PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 4, Pages 1233–1246
S 0002-9939(00)05630-6
Article electronically published on October 10, 2000

ON THE GRUSHIN OPERATOR AND HYPERBOLIC SYMMETRY

WILLIAM BECKNER

(Communicated by Christopher D. Sogge)

ABSTRACT. Complexity of geometric symmetry for differential operators with mixed homogeniety is examined here. Sharp Sobolev estimates are calculated for the Grushin operator in low dimensions using hyperbolic symmetry and conformal geometry.

Considerable interest exists in understanding differential operators with mixed homogeneity. A simple example is the Grushin operator on $\mathbb{R}^2$

$$\Delta_G = \frac{\partial^2}{\partial t^2} + 4t^2 \frac{\partial^2}{\partial x^2} \ .$$

The purpose of this note is to demonstrate the complexity of geometric symmetry that may exist for operators defined on Lie groups. Here the existence of an underlying $SL(2,R)$ symmetry for Δ_G is used to compute the sharp constant for the associated L^2 Sobolev inequality.

Theorem 1. *For $f \in C^1(\mathbb{R}^2)$*

$$\Big[\|f\|_{L^6(\mathbb{R}^2)}\Big]^2 \le \pi^{-2/3} \int_{\mathbb{R}^2} \left[\Big(\frac{\partial f}{\partial t}\Big)^2 + 4t^2 \Big(\frac{\partial f}{\partial x}\Big)^2 \right] dx\, dt \ . \tag{1}$$

This inequality is sharp, and an extremal is given by $\big[(1+|t|^2)^2 + |x|^2\big]^{-1/4}$.

This result follows from the analysis of a Sobolev inequality on $SL(2,R)/SO(2)$. But the hyperbolic embedding estimate requires some interpretation to take into account cancellation effects. It will be essential to include contibutions to the hyperbolic Dirichlet form from non-L^2 functions. Let $z = x + iy$ denote a point in the upper half-plane $\mathbb{R}^2_+ \simeq \mathbb{H}^2 \simeq M \simeq SL(2,R)/SO(2)$. Here the invariant distance is given by the Poincaré metric

$$d(z,z') = \frac{|z-z'|}{2\sqrt{yy'}}$$

with the corresponding invariant gradient $D = y\nabla$ and left-invariant Haar measure $d\nu = y^{-2}\, dy\, dx$.

Received by the editors March 19, 1999 and, in revised form, July 2, 1999.
2000 *Mathematics Subject Classification.* Primary 58J70, 35A15.
This work was partially supported by the National Science Foundation.

Theorem 2. *For $F \in C_c^1(M)$*

$$\Big[\|F\|_{L^6(M)}\Big]^2 \le 4\pi^{-2/3}\Big[\int_M |DF|^2\,d\nu - \frac{3}{16}\int_M |F|^2\,d\nu\Big], \tag{2}$$

$$\Big[\|F\|_{L^6(M)}\Big]^2 \le \frac{4}{3}\pi^{-2/3}\Big[\int_M |DF|^2\,d\nu - \frac{1}{4}\int_M |F|^2\,d\nu\Big]\ . \tag{3}$$

Both estimates are sharp as limiting forms.

These two estimates would seem to be contradictory, but it must be understood that the right-hand sides are to be evaluated as limiting forms for functions that may not be in $L^2(M)$. So the issue of which is the sharper Sobolev inequality must be studied carefully. On the hyperbolic manifold the Dirichlet form can be represented as a weighted Sobolev form so that for $\alpha > 0$

$$\int_M y^{2\alpha}|\nabla(y^{-\alpha}f)|^2\,dx\,dy = \int_M |Df|^2\,d\nu + \alpha(\alpha-1)\int_M |f|^2\,d\nu\ .$$

On the right-hand side of equation (3) observe the appearance of the spectral limit $\frac{1}{4}$ for the hyperbolic Laplacian $-y^2\Delta$:

$$\frac{1}{4}\int_M |F|^2\,d\nu \le \int_M |DF|^2\,d\nu\ . \tag{4}$$

Proof of Theorem 1. Let $\tilde{f}$ denote the Fourier transform of f in the first variable. That is, for integrable functions

$$\tilde{f}(\xi,t) = \int_{\mathbb{R}} e^{2\pi i\xi x} f(x,t)\,dx$$

so that by using the Plancherel identity, inequality (1) for some constant A_0 is equivalent to

$$\Big[\int_{\mathbb{R}^2} |\tilde{f} * \tilde{f} * \tilde{f}|^2\,d\xi\,dt\Big]^{1/3} \le A_0\int_{\mathbb{R}^2}\Big[\Big(\frac{\partial\tilde{f}}{\partial t}\Big)^2 + 16\pi^2|t|^2|\xi|^2|\tilde{f}|^2\Big]\,d\xi\,dt$$

where here convolution is only with respect to the first variable. Now one can apply standard rearrangement arguments of Riesz-Sobolev type to see that it suffices to consider this inequality only for non-negative functions $\tilde{f}(\xi,t)$ that are symmetric decreasing in each of the two variables separately. Hence, the function $f(x,t)$ in (1) can be taken to be symmetric in t. Moreover, $f(x,t)$ can be taken to be symmetric decreasing in x because the Dirichlet form in (1) taken only with respect to integration in x is diminished by a symmetric decreasing equimeasurable rearrangement in the first variable.

