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A CRITERION ON WEIGHTED L^p BOUNDEDNESS FOR ROUGH MULTILINEAR OSCILLATORY SINGULAR INTEGRALS

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Abstract. In this paper the authors give a criterion on the weighted L^p boundedness of the multilinear oscillatory singular integral operators with rough kernels.

§1. Introduction

It is well known that in 1987, Ricci and Stein [12] proved that a class of oscillatory singular integral operator with polynomial phases and smooth kernel is bounded on L^p. In 1992, Lu and Zhang [11] extended the above result to the case of the rough kernel; see also [7]. Moreover, in [11] they established a simple criterion on L^p boundedness of the rough oscillatory singular integral operator. In 1996, Ding and Lu [4] gave a criterion on weighted L^p boundedness for the higher order commutator of the oscillatory singular integral operator with rough kernel. Recently, Chen, Hu and Lu [2] obtained a criterion on L^p boundedness for a class of the multilinear oscillatory singular integral operator with rough kernel.

In this paper we will extend the result in [2] to the weighted case. However, the extension is not trivial. In fact, we will see that the proof of our main theorem depends strongly on the weighted boundedness of multilinear maximal operator with rough kernel. While the latter itself is also very interesting, since it is also an extension of known results.

Before stating the results in this paper, let us first give some definitions. Suppose that $n \geq 2$, $P(x,y)$ is a real-valued polynomial defined on $\mathbb{R}^n \times \mathbb{R}^n$, $\Omega \in L^q(S^{n-1})$ $(q > 1)$ is homogeneous of degree zero on $\mathbb{R}^n$. Then the multilinear oscillatory singular integral operator T^A is defined by

$$T^A f(x) = p.v. \int_{\mathbb{R}^n} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_m(A;x,y) f(y)\, dy,$$

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where $R_m(A;x,y)$ denotes the m-th ($m \geq 2$) remainder of Taylor series of A at x about y, more precisely,

$$R_m(A;x,y) = A(x) - \sum_{|\alpha|<m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha,$$

and $D^\alpha A \in BMO(\mathbb{R}^n)$ for all multi-indices $|\alpha| = m-1$.

A locally integrable nonnegative function ω is said to belong to A_p $(1 < p < \infty)$ if there is a constant $C > 0$ such that

$$\sup_{Q\subset\mathbb{R}^n} \left(\frac{1}{|Q|}\int_Q \omega(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{-1/(p-1)}\,dx\right)^{p-1} \leq C,$$

where Q denotes a cube in $\mathbb{R}^n$ with its sides parallel to the coordinate axes. Moreover, the minimum constant such that the above inequality holds is called the A_p constant of ω.

A real-valued polynomial $P(x,y)$ is said to be *non-trivial* if $P(x,y)$ cannot be written as $P_0(x) + P_1(y)$, where P_0 and P_1 are both polynomials defined on $\mathbb{R}^n$.

A non-trivial polynomial $P(x,y)$ is said to have *property* $\mathcal{P}$, if $P(x,y)$ satisfies

$$P(x,y) = P(x-h, y-h) + R_0(x,h) + R_1(y,h), \quad h \in \mathbb{R}^n,$$

where R_0 and R_1 are both real polynomials defined on $\mathbb{R}^n \times \mathbb{R}^n$.

A non-trivial polynomial $P(x,y)$ is said to be *non-degenerate* if for positive integers k and l

$$P(x,y) = \sum_{|\alpha|\leq k,\, |\beta|\leq l} a_{\alpha\beta} x^\alpha y^\beta, \quad \text{and} \quad \sum_{|\alpha|=k,\, |\beta|=l} |a_{\alpha\beta}| > 0.$$

Now, let us state our main result as follows.

Theorem. *Suppose that $\Omega \in L^q(S^{n-1})$ is homogeneous of degree zero on $\mathbb{R}^n$ and A has derivatives of order $m-1$ $(m \geq 2)$ in $BMO(\mathbb{R}^n)$. If p, q and ω satisfy one of the following conditions:*

(a) $q' < p < \infty$, *and* $\omega \in A_{p/q'}$;
(b) $1 < p < q$, *and* $\omega^{-1/(p-1)} \in A_{p'/q'}$;
(c) $1 < p < \infty$, *and* $\omega^{q'} \in A_p$,

then the following two statements are equivalent:

(i) *If $P(x,y)$ is a non-degenerate polynomial having property $\mathcal{P}$, then*

$$\|T^A f\|_{p,\omega} \leq C \sum_{|\alpha|=m-1} \|D^\alpha A\|_* \|f\|_{p,\omega}. \tag{1.1}$$

(ii) *The truncated operator*

$$S^A f(x) = p.v. \int_{|x-y|<1} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_m(A;x,y) f(y)\,dy$$

satisfies

$$\|S^A f\|_{p,\omega} \leq C \sum_{|\alpha|=m-1} \|D^\alpha A\|_* \|f\|_{p,\omega}. \tag{1.2}$$

The constants C in (1.1) and (1.2) depend only on n, p, q and the A_p constant of ω, the total degree $degP$ of the polynomial $P(x, y)$. Moreover, $\|b\|_*$ denotes the norm of the function b in BMO.

