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A RESULT ABOUT A SELECTION PROBLEM OF MICHAEL

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Abstract. It is shown that a continuum that is an S_4 space in the sense of Michael must be hereditarily decomposable. This improves known results, thereby providing more evidence that such continua must be dendrites.

1. Introduction

We let 2^X denote the space of all non-empty, compact subsets of a metric space X; 2^X has the Vietoris topology or, equivalently, the Hausdorff metric topology [7].

Let $\mathcal{F} \subseteq 2^X$. A *continuous selection for* $\mathcal{F}$ is a continuous function $\sigma\colon \mathcal{F} \to X$ such that $\sigma(F) \in F$ for each $F \in \mathcal{F}$.

Michael [6, p.178] defined an S_4 *space* to be a space X such that there is a continuous selection for every $\mathcal{F} \subseteq 2^X$ such that the members of $\mathcal{F}$ are mutually disjoint and $\bigcup \mathcal{F} = X$. The question of which spaces are S_4 spaces is asked in [6, p.155], and some partial answers are in [6, pp.178–179].

It is known that S_4 spaces cannot contain a nondegenerate hereditarily indecomposable continuum (5.14 of [7, p.260]). We show that a nondegenerate, separable metric space that is an S_4 space can be separated by a countable set (Theorem 7.1 in section 7, which is somewhat stronger). Thus, a nondegenerate continuum that is an S_4 space must be hereditarily decomposable.

In the proof of Theorem 7.1 we use an idea that is new to the theory of continuous selections; namely, we use connectivity functions. Our results about connectivity functions seem to be of independent interest. In particular, we prove some results about connectivity bijections.

2. Terminology

We denote the real numbers by $\mathbb{R}$. The cardinality $\mathbb{R}$ is denoted by $\mathfrak{c}$. We denote the unit circle by S^1.

A continuum is a non-empty compact connected metric space.

If X is a topological space and $A \subseteq X$ is such that $X \setminus A$ is separated (i.e., not connected), we say A is a separator of X.

The 2-fold symmetric product of X with the Vietoris topology is denoted by $F_2(X)$, as in [7].

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We say that a space X is $\mathfrak{c}$-connected provided that X has no separator of cardinality less than continuum.

We say X is *weak S_4 space* provided that there is a continuous selection for every $\mathcal{F} \subseteq F_2(X)$ such that the members of $\mathcal{F}$ are mutually disjoint and $\bigcup \mathcal{F} = X$. Clearly, every S_4 space is a weak S_4 space.

We use cl to denote closure.

3. S_4 spaces and weak S_4 spaces

In [6, p.178] it is stated that S^1 is not an S_4 space and that finite trees are S_4 spaces. It is in fact easy to check that there is no continuous selection for $\mathcal{E} = \{\{x, -x\} : x \in S^1\} \subseteq 2^{S^1}$, where $-x$ denotes the point antipodal to x in S^1.

We verify two basic facts about S_4 and weak S_4 spaces.

Proposition 3.1. *Both properties S_4 and weak S_4 are hereditary with respect to subsets.*

Proof. Suppose X is an S_4 space and $A \subseteq X$. Let $\mathcal{E}$ be a cover of A by mutually disjoint non-empty compact sets. To find a continuous selection for $\mathcal{E}$ consider the cover of X by mutually disjoint non-empty compact sets given by $\mathcal{E}^* = \mathcal{E} \cup \{\{x\} : x \in X \setminus A\}$. Since X is S_4, there is a continuous selection $\Psi : \mathcal{E}^* \to X$. The restriction $\Psi|\mathcal{E} : \mathcal{E} \to A$ is a continuous selection for $\mathcal{E}$. Thus, A is an S_4 space. The argument for weak S_4 spaces is similar. □

Proposition 3.1 implies that if a space X can be homeomorphically embedded into an S_4 or weak S_4 space, then X is S_4 or weak S_4. In fact, we can say a little bit more:

Proposition 3.2. *Suppose Y is an S_4 (weak S_4) space and $f : X \to Y$ is continuous and injective. Then X is an S_4 (weak S_4) space.*