Since $f(x,t)$ is even in t, set $y = t^2$ and let $f(x,|t|) = y^{-1/4}F(x,y)$. Then

$$\|f\|_{L^6(\mathbb{R}^2)} = \|F\|_{L^6(M)}$$

and inequality (1) is now equivalent to

$$\Big[\|F\|_{L^6(M)}\Big]^2 \le 4A_0\Big[\int_M |DF|^2\,d\nu - \frac{3}{16}\int_M |F|^2\,d\nu\Big]\ .$$

This is an *a priori* inequality where the function F can be taken to be smooth but still the form will extend to functions that are not in $L^2(M)$. One can also restrict this result to consideration of functions that are radial decreasing in the Poincaré distance from the origin $\hat{0} = (0,1) = i$. Now Theorem 1 will follow from the first

part of Theorem 2 with $A_0 = \pi^{-2/3}$. To make this argument clearer, observe that estimates for the Grushin operator are maximized on the class of even functions. Every even function of compact support in t can be smoothly approximated by a Schwartz class function that is analytic in t^2 through convolution with a Gaussian function. This class of functions suffices to determine the Grushin bound (1). But by using the functional transformation $F(x,y) = y^{1/4} f(x,|t|)$ to transfer the estimate to hyperbolic space, one finds that both the L^6 norm and the Dirichlet form for the elliptic operator $L = -y^2\Delta - (3/16)\mathbf{1}$ are finite. Every smooth function on M with suitable decay at infinity which vanishes to order one-quarter at the boundary $y = 0$ has finite Dirichlet form for inequality (2).

Proof of equation (2) *in Theorem* 2. By using equimeasurable radial decreasing rearrangement corresponding to the metric on hyperbolic space, it suffices to consider this inequality for radial decreasing functions of the distance from the origin. Let $u = [d(z,i)]^2$. Then for functions depending on distance the gradient is given by

$$|DF| = \sqrt{u+u^2}\left|\frac{dF}{du}\right|$$

and the volume form restricted to integration for radial function is given by $d\nu = 4\pi\, du$. Then (2) is equivalent to

$$\left[\int_0^\infty |F|^6\, du\right]^{1/3} \le 2^{10/3}\left[\int_0^\infty (u+u^2)|F'|^2\, du - \frac{3}{16}\int_0^\infty |F|^2\, du\right].$$

Let $G \in C_c^2([0,\infty))$ and set $F(u) = (1+u)^{-1/4}G(u)$. Then inequality (2) is equivalent to

$$\begin{aligned}&\left[\int_0^\infty |G|^6(1+u)^{-3/2}\, du\right]^{1/3}\\ &\qquad \le 2^{10/3}\left[\int_0^\infty u\sqrt{1+u}\,|G'|^2\, du + \frac{1}{16}\int_0^\infty |G|^2(1+u)^{-3/2}\, du\right].\end{aligned}$$

But now this estimate will be considered for all Lipschitz functions G such that the right-hand side is finite. Make the change of variables $u \to 1/u$ with $H(u) = G(1/u)$;

$$\begin{aligned}&\left[\int_0^\infty |H|^6(1+u)^{-3/2}u^{-1/2}\, du\right]^{1/3}\\ &\qquad \le 2^{10/3}\left[\int_0^\infty \sqrt{u(1+u)}\,|H'|^2\, du + \frac{1}{16}\int_0^\infty |H|^2(1+u)^{-3/2}u^{-1/2}\, du\right].\end{aligned} \tag{5}$$

By evaluating this estimate for $H(u) = (1+u)^{-\varepsilon}$ as $\varepsilon \to 0$, one sees that the constant cannot be smaller than $2^{10/3}$. This calculation also suggests that the inequality should be associated with sharp Sobolev embedding on S^2. Such intuition is realized by the following argument.

Define a new variable w by setting

$$(1+w)^{-2}\, dw = \frac{1}{2}u^{-1/2}(1+u)^{-3/2}\, du$$

so that

$$\sqrt{\frac{u}{1+u}} = \frac{w}{1+w}$$

and $w = u + \sqrt{u(1+u)}$. With this change of variables (5) becomes

$$\left[\int_0^\infty |G|^6(1+w)^{-2}\,dw\right]^{1/3} \le 4\int_0^\infty (2w+1)|G'|^2\,dw + \int_0^\infty |G|^2(1+w)^{-2}\,dw\ . \tag{6}$$

This inequality is controlled by sharp Sobolev embedding on S^2; more precisely, the family of sharp Sobolev inequalities on S^2 that are determined by the Hardy-Littlewood-Sobolev inequality (see Theorem 4 in [2])

$$\left[\int_{S^2} |F|^p\,d\xi\right]^{2/p} \le \frac{p-2}{2}\int_{S^2} |\nabla F|^2\,d\xi + \int_{S^2} |F|^2\,d\xi \tag{7}$$

for $2 \le p < \infty$ and $d\xi$ is a normalized surface measure on S^2. Inequality (6) follows from the case $p = 6$. Observe that the change of variables defined by stereographic projection between $\mathbb{R}^2$ and S^2-{pole} can be realized for the polar angle on S^2 by $\cos\theta = (1-|x|^2)/(1+|x|^2)$ and $w = |x|^2$ in (6). Since inequality (6) then corresponds to functions of the polar angle, it suffices simply to match up the "radial coordinates" in each domain. Then

$$w\Big(\frac{dG}{dw}\Big)^2\,dw = \frac{1}{2}\Big(\frac{dG}{d\theta}\Big)^2 \sin\theta\,d\theta$$

so that (7) for $p = 6$ and radial variables give a stronger inequality than (6):

$$\left[\int_0^\infty |G|^6(1+w)^{-2}\,dw\right]^{1/3} \le 2\int_0^\infty w|G'|^2\,dw + \int_0^\infty |G|^2(1+w)^{-2}\,dw\ . \tag{8}$$

This shows that equality is achieved in (6) only for constants.