The multilinear oscillatory singular integral operator T^A and the multilinear maximal operator are closely related to the multilinear singular integral operator defined by

$$T_A f(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_m(A; x, y) f(y)\, dy.$$

In 1986, Cohen and Gosselin [3] proved that if $\Omega \in Lip_1(S^{n-1})$ and $D^\alpha A \in BMO(\mathbb{R}^n)$ for $|\alpha| = m-1$, then the multilinear singular integral T_A is bounded on $L^p(\mathbb{R}^n)$. In 1994, after removing the smoothness condition on Ω, Hofmann [9] considered the weighted $L^p(\mathbb{R}^n)$ boundedness for the rough multilinear singular integral T_A. He proved that if $\Omega \in L^\infty(S^{n-1})$ and $D^\alpha A \in BMO(\mathbb{R}^n)$ for $|\alpha| = m-1$, then T_A is bounded on $L^p(\omega)$ for $\omega \in A_p$. Here and in what follows, $L^p(\omega)$ denotes the weighted L^p spaces.

The proof of our theorem is based on the weighted L^p boundedness of the multilinear maximal operator with rough kernel which is defined by

$$M_{\Omega,A} f(x) = \sup_{r>0} \frac{1}{r^{n+m-1}} \int_{|x-y|<r} |\Omega(x-y)||R_m(A; x, y)||f(y)|\, dy,$$

where Ω is homogeneous of degree zero on $\mathbb{R}^n$. Using this and the Stein-Weiss interpolation theorem with change of measures (see [13]), we deduce a family of weighted estimates with the geometric decay which can be summed. From this, we can easily deduce that (ii) of our theorem implies (i). It is worth pointing out that this way of applying the Stein-Weiss interpolation theorem first appears in Duoandikoetxea and Rubio de Francia [6].

§2. Proof of the Theorem

We first establish the weighted L^p boundedness of $M_{\Omega,A}$, and it is also interesting by itself.

Lemma 1. *Let Ω be homogeneous of degree zero on $\mathbb{R}^n$ with $\Omega \in L^q(S^{n-1})$ $(q > 1)$. Moreover, let $|\alpha| = m-1$, $m \geq 2$, and $D^\alpha A \in BMO(\mathbb{R}^n)$. If p, q and ω satisfy one of the following conditions:*

(a) $q' < p < \infty$, *and* $\omega \in A_{p/q'}$;
(b) $1 < p < q$, *and* $\omega^{-1/(p-1)} \in A_{p'/q'}$;
(c) $1 < p < \infty$, *and* $\omega^{q'} \in A_p$,

then the maximal operator $M_{\Omega,A}$ is bounded on $L^p(\omega)$. That is, there is a constant $C > 0$, independent of f, such that

$$\|M_{\Omega,A} f\|_{p,\omega} \leq C \sum_{|\alpha|=m-1} \|D^\alpha A\|_* \|f\|_{p,\omega}. \tag{2.1}$$

To prove Lemma 1, we need the following relation between the maximal operators $M_{\Omega,A}$ and M_Ω, where M_Ω denotes the rough maximal operator defined by

$$M_\Omega f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|x-y|<r} |\Omega(x-y)||f(y)|\, dy.$$

Lemma 2. *Suppose that for $|\alpha| = m-1$ and $D^\alpha A \in BMO(\mathbb{R}^n)$. Then for any $1 < t < \infty$, we have*

$$M_{\Omega,A}f(x) \le C_t \sum_{|\alpha|=m-1} \|D^\alpha A_1\|_* \big[M_\Omega f(x) + \big(M_{\Omega^t}(|f|^t)(x)\big)^{1/t}\big],$$

where C is independent of f.

To prove Lemma 2, we only need to use the following well-known lemma, Hölder's inequality and a trick from [3] which is standard. We omit the details here.

Lemma 3 (see [3])**.** *Let $b(x)$ be a function on $\mathbb{R}^n$ with derivatives of order m in $L^s(\mathbb{R}^n)$, $n < s$. Then*

$$|R_m(b;x,y)| \le C_{m,n}|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|I_x^y|}\int_{|I_x^y|} |D^\alpha b(z)|^s\,dz\right)^{1/s},$$

where I_x^y is the cube centered at x with sides paralled to the axes and whose diameter is $2n^{1/2}|x-y|$.

To prove Lemma 1, we also need the following weighted L^p boundedness of M_Ω.