Proof. Assume Y is an S_4 space. Let $F : 2^X \to 2^Y$ be the map induced by f, that is, $F(A) = f[A]$ for each $A \in 2^X$. Note that F is continuous. Let $\mathcal{E}$ be a cover of X by mutually disjoint non-empty compact sets. Since f is injective and continuous, $\mathcal{E}^* = \{f[E] : E \in \mathcal{E}\}$ is a cover of $f[X]$ by mutually disjoint non-empty compact sets. By Proposition 3.1 $f[X] \subseteq Y$ is an S_4 space. So, there is a continuous selector $\Psi^* : \mathcal{E}^* \to f[X]$. Define $\Psi : \mathcal{E} \to X$ by $\Psi(E) = (f^{-1} \circ \Psi^* \circ F)(E)$. Clearly, Ψ is a selection. We show Ψ is continuous. Let $E_n \in \mathcal{E}$ and $\lim_{n\to\infty} E_n = E$. We show that $\lim_{n\to\infty} \Psi(E_n) = \Psi(E)$. Let $\mathcal{K} = \{E_n : 1 \le n < \infty\} \cup \{E\}$ and let $K = \bigcup \mathcal{K}$. Notice that

$$(\Psi^* \circ F)[\mathcal{K}] = \Psi^*[\{f[E]\} \cup \{\{f[E_n]\} : 1 \le n < \infty\}] \subseteq f[K].$$

It follows that $\Psi|\mathcal{K} = (f^{-1}|f[K] \circ \Psi^* \circ F)|\mathcal{K}$. Since K is compact (by 5.6 of [6, p. 168]), $f|K$ is a homeomorphism. Since $f|K$ is a homeomorphism, $f^{-1}|f[K]$ is continuous. Thus,

$$\Psi|(\{E\} \cup \{\{E_n\} : 1 \le n < \infty\}) = (f^{-1}|f[K] \circ \Psi^* \circ F)|(\{E\} \cup \{\{E_n\} : 1 \le n < \infty\})$$

is continuous. Therefore, $\lim_{n\to\infty} \Psi(E_n) = \Psi(E)$.

The proof for the case when X is a weak S_4 space is similar. □

4. Connected spaces

We prove some useful facts about $\mathfrak{c}$-connected spaces and connected spaces in general. The following result may be known, but we include a proof for completeness.

Lemma 4.1. *Let X be a metric space, and let $D \subseteq X$. Then $D \subseteq X$ is a dense connected subspace of X if and only if D has non-empty intersection with every closed separator of X.*

Proof. Suppose D is a dense connected subset of X. By way of contradiction, assume there is a closed separator F of X such that $F \cap D = \emptyset$. Let $X \setminus F = U \cup V$ where U and V are disjoint non-empty open subsets of X. Since D is connected, $D \cap U = \emptyset$ or $D \cap V = \emptyset$; in either case the density of D is violated. So $F \cap D \neq \emptyset$.

Suppose D has non-empty intersection with every closed separator of X. We show D is connected. If D were separated, then $X \setminus D$ would be a separator of X. By [5, Theorem 3, p. 155], $X \setminus D$ contains a closed separator of X, but such a separator would have empty intersection with D contradicting our assumptions about D. Thus, D is connected. To see that D is dense it is enough to notice that for any $x \in X$ and sufficiently small open neighborhoods U of x, the boundary of U is a closed separator of X. □

Lemma 4.2. *Let Y be a $\mathfrak{c}$-connected nondegenerate separable metric space. Then there exist $\mathfrak{c}$-connected dense sets $Y_1, Y_2 \subseteq Y$ such that $Y = Y_1 \cup Y_2$ and $Y_1 \cap Y_2 = \emptyset$.*

Proof. Recall that separable metric spaces have at most $\mathfrak{c}$-many closed subsets. Let $\{T_\alpha\}_{\alpha \in \mathfrak{c}}$ be an enumeration of the closed separators of Y such that each closed separator appears $\mathfrak{c}$-many times in the enumeration. We construct two sequences $\{a_\alpha\}_{\alpha \in \mathfrak{c}}$ and $\{b_\alpha\}_{\alpha \in \mathfrak{c}}$ of points in Y such that for each $\alpha \in \mathfrak{c}$ we have

(1) $a_\alpha, b_\alpha \in T_\alpha$,
(2) $a_\alpha \notin \{a_\beta \colon \beta < \alpha\} \cup \{b_\beta \colon \beta \leq \alpha\}$, and
(3) $b_\alpha \notin \{b_\beta \colon \beta < \alpha\} \cup \{a_\beta \colon \beta \leq \alpha\}$.

Suppose $\alpha < \mathfrak{c}$ and we have constructed $\{a_\beta \colon \beta < \alpha\}$ and $\{b_\beta \colon \beta < \alpha\}$ satisfying (1), (2), and (3). We show how to pick a_α and b_α so that (1), (2), and (3) are satisfied. Since T_α separates Y, we have $|T_\alpha| = \mathfrak{c}$. Pick distinct points $a_\alpha, b_\alpha \in T_\alpha \setminus (\{a_\beta \colon \beta < \alpha\} \cup \{b_\beta \colon \beta < \alpha\})$. Clearly, a_α and b_α are as desired.