Proof of equation (3) *in Theorem* 2. This result is a special case of an argument in [4] that uses axial symmetry and $SL(2,R)$ to derive the sharp Sobolev embedding constant on $\mathbb{R}^n$ and characterize the extremals for that problem. The motivation for this approach came from problems in fluid mechanics and vortex dynamics (see the discussion concerning the Stokes stream function on page 829 in [4] and [10]). For $n > 2$ and $1/p = 1/2 - 1/n$

$$\|f\|_{L^p(\mathbb{R}^n)} \le A_p\|\nabla f\|_{L^2(\mathbb{R}^n)}, \qquad A_p = [\pi n(n-2)]^{-1/2}[\Gamma(n)/\Gamma(n/2)]^{1/n} \tag{9}$$

and up to the action of the conformal group, the sharp constant is only attained for functions of the form $A(1+|x|^2)^{-n/p}$. By using the technique of symmetrization (equimeasurable radial decreasing rearrangement), it suffices to consider this inequality for non-negative radial decreasing functions. For radial functions use the product structure for Euclidean space $\mathbb{R}^n \simeq \mathbb{R}\times\mathbb{R}^{n-1}$ with $x = (t, x')$ and set $y = |x'|$. Being radial in x means that the function is also radial in x'. Let $g(t,y) = y^{n/p}f(t,x')$ and inequality (9) becomes

$$\left[\int_M |g|^p\,d\nu\right]^{2/p} \le B_p\left[\int_M |Dg|^2\,d\nu + \frac{n}{p}\Big(\frac{n}{p}-1\Big)\int_M |g|^2\,d\nu\right] \tag{10}$$

where

$$B_p = \frac{4}{n(n-2)}\Big[\frac{n-1}{2\pi}\Big]^{2/n}\ .$$

For the case $n = 3$, $p = 6$ and $B_6 = (4/3)\pi^{-2/3}$, and equation (3) is proved. The argument in [4] to obtain extremals for the Sobolev inequality (9) is a nice application of the competing radial and cylindrical symmetry.

This result on sharp Grushin estimates is interesting because (a) the solution does match the pattern suggested by the Heisenberg group (see [3]), (b) the analysis is controlled by the two-dimensional sharp Hardy-Littlewood-Sobolev inequality, and (c) the identification of $SL(2,R)$ symmetry is related to the role of analyticity in the Lewy example.

For higher dimensions this problem has corresponding behavior. Consider $(x,t) \in \mathbb{R} \times \mathbb{R}^2 \simeq \mathbb{R}^3$ with

$$\Delta_G = \Delta_t + 4|t|^2 \frac{\partial^2}{\partial x^2} \ .$$

The homogeneous dimension of this operator is 4. Here one can also use the underlying $SL(2,R)$ symmetry to compute the sharp constant for the associated L^2 Sobolev inequality with a similar analysis.

Theorem 3. *For $f \in C^1(\mathbb{R}^3)$*

$$\Big[\|f\|_{L^4(\mathbb{R}^3)}\Big]^2 \le \frac{1}{2\pi} \int_{\mathbb{R}\times\mathbb{R}^2} \Big[|\nabla_t f|^2 + 4|t|^2 \Big(\frac{\partial f}{\partial x}\Big)^2\Big] \, dx\, dt \ . \tag{11}$$

This inequality is sharp, and an extremal is given by $\big[(1+|t|^2)^2 + |x|^2\big]^{-1/2}$.

Theorem 4. *For $F \in C_c^1(M)$*

$$\Big[\|F\|_{L^4(M)}\Big]^2 \le \frac{2}{\sqrt{\pi}} \Big[\int_M |DF|^2 \, d\nu - \frac{1}{4}\int_M |F|^2 \, d\nu\Big] \ . \tag{12}$$

Proof of Theorem 3. Let $\tilde{f}$ denote the Fourier transform of f in the first variable x. Using the Plancherel identity, inequality (11) for some constant A_0 is equivalent to

$$\Big[\int_{\mathbb{R}^3} |\tilde{f} * \tilde{f}|^2 \, d\xi\, dt\Big]^{1/2} \le A_0 \int_{\mathbb{R}^3} \Big[|\nabla_t \tilde{f}|^2 + 16\pi^2 |t|^2 |\xi|^2 |\tilde{f}|^2\Big] \, d\xi\, dt$$

where here convolution is only with respect to the first variable. By applying Riesz-Sobolev rearrangement arguments, it suffices to consider this inequality only for non-negative functions $\tilde{f}(\xi,t)$ that are radial decreasing in each of the two variables separately. Hence, the function $f(x,t)$ in (11) can be taken to be radial in t and symmetric decreasing in x. The second part of this remark follows from the fact that the Dirichlet form in (11) taken only with respect to integration in x is diminished by a symmetric decreasing equimeasurable rearrangement with respect to the first variable.

Since $f(x,t)$ is radial in t, set $y = |t|^2$ and let $f(x,|t|) = y^{-1/2}F(x,y)$. Then

$$\|f\|_{L^4(\mathbb{R}^3)} = \pi^{1/4}\|F\|_{L^4(M)}$$

and inequality (11) is now equivalent to

$$\Big[\|F\|_{L^4(M)}\Big]^2 \le 4\sqrt{\pi}\, A_0 \Big[\int_M |DF|^2 \, d\nu - \frac{1}{4}\int_M |F|^2 \, d\nu\Big] \ .$$

This is an *a priori* inequality where the function F can be taken to be smooth with compact support. Now Theorem 3 will follow from Theorem 4 with $A_0 = 1/(2\pi)$. One simply calculates that equality is attained for the indicated extremal. The

details of the argument to show equivalence follow the method outlined in the proof of Theorem 1. Estimates for the Grushin operator are maximized on the class of functions that are radial in the variable t. Every radial function of compact support in t can be smoothly approximated by a Schwartz class function that is analytic in $|t|^2$ through convolution with a Gaussian function. This class of functions suffices to determine the Grushin bound (11). By using the functional transformation $F(x,y) = y^{1/2} f(x,|t|)$ to transfer the estimate to hyperbolic space, one finds that both the L^4 norm and the Dirichlet form for the elliptic operator $L = -y^2\Delta - (1/4)\mathbf{1}$ are finite. Every smooth function on M with suitable decay at infinity which vanishes to order one-half at the boundary $y = 0$ has finite Dirichlet form for inequality (12).