Lemma 4. *Let Ω be homogeneous of degree zero on $\mathbb{R}^n$ with $\Omega \in L^q(S^{n-1})$ ($q > 1$). Then M_Ω is bounded from $L^p(\omega)$ to itself, when p, q and ω satisfy one of the following:*

(a) $q' \le p < \infty$, $p \ne 1$ *and* $\omega \in A_{p/q'}$;
(b) $1 < p \le q$, $p \ne \infty$, *and* $\omega^{1-p'} \in A_{p'/q'}$;
(c) $1 < p < \infty$ *and* $\omega^{q'} \in A_p$.

Lemma 4 with cases (a) and (b) was obtained by Duoandikoetxea in [5], while Lemma 4 with case (c) can be obtained by using the Stein-Weiss interpolation theorem with change measures (see [13]) between (a) and (b) and the method in [10].

In the proof of Lemma 1, we also need to use some elementary properties of A_p weight; see [8] for the proof.

Lemma 5. *Let $1 < p < \infty$. Then the following properties on A_p weights hold*:

(2.2) $A_{p_1} \subset A_{p_2}$, *if* $1 < p_1 < p_2 < \infty$.
(2.3) $\omega(x) \in A_p$, *if and only if* $\omega(x)^{1-p'} \in A_{p'}$.
(2.4) *If* $\omega(x) \in A_p$, *then there is an* ε $(0 < \varepsilon < 1)$ *such that* $p\varepsilon > 1$ *and* $\omega(x) \in A_{p\varepsilon}$.
(2.5) *If* $\omega(x) \in A_p$, *then there is an* $\varepsilon > 0$ *such that* $\omega(x)^{1+\varepsilon} \in A_p$.
(2.6) *If* $\omega(x) \in A_p$, *then for any* $0 < \varepsilon < 1$, $\omega(x)^\varepsilon \in A_p$.

Now let us turn to the proof of Lemma 1. By Lemma 4, we know that when one of the conditions (a), (b) and (c) of Lemma 1 is satisfied, M_Ω is bounded on $L^p(\omega)$. Therefore, in order to prove (2.1), by Lemma 2 we only need to show that under the conditions (a), (b) and (c) of Lemma 1, we may appropriately choose $t > 1$ such that

$$\left\|\big[M_{\Omega^t}(|f|^t)(\cdot)\big]^{1/t}\right\|_{p,\omega} \le C\|f\|_{p,\omega}.$$

Note that

$$\left\|\big[M_{\Omega^t}(|f|^t)(\cdot)\big]^{1/t}\right\|_{p,\omega} = \left\{\left\|M_{\Omega^t}(|f|^t)(\cdot)\right\|_{p/t,\omega}\right\}^{1/t}. \tag{2.7}$$

Thus, by (2.7) it is enough to show that under the conditions (a), (b) and (c) of Lemma 1, we can choose an appropriate $t > 1$ such that M_{Ω^t} is a bounded operator on $L^{p/t}(\omega)$, which can be done by using Lemmas 4 and 5. We omit the details.

This finishes the proof of Lemma 1.

Before proving our main theorem, we first establish the following lemma.

Lemma 6. *Suppose that $K(x,y)$ is a distribution which agrees with a function away from the diagonal $\{x = y\}$ satisfying*

$$|K(x,y)| \leq \frac{|\Omega(x-y)|}{|x-y|^{n+m-1}} |R_m(A;x,y)|.$$

Moreover, let Ω, A be the same as the assumption in Theorem 1. If one of the conditions (a), (b) and (c) in Theorem 1 is satisfied, and the operator defined by

$$Tf(x) = p.v. \int_{\mathbb{R}^n} K(x,y) f(y) dy$$

is bounded on $L^p(\omega)$, then the truncated operator

$$T_0 f(x) = p.v. \int_{|x-y|\leq 1} K(x,y) f(y) dy$$

is also bounded on $L^p(\omega)$ with bound $C(\|T\| + \|D^\alpha A\|_*)$, where C is independent of T, and $\|T\|$ denotes the operator norm of T from $L^p(\omega)$ to itself.

Proof. The main idea of the proof is taken from [12]. If we can prove

$$\int_{|x-h|<1/4} |T_0 f(x)|^p \omega(x) dx \leq C \int_{|y-h|<5/4} |f(y)|^p \omega(y) dy \tag{2.8}$$

holds for all $h \in \mathbb{R}^n$ with the bound independent of h, then integrating the above inequality with respect to h yields that

$$\int_{\mathbb{R}^n} |T_0 f(x)|^p \omega(x) dx \leq C \int_{\mathbb{R}^n} |f(y)|^p \omega(y) dy.$$