Let $Y_1 = \{a_\alpha \colon \alpha < \mathfrak{c}\}$ and $Y_2 = Y \setminus Y_1$. Clearly, $Y = Y_1 \cup Y_2$ and $Y_1 \cap Y_2 = \emptyset$, and by (2) and (3) we have $\{b_\alpha \colon \alpha < \mathfrak{c}\} \subseteq Y_2$. Let $i \in \{1, 2\}$. Suppose $S \subseteq Y_i$ and $|S| < \mathfrak{c}$. By (1) and the fact that each closed separator appears $\mathfrak{c}$-many times in the enumeration, $|T \cap Y_i| = \mathfrak{c}$ for every closed separator T of Y. So, $T \cap (Y_i \setminus S) \neq \emptyset$ for any closed separator T of Y. By Lemma 4.1 we have that $Y_i \setminus S$ is connected and dense in Y. Thus, Y_i is dense in Y and $\mathfrak{c}$-connected. □

5. Type 1 and type 2 separators

To prove Theorem 7.1 we will need to notice that there are two basic types of closed separators of the product of two spaces:

Lemma 5.1. *Let X and Y be connected spaces, let S be a closed separator of $X \times Y$, and let*

$$H_S = \{x \in X \colon \pi_X^{-1}(x) \subseteq S\}.$$

Then one of the following holds:

(1) *H_S is a separator of X;*
(2) *there is a non-empty open set U in X such that $\pi_X^{-1}(x) \cap S$ is a separator of $\{x\} \times Y$ for all $x \in U$.*

Proof. Let $(X \times Y) \setminus S = W \cup V$ where $W \cap V = \emptyset$ and both W and V are non-empty and open in $X \times Y$. Let $W_1 = \pi_X(W)$ and $V_1 = \pi_X(V)$. There two possible cases.

Case 1: $W_1 \cap V_1 = \emptyset$. We show that (1) holds. Notice that $X \setminus (W_1 \cup V_1)$ separates X. Clearly, $H_S = X \setminus (W_1 \cup V_1)$. So, H_S is a separator of X.

Case 2: $W_1 \cap V_1 \neq \emptyset$. We show that (2) holds with $W_1 \cap V_1 = U$. Let $x \in U$. Since $(\{x\} \times Y) \cap W$ and $(\{x\} \times Y) \cap V$ are non-empty disjoint open subsets of $\{x\} \times Y$, it follows that $(\{x\} \times Y) \cap S$ separates $\{x\} \times Y$. So, $\pi_X^{-1}(x) \cap S$ is a separator of $\{x\} \times Y$. Thus, (2) holds with $U = W_1 \cap V_1$. □

For $i \in \{1, 2\}$, we say that a closed separator S of $X \times Y$ is a type i separator in $X \times Y$ if S satisfies (i) of Lemma 5.1. Note that a type i separator of $X \times Y$ may not be a type i separator of $Y \times X$, where $i \in \{1, 2\}$.

Notice that the proof of Lemma 5.1 works for connected spaces in general.

6. Connectivity functions

Throughout the remainder of this paper do not distinguish between a function and its graph.

A function $f\colon X \to Y$ is called a connectivity function provided that the graph of $f|C$ is connected for every connected subset C of X. Connectivity functions have been studied extensively (e.g., [1]–[3]). We prove some results about connectivity functions and functions with connected graphs.

Lemma 6.1. *Let X,Y_1, and Y_2 be nondegenerate connected separable metric spaces such that Y_1 and Y_2 are $\mathfrak{c}$-connected. Let $\mathcal{C}$ be a collection of $\leq \mathfrak{c}$ nondegenerate connected subsets of X. Then there is a function $F\colon X \to Y_1 \times Y_2$ such that*

(i) *$|\{x \in X\colon y_i = \pi_{Y_i}(F(x))\}| = 1$ for all $y_i \in Y_i$ and $i = 1, 2$;*
(ii) *$F|C$ is connected and dense in $C \times (Y_1 \times Y_2)$ for each $C \in \mathcal{C}$.*

Proof. Let $\{T_\alpha\}_{\alpha \in \mathfrak{c}}$ be an enumeration of all subsets of $X \times (Y_1 \times Y_2)$ each of which is a type 2 (closed) separator of $C \times (Y_1 \times Y_2)$ for some $C \in \mathcal{C}$. (Indeed, there are at most $\mathfrak{c}$ such sets T_α since $|\mathcal{C}| \leq \mathfrak{c}$ and $C \times (Y_1 \times Y_2)$ has only $\mathfrak{c}$ closed sets for each $C \in \mathcal{C}$.)