Proof of Theorem 4. Using equimeasurable radial decreasing rearrangement corresponding to the metric on hyperbolic space, it suffices to consider this inequality for radial decreasing functions of the distance from the origin. Set $u = d^2(z,i)$. Then the volume form restricted to integration for radial functions is given by $d\nu = 4\pi\, du$ and inequality (12) becomes (see [5])

$$\left[\int_0^\infty |F|^4\, du\right]^{1/2} \le 4\left[\int_0^\infty (u^2+u)\Big|\frac{dF}{du}\Big|^2\, du - \frac{1}{4}\int_0^\infty |F|^2\, du\right]. \tag{13}$$

If one can show that this is a good upper bound, then the sequence of functions $F_\varepsilon(u) = (1+u)^{-\varepsilon}$ for $\varepsilon > \frac{1}{2}$ shows that the estimate is sharp. Let $G \in C_c^2([0,\infty))$ and set $F(u) = (1+u)^{-1/2}G(u)$. Then inequality (13) takes the form

$$\left[\int_0^\infty |G|^4 \frac{1}{(1+u)^2}\, du\right]^{1/2} \le 4\int_0^\infty u(G')^2\, du + \int_0^\infty |G|^2 \frac{1}{(1+u)^2}\, du\,. \tag{14}$$

This inequality is controlled by sharp Sobolev embedding on S^2; more precisely, the family of sharp Sobolev inequalities on S^2 that are determined by the Hardy-Littlewood-Sobolev inequality (see Theorem 4 in [2])

$$\left[\int_{S^2} |F|^p\, d\xi\right]^{2/p} \le \frac{p-2}{2}\int_{S^2} |\nabla F|^2\, d\xi + \int_{S^2} |F|^2\, d\xi \tag{15}$$

for $2 \le p < \infty$ and $d\xi$ is a normalized surface measure on S^2. Inequality (14) follows from the case $p = 4$. Observe that change of variables defined by stereographic projection between $\mathbb{R}^2$ and S^2-{pole} can be realized for the polar angle on S^2 by $\cos\theta = (1-|x|^2)/(1+|x|^2)$ and $u = |x|^2$ in (14). Since inequality (14) corresponds to functions of the polar angle, it suffices simply to match up the "radial coordinates" in each domain. Then

$$u\Big(\frac{dG}{du}\Big)^2\, du = \frac{1}{2}\Big(\frac{dG}{d\theta}\Big)^2 \sin\theta\, d\theta$$

so that (15) for $p = 4$ and radial variables gives a stronger inequality than (14):

$$\left[\int_0^\infty |G|^4 \frac{1}{(1+u)^2}\, du\right]^{1/2} \le \int_0^\infty u(G')^2\, du + \int_0^\infty |G|^2 \frac{1}{(1+u)^2}\, du\,. \tag{16}$$

This shows that Theorem 4 is sharp as a limiting form. However, the limit "extremal"

$$F(x,y) = [1 + d^2(z,i)]^{-1/2}$$

is not in $L^2(M)$. This observation emphasizes that the appropriate Dirichlet form for Sobolev embedding on hyperbolic space $\mathbb{H}^2$ should correspond to the intrinsic positive elliptic differential operator

$$L_s = -y^2\Delta + s(s-1)\mathbf{1}.$$

These two results illustrate the complexity and interdependence of Sobolev estimates on Lie groups and symmetric spaces, and demonstrate that there is still much to understand about the geometry of Grushin operators. The elementary nature of these calculations was facilitated by the capability to use rearrangement arguments which here depended on the Sobolev index being an even integer. An interesting aspect of the analysis is that the intermediate estimate on hyperbolic space must be defined as a limiting form using the positive elliptic operator L_s at the critical exponent for the Grushin embedding estimate.

Appendix

The argument used here to relate sharp Sobolev embedding on S^2 to embedding estimates on hyperbolic space determines a more general family of such estimates.

Theorem 5. *For $F \in C_c^1(M)$, $0 < s \le \frac{1}{2}$ and $p = 2 + \frac{1}{s} \ge 4$*

$$\left[\|F\|_{L^p(M)}\right]^2 \le A_p\left[\int_M |DF|^2\,d\nu + s(s-1)\int_M |F|^2\,d\nu\right], \tag{17}$$
$$A_p = (2\pi)^{\frac{2}{p}-1}s^{-1-\frac{2}{p}}\ .$$

Proof. By using equimeasurable radial decreasing rearrangement corresponding to the metric on hyperbolic space, it suffices to consider this inequality for radial decreasing functions of the distance from the origin. Let $u = [d(z,i)]^2$. Then (17) is equivalent to

$$\left[\int_0^\infty |g|^p\,du\right]^{2/p} \le (4\pi)^{1-\frac{2}{p}}A_p\left[\int_0^\infty (u+u^2)|g'|^2\,du + s(s-1)\int_0^\infty |g|^2\,du\right].$$

