) for some \( p \in S(V) \cap \tilde{T}(S(V)) \) of the form

\[
\tilde{T} = (\alpha \tilde{\phi}_1 + \beta \tilde{\phi}_2) \otimes (1, 0) + (\gamma \tilde{\phi}_1 + \delta \tilde{\phi}_2) \otimes (0, 1),
\]

all of which have two real eigenvalues (1 and \( e \), where \( |e| \leq 1 \)). In [4] we show that, if the spectrum of \( \tilde{T} \neq I \) lies on the unit circle, then the Hahn-Banach operator \( \tilde{T} \) has norm \( > 1 \).

**Corollary 2.3.** If the two-dimensional real Banach space \( V \) possesses a rank-two Hahn-Banach operator \( \tilde{T} \neq I \) whose spectrum lies on the unit circle, then \( V \) also possesses a rank-two Hahn-Banach operator with eigenvalues 1, \( d \), |\( d \)| < 1.

**Corollary 2.4.** The two-dimensional real Banach space \( V \) possesses a rank-two Hahn-Banach operator if and only if its dual space \( V^* \) possesses a rank-two Hahn-Banach operator.

**Corollary 2.5.** For every two-dimensional subspace \( V \) of \( \ell_{1} \) or \( \ell_{\infty} \) there exists a rank-two operator \( \tilde{T} : V \to V \) such that \( e(\tilde{T}) = \| \tilde{T} \| \).

For every two-dimensional subspace \( V \) of \( \ell_{p} \) \((p \in (1, \infty))\) and for every rank-two operator \( \tilde{T} : V \to V \) we have \( e(\tilde{T}) > \| \tilde{T} \| \).

**Example 2.1.** Let \( V = [\tilde{v}] \subset L^{1}[-1, 1] \), where \( (v_1, v_2) = (\tfrac{1}{2}, t) \) and let \( \tilde{T}v_1 = v_1 \) and \( \tilde{T}v_2 = av_2 \), \( a \geq \tfrac{1}{2} \). Then Theorem 1.2 yields that the minimal extension operator \( T = \sum_{i=1}^{2} u_i \otimes v_i : L^{1}[-1, 1] \to V \) is given by

\[
\tilde{u} = \kappa \left[ \left( \frac{1, 2mt}{\sqrt{1 + 4m^{2}t^{2}}} \right) + \left( 0, \frac{\text{sgn} t}{2} \right) \right],
\]

where \( \kappa = \| T \|, m \) is a non-negative constant, and \( M = (\begin{smallmatrix} 1 & 0 \\ 0 & m \end{smallmatrix}) \). (In fact \( a = (\tfrac{1}{2} + \int_{0}^{1} \frac{4mt^{2}dt}{\sqrt{1 + 4m^{2}t^{2}}} ))/ \int_{0}^{1} \frac{dt}{\sqrt{1 + 4m^{2}t^{2}}} \). Since \( V \) is smooth, Theorem 2.2 provides that \( \| T \| = \| \tilde{T} \| = 1 \) precisely when \( m = 0 \). Note that then \( u_1 = 1 \) and \( u_2 = \tfrac{1}{2} \text{sgn} t \) shows that \( u_i \in L_{\infty}[-1, 1] \) is the Hahn-Banach extension of \( \tilde{u}_i, i = 1, 2 \). (See Figure 2.)
As a further example, following Corollary 1.1, note that the space $V$ is also realized as $V = \left[\frac{1}{2}, t\right] \subset L^1(\mathbb{R}, \nu)$, where $\nu$ is Lebesgue measure on $[-1, 1]$ and zero off $[-1, 1]$. Thus replacing $[-1, 1]$ by $\mathbb{R}$ in the above, we see that, for $m > 0$, $\bar{\nu}/\|T\|$ now covers all of $S(V)$ and, for $m = 0$, let $\bar{\nu}(t)/\|T\| = \bar{\nu}(t)$, $|t| > 1$, be any convenient parameterization of the curved portion of $S(V)$. Thus for $m = 0$ the adjoint operator $T^*$ (with same-norm extension $T^*$) provides a Hahn-Banach operator for the non-smooth, strictly convex space $V^* = [t, 1 - t^2] \subset L_\infty[-1, 1]$ with a “corner” at $(1, 0)$ ($V^* \cong [\bar{\nu}/\|T\|] \subset L_\infty[\mathbb{R}]$).

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REFERENCES


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