We first show how to define six injective sequences

$$R_1 = \{x_\alpha \in X\}_{\alpha \in \mathfrak{c}},\ R_2 = \{z_\alpha \in X\}_{\alpha \in \mathfrak{c}},\ R_3 = \{y_\alpha^1 \in Y_1\}_{\alpha \in \mathfrak{c}},\ R_4 = \{y_\alpha^2 \in Y_2\}_{\alpha \in \mathfrak{c}},$$
$$R_5 = \{w_\alpha^1 \in Y_1\}_{\alpha \in \mathfrak{c}},\ R_6 = \{w_\alpha^2 \in Y_2\}_{\alpha \in \mathfrak{c}}$$

satisfying the following three conditions for all $\alpha \in \mathfrak{c}$:

(1) $(x_\alpha, y_\alpha^1, y_\alpha^2) \in T_\alpha$;
(2) $x_\beta \neq z_\lambda$ for all $\beta, \lambda \leq \alpha$;
(3) $y_\beta^i \neq w_\lambda^i$ for all $\beta, \lambda \leq \alpha$ and $i = 1$ and 2.

We use transfinite induction, as follows: Assume that we have defined the terms of the sequences for all $\beta < \alpha$.

(a) Define z_α, w_α^1, and w_α^2 as follows: Pick

$$z_\alpha \in X \setminus \{x_\beta, z_\beta \in X\colon \beta < \alpha\}$$

and, for $i = 1$ and 2, pick

$$w^i_\alpha \in Y_i \setminus \{y^i_\beta, w^i_\beta \in Y_i \colon \beta < \alpha\}.$$

(b) Define x_α as follows: Since T_α is a type 2 separator of $C \times (Y_1 \times Y_2)$ for some $C \in \mathcal{C}$, there is a nonempty open set U in C such that $\pi_X^{-1}(x) \cap T_\alpha$ is a separator of $\{x\} \times (Y_1 \times Y_2)$ for each $x \in U$. Now, noting that $|U| = \mathfrak{c}$, pick

$$x_\alpha \in U \setminus \{x_\beta, z_\gamma \in X \colon \beta < \alpha \text{ and } \gamma \leq \alpha\}.$$

(c) Finally, define y^1_α and y^2_α as follows: Let T_α, U, and x_α be as in (b), and let

$$Q = \pi_{Y_1 \times Y_2}[\pi_X^{-1}(x_\alpha) \cap T_\alpha].$$

Since $x_\alpha \in U$, $\pi_X^{-1}(x_\alpha) \cap T_\alpha$ is a closed separator of $\{x_\alpha\} \times (Y_1 \times Y_2)$; hence, Q is a closed separator of $Y_1 \times Y_2$. We pick y^1_α and y^2_α according to two cases (which exhaust all possibilities by Lemma 5.1):

Case 1: Q is a type 1 separator of $Y_1 \times Y_2$. Then,

$$H_Q = \{y \in Y_1 \colon \pi_{Y_1}^{-1}(y) \subseteq Q\} \text{ is a separator of } Y_1.$$

Thus, since Y_1 is $\mathfrak{c}$-connected, $|H_Q| = \mathfrak{c}$. Hence, we can pick

$$y^1_\alpha \in H_Q \setminus \{y^1_\beta, w^1_\gamma \in Y_1 \colon \beta < \alpha \text{ and } \gamma \leq \alpha\}.$$

Pick

$$y^2_\alpha \in Y_2 \setminus \{y^2_\beta, w^2_\gamma \in Y_2 \colon \beta < \alpha \text{ and } \gamma \leq \alpha\}.$$

Case 2: Q is a type 2 separator of $Y_1 \times Y_2$. Then there is a nonempty open set V in Y_1 such that $\pi_{Y_1}^{-1}(y) \cap Q$ is a separator of $\{y\} \times Y_2$ for each $y \in V$. Thus, since $|V| = \mathfrak{c}$, we can pick

$$y^1_\alpha \in V \setminus \{y^1_\beta, w^1_\gamma \in Y_1 \colon \beta < \alpha \text{ and } \gamma \leq \alpha\}.$$

Since Y_2 is $\mathfrak{c}$-connected and $y^1_\alpha \in V$, $|\pi_{Y_1}^{-1}(y^1_\alpha) \cap Q| = \mathfrak{c}$; hence, we can pick

$$y^2_\alpha \in \pi_{Y_2}[\pi_{Y_1}^{-1}(y^1_\alpha) \cap Q] \setminus \{y^2_\beta, w^2_\gamma \in Y_2 \colon \beta < \alpha \text{ and } \gamma \leq \alpha\}.$$