Set $g = (1+u)^{-\alpha}h$ and $C_p = (4\pi)^{(p-2)/p}A_p$. Then

$$\begin{aligned}\left[\int_0^\infty |h|^p(1+u)^{-p\alpha}\,du\right]^{2/p} \le C_p\Bigg[&\int_0^\infty u(1+u)^{1-2\alpha}|h'|^2\,du\\ &+\alpha^2\int_0^\infty |h|^2(1+u)^{-1-2\alpha}\,du\\ &+(s^2-s+\alpha-\alpha^2)\int_0^\infty |h|^2(1+u)^{-2\alpha}\,du\Bigg].\end{aligned}$$

Set $\alpha = s$, $p\alpha = 2\alpha+1$ and $\beta = 2\alpha > 0$. Then

$$\begin{aligned}\left[\int_0^\infty |h|^p(1+u)^{-\beta-1}\,du\right]^{2/p} \le C_p\Bigg[&\int_0^\infty u(1+u)^{1-\beta}|h'|^2\,du\\ &+\frac{\beta^2}{4}\int_0^\infty |h|^2(1+u)^{-1-\beta}\,du\Bigg].\end{aligned}$$

Make the change of variables $u \to 1/u$ with $H(u) = h(1/u)$ so that $|h'(1/u)| = u^2|H'(u)|$ and

$$\left[\int_0^\infty |H|^p(1+u)^{-\beta-1}u^{\beta-1}\,du\right]^{2/p} \le C_p\left[\int_0^\infty u^\beta(1+u)^{1-\beta}|H'|^2\,du + \frac{\beta^2}{4}\int_0^\infty |H|^2(1+u)^{-\beta-1}u^{\beta-1}\,du\right].$$

Now set $(1+w)^{-2}\,dw = \beta(1+u)^{-\beta-1}u^{\beta-1}\,du$ so that

$$\frac{w}{1+w} = \left(\frac{u}{1+u}\right)^\beta$$

which gives for $G(w) = H(u)$, $2/p = \beta/(1+\beta)$ and $B_p = \frac{1}{4}\beta^{(1+2\beta)/(1+\beta)}C_p$

$$\begin{aligned}\left[\int_0^\infty |G|^p(1+w)^{-2}\,dw\right]^{\beta/(1+\beta)} &\le B_p\left[4\int_0^\infty w^{2-\frac{1}{\beta}}\left[(1+w)^{1/\beta} - w^{1/\beta}\right]|G'|^2\,dw \right.\\ &\qquad \left. + \int_0^\infty |G|^2(1+w)^{-2}\,dw\right].\end{aligned} \tag{18}$$

This equation can be simplified using the change of variables $w \to 1/w$ and setting $\widetilde{G}(w) = G(1/w)$:

$$\begin{aligned}\left[\int_0^\infty |\widetilde{G}|^p(1+w)^{-2}\,dw\right]^{\beta/(1+\beta)} &\le B_p\left[4\int_0^\infty \left[(1+w)^{1/\beta} - 1\right]|\widetilde{G}'|^2\,dw + \int_0^\infty |\widetilde{G}|^2(1+w)^{-2}\,dw\right].\end{aligned} \tag{19}$$

Now this estimate should be compared with the sharp Sobolev embedding on S^2 that is determined by the Hardy-Littlewood-Sobolev inequality:

$$\left[\int_{S^2} |F|^p\,d\xi\right]^{2/p} \le \frac{p-2}{2}\int_{S^2} |\nabla F|^2\,d\xi + \int_{S^2} |F|^2\,d\xi$$

where $d\xi$ denotes normalized surface measure and $p > 2$, and in turn gives for radial functions and $p = 2(1+1/\beta)$

$$\begin{aligned}\left[\int_0^\infty |F|^p(1+w)^{-2}\,dw\right]^{\beta/(1+\beta)} &\le \frac{1}{\beta}\int_0^\infty w|F'|^2\,dw + \int_0^\infty |F|^2(1+w)^{-2}\,dw\,.\end{aligned} \tag{20}$$

Now set $B_p = 1$ in (19) which corresponds to the value of A_p in (17) and observe that for $r = 1/\beta$ and $w \ge 0$, then $(1+w)^r \ge 1 + rw$ for $r \ge 1$. Hence the estimate (20) derived from a Sobolev embedding on S^2 implies that (19) holds for $B_p = 1$ and $\beta \ge 1$. The proof of Theorem 5 is then complete for $0 < s \le 1/2$.

A discrete set of hyperbolic embedding estimates can be obtained in the analysis of a Sobolev embedding on the Heisenberg group $\mathcal{H}_n$ realized as the manifold $\mathbb{C}^n \times \mathbb{R}$ and restricted to radial symmetry in the complex variables (see Theorem 18 in [3]).

Theorem 6. *For $F \in C^1(M) \cap L^2(M)$, $s = n/2$ for $n \in \mathbb{N}$ and $p = 2 + \frac{1}{s} \leq 4$*

$$\Big[\|F\|_{L^p(M)}\Big]^2 \leq A_p \Big[\int_M |DF|^2\, d\nu + s(s-1)\int_M |F|^2\, d\nu,$$
$$A_p = (2\pi)^{\frac{2}{p}-1} s^{-1-\frac{2}{p}} \ .$$

For $n > 1$ and up to the "conformal structure" of M, an extremal is given by

$$F(z) = \Big[1 + d^2(z,i)\Big]^{-s} \ .$$

This family of hyperbolic embedding estimates can be extended to include values of $s \geq 1$ by using duality and the fundamental solution corresponding to the differential operator L_s. Note that in this case an L^2 extremal function will exist. The fundamental solution for $L_s = -y^2\Delta + s(s-1)\mathbf{1}$ for $s \geq 1$ is given by

$$\begin{aligned}\psi_s(u) &= \frac{1}{4\pi}\int_0^1 \big[t(1-t)\big]^{s-1}(t+u)^{-s}\, dt \\ &= \frac{\Gamma(s)\Gamma(s)}{4\pi\Gamma(2s)}(1+u)^{-s} F\left(s, s, 2s; \frac{1}{1+u}\right)\end{aligned}$$

where $u = [d(z,i)]^2$ and F is the hypergeometric function. The transition from Sobolev embedding estimates to a Hardy-Littlewood-Sobolev convolution inequality is made using the following lemma.