We have finished defining the six injective sequences

$$R_1 = \{x_\alpha \in X\}_{\alpha \in \mathfrak{c}},\ R_2 = \{z_\alpha \in X\}_{\alpha \in \mathfrak{c}},\ R_3 = \{y^1_\alpha \in Y_1\}_{\alpha \in \mathfrak{c}},\ R_4 = \{y^2_\alpha \in Y_2\}_{\alpha \in \mathfrak{c}},$$
$$R_5 = \{w^1_\alpha \in Y_1\}_{\alpha \in \mathfrak{c}},\ R_6 = \{w^2_\alpha \in Y_2\}_{\alpha \in \mathfrak{c}}.$$

We show that the sequences satisfy (1): $(x_\alpha, y^1_\alpha, y^2_\alpha) \in T_\alpha$ for all $\alpha \in \mathfrak{c}$. In Case 1 we picked $y^1_\alpha \in H_Q = \{y \in Y_1 \colon \pi_{Y_1}^{-1}(y) \subseteq Q\}$ and we picked $y^2_\alpha \in Y_2$. Hence, $(y^1_\alpha, y^2_\alpha) \in Q$. Therefore, since

$$Q = \pi_{Y_1 \times Y_2}[\pi_X^{-1}(x_\alpha) \cap T_\alpha],$$

we have that $(x_\alpha, y^1_\alpha, y^2_\alpha) \in T_\alpha$. In Case 2, we picked $y^1_\alpha \in V \subseteq Y_1$ and we picked $y^2_\alpha \in \pi_{Y_2}[\pi_{Y_1}^{-1}(y^1_\alpha) \cap Q]$. Hence, $(y^1_\alpha, y^2_\alpha) \in Q$. Therefore, again we have that $(x_\alpha, y^1_\alpha, y^2_\alpha) \in T_\alpha$. This completes the proof of (1).

It is clear that the sequences $R_1, ..., R_6$ satisfy conditions (2) and (3).

Next, we define the function $F \colon X \to Y_1 \times Y_2$. Note that $|R_i| = \mathfrak{c}$ for each $i = 1, ..., 6$. Also, note that

$$X \setminus R_1 \supset R_2,\ Y_1 \setminus R_3 \supset R_5,\ Y_2 \setminus R_4 \supset R_6.$$

Hence, there are bijections $g\colon X \setminus R_1 \twoheadrightarrow Y_1 \setminus R_3$ and $h\colon X \setminus R_1 \twoheadrightarrow Y_2 \setminus R_4$. Let

$$F(x) = \begin{cases} (y^1_\alpha, y^2_\alpha), & x = x_\alpha \in R_1, \\ (g(x), h(x)), & x \in X \setminus R_1. \end{cases}$$

Finally, we show that F satisfies conditions (i) and (ii) of our lemma.

We prove (i) for $y_1 \in Y_1$ (the proof for $y_2 \in Y_2$ is similar). Fix $y_1 \in Y_1$, and let

$$n = |\{x \in X : y_1 = \pi_{Y_1}(F(x))\}|.$$

We first show that $n \geq 1$. If $y_1 \in R_3$, say $y_1 = y^1_\alpha$, then $F(x_\alpha) = (y^1_\alpha, y^2_\alpha)$ and, hence,

$$\pi_{Y_1}(F(x_\alpha)) = y^1_\alpha = y_1.$$

If $y_1 \in Y_1 \setminus R_3$, then, since $g(X \setminus R_1) = Y_1 \setminus R_3$, there exists $x \in X \setminus R_1$ such that $g(x) = y_1$; hence.

$$\pi_{Y_1}(F(x)) = g(x) = y_1.$$

Therefore, we have shown that $n \geq 1$. Now, suppose by way of contradiction that $n > 1$. Then, there exist $a \neq b$ in X such that

$$\pi_{Y_1}(F(a)) = y_1 = \pi_{Y_1}(F(b)).$$

We obtain a contradiction in each of the following three cases (which take care of all posssibilities).

Case 1: $a, b \in R_1$. Then, $a = x_\alpha$ and $b = x_\beta$ with $\alpha \neq \beta$; hence,

$$y^1_\alpha = \pi_{Y_1}(F(a)) = \pi_{Y_1}(F(b)) = y^1_\beta;$$

however, since $\alpha \neq \beta$, $y^1_\alpha \neq y^1_\beta$.

Case 2: $a, b \in X \setminus R_1$. Then,

$$g(a) = \pi_{Y_1}(F(a)) = \pi_{Y_1}(F(b)) = g(b),$$

which contradicts that g is one-to-one.