Lemma. *Let K and Λ be densely defined, positive-definite, self-adjoint operators acting on functions defined on a σ-finite measure space M and satisfying the relation*

$$\Lambda K = K\Lambda = \mathbf{1} \ .$$

Then the following two inequalities are equivalent:

$$(*) \qquad \|Kf\|_{L^{p'}(M)} \leq C_p \|f\|_{L^p(M)},$$

$$(**) \qquad \|g\|_{L^{p'}(M)} \leq \sqrt{C_p}\|\Lambda^{1/2} g\|_{L^2(M)} \ .$$

Here $1 < p < 2$ and $1/p + 1/p' = 1$. Extremal functions for one inequality will determine extremal functions for the other inequality if the operator forms are well-defined.

Proof. In $(**)$ substitute $g = Kf$ so that

$$\begin{aligned}\Big[\|Kf\|_{L^{p'}(M)}\Big]^2 &\leq C_p \int_M (Kf)\Lambda(Kf)\, dm = C_p \int_M (Kf) f\, dm \\ &\leq C_p \|Kf\|_{L^{p'}(M)} \|f\|_{L^p(M)},\end{aligned}$$

which is now $(*)$. For equivalence in the reverse direction, K is a positive-definite self-adjoint operator and notice that $(*)$ implies

$$\|K^{1/2} f\|_{L^2(M)} \leq \sqrt{C_p}\|f\|_{L^p(M)}$$

which by duality implies

$$\|K^{1/2} h\|_{L^{p'}(M)} \leq \sqrt{C_p}\|h\|_{L^2(M)} \ .$$

Now substitute $h = K^{1/2}(\Lambda g)$ which results in $(**)$. The full equivalence is obtained by taking limits on dense domains.

For $s > 0$ define the fractional integral operator

$$(I_s G)(z) = \int_M \psi_s\big[d^2(z,w)\big] G(w)\, d\nu \ . \tag{21}$$

The symmetric space $SL(2,R)/SO(2) \simeq \mathbb{H}^2$ can be identified with the subgroup of $SL(2,R)$ given by all matrices of the form

$$\begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}$$

with $y > 0$ and $x \in \mathbb{R}$ which act via fractional linear transformations on $\mathbb{R}^2_+ \simeq \mathbb{H}^2$.

$$z = x + iy \in \mathbb{R}^2_+ \to \frac{az+b}{cz+d}$$

for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,R)$. The modular function is $\Delta(x,y) = 1/y$ and $d\nu = y^{-2}\, dx\, dy$ is a left-invariant Haar measure on the group. Observe that the group action here corresponds to the multiplication rule

$$(x,y)(u,v) = (x + yu, yv)$$

for $x, u \in \mathbb{R}$ and $y, v > 0$. This $SL(2,R)$ subgroup is the "$ax+b$ group", namely the group of all linear transformations of the real line to itself that preserve orientation. With this framework, the operator $I_s G$ can be represented as a convolution operator

$$I_s G = G * \psi_s \tag{22}$$

where convolution for the left-invariant Haar measure on a locally compact group is defined by

$$(f * g)(x) = \int_G f(y) g(y^{-1}x)\, dy \ .$$

Observe that $L_s(I_s G) = G$ for $s \geq 1$. The Riesz-Sobolev inequality and an extension of Young's inequality to non-unimodular groups provide good estimates for the fractional integral operator I_s.

Riesz-Sobolev Inequality on $SL(2,R)/SO(2)$.

$$\int_M (f * g)(w) h(w)\, d\nu \leq \int_M (f^* * g^*)(w) h^*(w)\, d\nu \tag{23}$$

where f, g and h are non-negative measurable functions with f^*, g^* and h^* denoting their respective equimeasurable, geodesically decreasing rearrangements on $M \simeq SL(2,R)/SO(2) \simeq \mathbb{H}^2$ and $d\nu$ is a left-invariant Haar measure on M.

Young's inequality. *Let G be a locally compact group with the left-invariant Haar measure denoted by m. For $1 \leq p \leq \infty$*

$$\begin{aligned} \|f * g\|_{L^p(G)} &\leq \|f\|_{L^p(G)} \|\Delta^{-1/p'} g\|_{L^1(G)}, \\ \|f * g\|_{L^p(G)} &\leq \|f\|_{L^1(G)} \|g\|_{L^p(G)}, \\ \|f * g\|_{L^r(G)} &\leq \|f\|_{L^p(G)} \|\Delta^{-1/p'} g\|_{L^q(G)} \end{aligned}$$

where Δ denotes the modular function defined by $m(Ey) = \Delta(y) m(E)$, $1/p + 1/p' = 1$ and $1/r = 1/p + 1/q - 1$.

Proof. Consider the form

$$\begin{aligned}\int_G h(x)(f*g)(x)\,dx &= \int_{G\times G} h(x)f(y)g(y^{-1}x)\,dx\,dy\\ &= \int_{G\times G} h(x)f(xy)g(y^{-1})\,dx\,dy\\ &= \int_{G\times G} h(x)f(xy^{-1})g(y)\Delta(y^{-1})\,dx\,dy\ .\end{aligned}$$

Then apply Hölder's inequality.

It is natural here to consider fractional integration as a map from a space to its dual. The asymptotic behavior of ψ_s combined with Young's inequality provide the necessary estimates to show that I_s is a bounded map from $L^q(M)$ to $L^p(M)$ where $1/q+1/p=1$, $p=2+1/s$ and $q=2-1/(1+s)$:

$$\begin{aligned}\psi_s(u) &\simeq \frac{\Gamma(s)\Gamma(s)}{4\pi\Gamma(2s)}u^{-s} && \text{as } u\to\infty\\ &\simeq -\frac{1}{4\pi}\ln u && \text{as } u\to 0\ .\end{aligned}$$

Hence, any power of ψ_s is locally integrable and using Young's inequality

$$\|f*\psi_s\|_{L^p(M)} \le \|f\|_{L^q(M)}\|\Delta^{-1/p}\psi_s\|_{L^{p/2}(M)} = \|f\|_{L^q(M)}\|y^{1/p}\psi_s\|_{L^{p/2}(M)}\ . \tag{24}$$

The critical estimate is now reduced to the fact that $y^{s-1}(y+1)^{-2s}$ is integrable on $[0,\infty)$ for $s>0$. So the map I_s is bounded from $L^q(M)$ to $L^p(M)$. The sharp constant for this estimate will be obtained using duality.

Theorem 7. *For $s\ge 1$, $p=2+1/s$, $q=2-1/(1+s)$*

$$\begin{gathered}\|I_sG\|_{L^p(M)} \le A_p\|G\|_{L^q(M)},\\ A_p = (2\pi)^{\frac{2}{p}-1}s^{-1-\frac{2}{p}}\ .\end{gathered} \tag{25}$$

This inequality is sharp and an extremal is given by $[1+d^2(z,i)]^{-1-s}$. For $F\in C^2(M)$

$$\Big[\|F\|_{L^p(M)}\Big]^2 \le A_p\int_M F(L_sF)\,d\nu\ . \tag{26}$$

Here the extremal is $[1+d^2(z,i)]^{-s}$. Because $s\ge 1$, this latter result can be represented for $F\in C^2(M)\cap L^2(M)$ as

$$\Big[\|F\|_{L^p(M)}\Big]^2 \le A_p\Big[\int_M |DF|^2\,d\nu + s(s-1)\int_M |F|^2\,d\nu\Big]\ . \tag{27}$$

Proof. The plan of the argument is to use the Riesz-Sobolev inequality to show that an extremal function exists for (25) and hence by duality an extremal function exists for (26) which can be calculated using the Euler-Lagrange variational equation. To show the existence of an extremal for (25), it suffices to consider the functional

$$\int_{M\times M} F(a)\psi_s[d^2(z,w)]G(w)\,d\nu\,d\nu$$

for $F,G\ge 0$ and $\|F\|_q=\|G\|_q=1$. By (24) this form is bounded above and by applying the Riesz-Sobolev inequality one can restrict attention to the case where

F and G are geodesically radial decreasing functions. Then consider sequences of functions $\{F_n, G_n\}$ with $\|F_n\|_q = \|G_n\|_q = 1$ so that

$$\int_{M\times M} F_n(z)\psi_s[d^2(z,w)]G_n(w)\,d\nu\,d\nu$$

converges to its maximum value. Since these functions are decreasing, one can use the Helly selection principle to choose subsequences that converge almost everywhere to functions $F, G \in L^q(M)$. By Fatou's lemma $\|F\|_q \le 1$, $\|G\|_q \le 1$. Notice that $F_n(z) \le (4\pi u)^{-1/q}$, $G_n(z) \le (4\pi u)^{-1/q}$ using the radial variable $u = d^2(z,i)$ since Haar measure restricted to the radial variable is $d\nu = 4\pi\,du$. Observe that

$$[d(z,i)]^{-2/q}\psi_s[d^2(z,w)][d(w,i)]^{-2/q} \in L^1(M\times M)\ .$$

Re-label the subsequences to have index n. By the dominated convergence theorem

$$\int_{M\times M} F_n(z)\psi_s[d^2(z,w)]G_n(w)\,d\nu\,d\nu \longrightarrow \int_{M\times M} F(z)\psi_s[d^2(z,w)]G(w)\,d\nu\,d\nu,$$

and so $\|F\|_{L^q(M)} = \|G\|_{L^q(M)} = 1$ and F, G must be extremal functions for (25). A somewhat similar argument is given in [3], page 40.

From the lemma above, one sees that if G is an extremal for (25), then $F = I_sG$ is an extremal for (26). Moreover, if G is radial decreasing, then F will be radial decreasing since the convolution of two radial decreasing functions is radial decreasing. Hence, such an extremal F must satisfy the Euler-Lagrange variational equation for (26):

$$L_sF = \gamma F^{p-1}, \quad \|F\|_{L^p(M)} = 1\ .$$

For F a decreasing function of the radial variable $u = d^2(z,i)$, one looks for solutions of the differential equation

$$-\frac{d}{du}\left[u(u+1)\frac{dF}{du}\right] + s(s-1)F = cF^{p-1}, \qquad p = 2 + 1/s\ . \tag{28}$$

Note that if $F = I_sG$ for $G \in L^q(M)$ with $q = 2 - 1/(1+s)$, then F is bounded. Hence, there will be a unique solution to (28) that is bounded and monotonically decreasing on $[0,\infty)$. This solution is

$$F(u) = B(1+u)^{-s}$$

where the constant B is determined by the condition that $\|F\|_q = 1$. Now one can calculate the value of the sharp constant A_p. An extremal for (25) is obtained by

$$L_sF = L_s(I_sG) = G\ .$$

This calculation completes the proof of Theorem 7. The argument developed here complements the result of Theorem 5. Similar methods can also be applied for the case $0 < s < 1$ and will be discussed in a more comprehensive treatment of Riesz potentials and Sobolev embedding on hyperbolic space [6].