Case 3: $a \in R_1$, $b \in X \setminus R_1$. Since $a \in R_1$, $a = x_\alpha$ for some α, so

$$(*) \qquad \pi_{Y_1}(F(a)) = y^1_\alpha \in R_3.$$

Since $b \in X \setminus R_1$ and $g\colon X \setminus R_1 \to Y_1 \setminus R_3$,

$$(\#) \qquad \pi_{Y_1}(F(b)) = g(b) \in Y_1 \setminus R_3.$$

However, $(*)$ and $(\#)$ contradict that $\pi_{Y_1}(F(a)) = \pi_{Y_1}(F(b))$.

This completes the proof that F satisfies condition (i).

To show that F satisfies condition (ii) of our lemma, let $C \in \mathcal{C}$. By Lemma 4.1, it suffices to show that F intersects every closed separator of $C \times (Y_1 \times Y_2)$. Let T be a closed separator of $C \times (Y_1 \times Y_2)$. We take two cases (which represent all possibilities by Lemma 5.1):

Case 1: T is a type 1 separator of $C \times (Y_1 \times Y_2)$. Then

$$H_T = \{x \in C : \pi_X^{-1}(x) \subseteq T\} \textit{ is a separator of } C.$$

Hence, $H_T \neq \emptyset$. Thus, $\exists\, p \in H_T$. Therefore, since any function $k\colon C \to Y_1 \times Y_2$ intersects $\pi_C^{-1}(p)$, $F \cap T \neq \emptyset$.

Case 2: T is a type 2 separator of $C \times (Y_1 \times Y_2)$. Then, $T = T_\alpha$ for some α. Thus, since the sequences R_i satisfy condition (1), $(x_\alpha, y^1_{\alpha,} y^2_\alpha) \in T$. Therefore, since $F(x_\alpha) = (y^1_\alpha, y^2_\alpha)$, $F \cap T \neq \emptyset$. $\square$

We state two corollaries to Lemma 6.1; we do not use the corollaries later, but we feel that the corollaries are of interest in themselves.

Corollary 6.1. *If Y is a nondegenerate $\mathfrak{c}$-connected separable metric space and $I \subseteq \mathbb{R}$ is a nondegenerate interval, then there is a connectivity bijection $g\colon I \to Y$ with dense graph.*

Proof. Let $\mathcal{C} = \{C \subseteq I\colon C \text{ is connected and nondegenerate}\}$; notice that $|\mathcal{C}| = \mathfrak{c}$. Applying Lemma 6.1 with $Y_1 = Y_2 = Y$, there is a function $F\colon I \to (Y_1 \times Y_2)$ such that

(i) $|\{x \in X\colon y_i = \pi_{Y_i}(F(x))\}| = 1$ for every $y_i \in Y_i$ where $i \in \{1, 2\}$ and
(ii) $F|C$ is connected and dense in $C \times (Y_1 \times Y_2)$ for each $C \in \mathcal{C}$.

By (ii) and our choice of $\mathcal{C}$, F is a connectivity function. Let $g = \pi_{Y_1} \circ F$. Since g is the composition of a connectivity function followed by a continuous function, it follows from [4, Lemma 4.1] that g is connectivity. By (i) we know that g is a bijection. By (ii) g is dense in $I \times Y$. □

In connection with the corollary just proved note that the only connectivity bijections of $[0, 1]$ into itself are homeomorphisms.

Corollary 6.2. *If Y is a nondegenerate $\mathfrak{c}$-connected separable metric space and X is a nondegenerate continuum, then there is a bijection $g\colon X \to Y$ such that $g|C$ is connected and dense in $C \times Y$ for any nondegenerate subcontinuum C of X.*

Proof. The proof is similar to the proof of Corollary 6.1 with the modification that $\mathcal{C} = \{C \subseteq X\colon\ C \text{ is a nondegenerate subcontinuum of } X\}$. □

7. Theorem 7.1

We state Theorem 7.1 and two corollaries, after which we prove Theorem 7.1.

Theorem 7.1. *Let Y be a nondegenerate separable metric space. If Y is a weak S_4 space, then Y is not $\mathfrak{c}$-connected.*

Corollary 7.1. *If a nondegenerate continuum X is a weak S_4 space, then X can be separated by countably many points.*

Proof. For complete separable metric spaces, having no countable separator and being $\mathfrak{c}$-connected are equivalent. This follows from [5, Theorem 3 p. 155] and the fact that uncountable closed subsets of complete separable metric spaces have cardinality $\mathfrak{c}$. □

Corollary 7.2. *If a nondegenerate continuum X is a weak S_4 space, then X is hereditarily decomposable.*

Proof. If X is indecomposable, then X has uncountably many mutually disjoint composants, each of which is a dense connected subset of X [8, pp. 83 and 204]; so, no countable set can separate X, and hence X would not be weak S_4 by Corollary 7.1. Thus, no nondegenerate indecomposable continuum is a weak S_4 space. Since being weak S_4 is hereditary by Proposition 3.1, any weak S_4 continuum must be hereditarily decomposable. □

We use the following two lemmas in the proof of Theorem 7.1.