Postscript. After this work was completed, I received in April 1999 three interesting papers (see [12], [20] and [21]) that treat some issues discussed here using complementary methods. Mugelli and Talenti also recognize the relation of the Fraenkel inequality to Sobolev inequalities on hyperbolic space and the usefulness of geodesic rearrangement in the hyperbolic setting. Their approach emphasizes geometric aspects, and they do not use the Hardy-Littlewood-Sobolev framework

of duality. The Garofalo-Vassilev paper considers a broad class of nonlinear subelliptic problems that arise from geometric function theory on nilpotent Lie groups.

Acknowledgement

I would like to thank Tony Carbery for his invitation to participate in the Sussex Workshop on Fourier Analysis, and L.E. Fraenkel, Nicola Garofalo, David Jerison, Chris Sogge and Eli Stein for useful remarks. In addition, I would like to thank Giorgio Talenti for communicating with me his recent work.

References

[1] W. Beckner, *Sobolev inequalities, the Poisson semigroup and analysis on the sphere S^n*, Proc. Nat. Acad. Sci. **89** (1992), 4816–4819. MR **93d:**26018

[2] W. Beckner, *Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality*, Ann. Math. **138** (1993), 213–242. MR **94m:**58232

[3] W. Beckner, *Geometric inequalities in Fourier analysis*, Essays on Fourier Analysis in Honor of Elias M. Stein, Princeton University Press, 1995, pp. 36–68. MR **95m:**42004

[4] W. Beckner, *Sharp inequalities and geometric manifolds*, J. Fourier Anal. Appl. **3** (1997), 825–836. MR **2000c:**58059

[5] W. Beckner, *Geometric asymptotics and the logarithmic Sobolev inequality*, Forum Math. **11** (1999), 105–137. MR **2000a:**46049

[6] W. Beckner, *Riesz potentials and Sobolev embedding on hyperbolic space*, (in preparation).

[7] W. Beckner, *Sobolev inequalities, vortex dynamics and Lie groups*, (in preparation).

[8] H.J. Brascamp, E.H. Lieb and J.M. Luttinger, *A general rearrangement inequality for multiple integrals*, J. Funct. Anal. **17** (1974), 227–237. MR **49:**10835

[9] G.B. Folland, *A fundamental solution for a subelliptic operator*, Bull. Amer. Math. Soc. **79** (1973), 373–376. MR **47:**3816

[10] L.E. Fraenkel, *On steady vortex rings with swirl and a Sobolev inequality*, Progress in partial differential equations, calculus of variations and applications, Longman Scientific and Technical, 1992, pp. 13–26. MR **93k:**76020

[11] P.R. Garabedian, *Partial differential equations*, John Wiley, 1964. MR **28:**5247

[12] N. Garofalo and D. Vassilev, *Dirichlet problem and conformal geometry of sub-Laplacians on Carnot groups*, (preprint, April 1999).

[13] V.V. Grushin, *On a class of hypoelliptic operators*, Math. Sbornik **12** (1970), 458–475.

[14] S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, 1962. MR **26:**2986

[15] S. Helgason, *Fundamental solutions of invariant differential operators on symmetric spaces*, Amer. J. Math. **86** (1964), 565–601. MR **29:**2323

[16] D. Jerison and J.M. Lee, *Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem*, J. Amer. Math. Soc. **1** (1988), 1–13. MR **89b:**53063

[17] F. John, *The fundamental solution of linear elliptic differential equations with analytic coefficients*, Comm. Pure Appl. Math. **3** (1950), 273–304. MR **13:**40h

[18] S. Lang, $SL_2(R)$, Addison-Wesley, 1975. MR **55:**3170

[19] H. Lewy, *An example of a smooth linear partial differential equation without solution*, Ann. of Math. **66** (1957), 155–158. MR **19:**551d

[20] F. Mugelli and G. Talenti, *Sobolev inequalities in 2-dimensional hyperbolic space*, International series of numerical mathematics, Vol. 123, Birkhäuser Verlag, 1997, pp. 201–216. MR **98g:**46044

[21] F. Mugelli and G. Talenti, *Sobolev inequalities in 2-d hyperbolic space: a borderline case*, J. Inequal. and Appl. **2** (1998), 195–228. CMP 99:08

[22] A. Nagel and E.M. Stein, *Lectures on pseudo-differental operators*, Princeton University Press, 1979. MR **82f:**47059

[23] P.G. Saffman, *Vortex dynamics*, Cambridge University Press, 1992. MR **94c:**76015

[24] N. Varopoulos, L. Saloff-Coste and T. Coulhon, *Analysis and geometry on groups*, Cambridge University Press, 1992. MR **95f:**43008

[25] N.J. Vilenkin, *Special functions and the theory of group representations*, American Mathematical Society, 1968. MR **37:**5429

[26] A. Weil, *L'integration dans les groupes topologiques et ses applications*, Hermann, 1940. MR **3:**198b

[27] E.T. Whittaker and G.N. Watson, *A course of modern analysis*, Cambridge University Press, 1927. Reprint MR **97k:**01072

Department of Mathematics, University of Texas at Austin, Austin, Texas 78712-1082
E-mail address: beckner@math.utexas.edu