Lemma 7.1. *Let Y_1 and Y_2 be $\mathfrak{c}$-connected separable metric spaces. Then there is a dense connectivity function $h\colon [0,1) \to Y_1 \times Y_2$ such that*

$$|\{x \in [0,1)\colon y_i = \pi_{Y_i}(h(x))\}| = 1$$

for every $y_i \in Y_i$ where $i \in \{1,2\}$.

Proof. Let $\mathcal{C} = \{C \subseteq [0,1)\colon C \text{ is connected and nondegenerate}\}$. Notice that $|\mathcal{C}| = \mathfrak{c}$ and apply Lemma 6.1. □

In what follows, consider the unit circle S^1 as the disjoint union of

$$A = \{(\cos(\theta), \sin(\theta))\colon \theta \in [0,\pi)\} \quad \text{and} \quad B = \{(\cos(\theta), \sin(\theta))\colon \theta \in [\pi, 2\pi)\}.$$

Note that A and B are both homeomorphic to $[0,1)$. For each point $p \in S^1$, we let $-p$ denote the point antipodal to p. The homeomorphism of S^1 onto itself defined by $p \to -p$ is denoted by ρ. We will let Y be a fixed nondegenrate $\mathfrak{c}$-connected separable metric space, and we let $Y = Y_1 \cup Y_2$, where Y_1 and Y_2 are as in Lemma 4.2.

Lemma 7.2. *Identifying $[0,1)$ and A, let $F\colon A \to Y_1 \times Y_2$ be the dense connectivity function guaranteed by Lemma 7.1. Then the function $F^*\colon S^1 \to Y$ defined by*

$$F^*(x) = \begin{cases} \pi_{Y_1}(F(x)) & \text{if } x \in A; \\ \pi_{Y_2}(F(-x)) & \text{if } x \in B \end{cases} \tag{1}$$

is a bijection and for any connected subset C of S^1, the set

$$P(C) = \{\{(x, F^*(x)), (-x, F^*(-x))\}\colon x \in C\}$$

is a connected subset of the 2-fold symmetric product $F_2(S^1 \times Y)$.

Proof. That F^* is a bijection follows from (i) of Lemma 7.1 and the fact that Y is the disjoint union of Y_1 and Y_2.

Let $C \subseteq S^1$ be connected. We prove $P(C)$ is connected by considering four cases:

Case 1: $C \subseteq A$.

Since F is a connectivity function, $F|C$ is connected. Define $\Theta\colon F|C \to P(C)$ by $\Theta((x, F(x))) = \{(x, F^*(x)), (-x, F^*(-x))\}$. The function Θ is easily checked to be continuous, so $P(C) = \Theta(F|C)$ is connected.

Case 2: $C \subseteq B$.

There is a $D \subseteq A$ such that $-C = D$. By Case 1, $P(D)$ is connected. Notice that $P(D) = P(C)$, so $P(C)$ is connected.

Case 3: $C_1 = C \cap A$ and $C_2 = C \cap B$ are both connected and non-empty.

By the first two cases we know that $P(C_1)$ and $P(C_2)$ are both connected. Let $p \in \mathrm{cl}(C_1) \cap \mathrm{cl}(C_2)$. Assume $p \in C_2$. It is enough for us to show that $\{(p, F^*(p)), (-p, F^*(-p))\} \in \mathrm{cl}(P(C_1))$. Since $F|C_1$ is dense in $C_1 \times (Y_1 \times Y_2)$, it follows that there is a sequence of points $p_n \in C_1$ such that $\lim_{n\to\infty} p_n = p$ and $\lim_{n\to\infty} F(p_n) = F(p)$. In particular, we have $\lim_{n\to\infty} \pi_{Y_1}(F(p_n)) = \pi_{Y_1}(F(p))$ and $\lim_{n\to\infty} \pi_{Y_2}(F(p_n)) = \pi_{Y_2}(F(p))$. It follows that

$$\lim_{n\to\infty} \{(p_n, F^*(p_n)), (-p_n, F^*(-p_n))\} = \{(p, F^*(p)), (-p, F^*(-p))\}.$$

Thus, $\{(p, F^*(p)), (-p, F^*(-p))\} \in \mathrm{cl}(P(C_1))$ so $P(C)$ is connected. When $p \in C_1$, then we argue that $\{(p, F^*(p)), (-p, F^*(-p))\} \in \mathrm{cl}(P(C_2))$. Since $C_2 = -D$ for some $D \subseteq A$ and ρ is a homeomorphism, it follows that $(F \circ \rho)|C_2$ is dense in $C_2 \times (Y_1 \times Y_2)$; the argument now proceeds as in the case when $p \in C_2$.

Case 4: $C_1 = C \cap A$ and $C_2 = C \cap B$ are non-empty but not both are connected.

It is easy to check that either $C_1 = A$ or $C_2 = B$. Assume that $C_2 = B$; then C_1 is the union of two separated connected sets G_1 and G_2 such that $(-1, 0) \in \mathrm{cl}(G_1)$ and $(1, 0) \in G_2$. Arguing as in the Case 3, we have that $P(C_2 \cup G_1)$ and $P(C_2 \cup G_2)$ are connected. Since $P(C_2)$ is connected by Case 2, it follows that $P(C)$ is connected. The case when $C_1 = A$ is similar. □

Proof of Theorem 7.1. Suppose, to the contrary, that there is a $\mathfrak{c}$-connected weak S_4 space Y. By Lemma 4.2, there are disjoint $\mathfrak{c}$-connected dense sets, Y_1 and Y_2, in Y such that $Y = Y_1 \cup Y_2$. Let

$$F\colon A \to Y_1 \times Y_2 \quad , \quad F^*\colon S^1 \to Y$$

be as in Lemma 7.2.

Since $F^*\colon S^1 \to Y$ is one-to-one (Lemma 7.2), $\pi_Y\,|F^*\colon F^* \to Y$ is one-to-one. Thus, by Proposition 3.2, F^* is a weak S_4 space. Therefore, letting

$$Q = \Big\{\{(x, F^*(x)), (-x, F^*(-x))\}\colon x \in S^1\Big\}$$

there is a continuous selector $\Sigma\colon Q \to F^*$.

We use Σ to obtain a contradiction. Let

$$V = \{(1, \theta) \in S^1\colon -\frac{\pi}{2} < \theta < \frac{\pi}{2}\},$$

$$\Lambda = \Big\{\{(x, F^*(x)), (-x, F^*(-x))\}\colon x \in V\Big\}.$$

Since Λ is the set $P(V)$ in Lemma 7.2, Λ is connected. Thus, since $\Lambda \subseteq Q$ and $\Sigma\colon Q \to F^*$ is a continuous selector, we see that $\Sigma(\Lambda)$ is a connected subset of $(F^*|V) \cup (F^*| - V)$. Therefore, since $F^*|V$ and $F^*| - V$ are mutually separated, we see that $\Sigma(\Lambda) \subseteq F^*|V$ or $\Sigma(\Lambda) \subseteq F^*| - V$. Hence, without loss of generality, we can assume that

$$\Sigma(\Lambda) \subseteq F^*|V.$$

Now, let $W = \{(1, \theta) \in S^1\colon 0 < \theta < \pi\}$. Let

$$\Gamma = \Big\{\{(x, F^*(x)), (-x, F^*(-x))\}\colon x \in W\Big\}.$$

By the argument above, we see that

$$\Sigma(\Gamma) \subseteq F^*\,|W \quad \text{or} \quad \Sigma(\Gamma) \subseteq F^*\,|{-W}.$$

Case 1: $\Sigma(\Gamma) \subseteq F^*\,|W$. Let $p = (1, \frac{3\pi}{4}) \in S^1$ (note that $p \in W \cap (-V)$), and let

$$D = \{(p, F^*(p)), (-p, F^*(-p))\}.$$

Since $D \in \Gamma$ and since $\Sigma(\Gamma) \subseteq F^*\,|W$, $\Sigma(D) \in F^*\,|W$. Thus, since $-p \notin W$, we see that

$$\Sigma(D) = (p, F^*(p)). \tag{1}$$

Since $-p \in V$, we know that $D \in \Lambda$; thus, since $\Sigma(\Lambda) \subseteq F^*\,|V$, $\Sigma(D) \in F^*\,|V$. Therefore, since $p \notin V$, we see that

$$\Sigma(D) = (-p, F^*(-p)). \tag{2}$$

By (1) and (2), we have a contradiction.

Case 2: $\Sigma(\Gamma) \subseteq F^*| - W$. Let $q = (1, \frac{\pi}{4}) \in S^1$ (note that $q \in W \cap V$). Then, letting

$$E = \{(q, F^*(q)), (-q, F^*(-q))\},$$

we can apply the ideas used in Case 1 to obtain a contradiction.

Since Case 1 and Case 2 each led to a contradiction, we have a contradiction to our supposition at the beginning of the proof of the theorem. □

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