To prove (2.8), for any fixed $h \in \mathbb{R}^n$, we split f into three parts $f = f_1 + f_2 + f_3$, where

$$f_1(y) = f(y) \chi_{\{|y-h|<1/2\}}(y),$$

$$f_2(y) = f(y) \chi_{\{1/2 \leq |y-h| < 5/4\}}(y)$$

and

$$f_3(y) = f(y) \chi_{\{|y-h| \geq 5/4\}}(y).$$

Since $|x-h| < 1/4$ and $|y-h| < 1/2$ imply $|x-y| < 1$, it is obvious that $T_0 f_1(x) = T f_1(x)$ when $|x-h| < 1/4$. By the weighted L^p boundedness of T, we obtain

$$\begin{aligned}
\int_{|x-h|<1/4} |T_0 f_1(x)|^p \omega(x) dx &= \int_{|x-h|<1/4} |T f_1(x)|^p \omega(x) dx \\
&\leq \|T\|^p \int_{\mathbb{R}^n} |f_1(y)|^p \omega(y) dy \\
&\leq \|T\|^p \int_{|y-h|<5/4} |f(y)|^p \omega(y) dy.
\end{aligned}$$

When $|x-h|<1/4$ and $1/2\le|y-h|<5/4$, we have $|x-y|>1/4$. Thus

$$|T_0f_2(x)|\le\int_{|x-y|<1}\frac{|\Omega(x-y)|}{|x-y|^{n+m-1}}|R_m(A;x,y)||f_2(y)|dy\le CM_{\Omega,A}f_2(x).$$

When ω satisfies one of the conditions (a), (b) and (c) in the theorem, by Lemma 1 we have

$$\begin{aligned}\int_{|x-h|<1/4}|T_0f_2(x)|^p\omega(x)dx&\le C\int_{\mathbb{R}^n}|M_{\Omega,A}f_2(x)|^p\omega(x)dx\\&\le C\Big(\sum_{|\alpha|=m-1}\|D^\alpha A\|_*\Big)^p\int_{|y-h|<5/4}|f(y)|^p\omega(y)dy.\end{aligned}$$

Finally, note that if $|x-h|<1/4$ and $|y-h|\ge5/4$, then $|x-y|>1$. Hence $T_0f_3(x)=0$ when $|x-h|<1/4$. Thus, we complete the proof of Lemma 6. □

Now let us turn to the proof the theorem. In the following proof we use some basic ideas in [11] and [12].

The proof that (ii) implies (i). Let k and l be two positive integers and $P(x,y)$ be a non-degenarate real-valued polynomial with degree k in x and l in y. Write $P(x,y)=\sum_{|\gamma|\le k,|\beta|\le l}a_{\gamma\beta}x^\gamma y^\beta$. By the dilation invariance, we may assume that $\sum_{|\gamma|=k,|\beta|=l}|a_{\gamma\beta}|=1$. Decompose T^A as

$$\begin{aligned}T^Af(x)&=\int_{|x-y|<1}e^{iP(x,y)}\frac{\Omega(x-y)}{|x-y|^{n+m-1}}R_m(A;x,y)f(y)\,dy\\&\quad+\sum_{j=1}^\infty\int_{2^{j-1}<|x-y|\le2^j}e^{iP(x,y)}\frac{\Omega(x-y)}{|x-y|^{n+m-1}}R_m(A;x,y)f(y)\,dy\\&:=T_0^Af(x)+\sum_{j=1}^\infty T_j^Af(x).\end{aligned}$$

Below we will estimate the operator T_0^A and T_j^A $(j\ge1)$, respectively. First we prove

$$\|T_0^Af\|_{p,\omega}\le C(degP,n)\sum_{|\alpha|=m-1}\|D^\alpha A\|_*\|f\|_{p,\omega}. \tag{2.9}$$

Let us begin with a double induction on the degree in x and y of the polynomial. If $P(x,y)$ depends only on x or only on y, it is obvious that the condition (ii) implies (2.9). We assume that (2.9) holds for all polynomials which are sums of monomials of degree less than k in x times monomials of any degree in y, together with monomials which are of degree k in x times monomials which are of degree less than l in y. Rewrite

$$P(x,y)=\sum_{|\gamma|=k,|\beta|=l}a_{\gamma\beta}\big(x^\gamma y^\beta-y^{\gamma+\beta}\big)+P_0(x,y),$$

where $P_0(x,y)$ satisfies the inductive assumption. We split T_0^A into

$$\begin{aligned} T_0^A f(x) &= \int_{|x-y|<1} e^{iP_0(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_m(A;x,y) f(y)\,dy \\ &\quad + \int_{|x-y|<1} \big(e^{iP(x,y)} - e^{iP_0(x,y)}\big) \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_m(A;x,y) f(y)\,dy \\ &:= T_{0,1}^A f(x) + T_{0,2}^A f(x). \end{aligned}$$

Now our induction assumption states that

$$\|T_{0,1}^A f\|_{p,\omega} \le C \sum_{|\alpha|=m-1} \|D^\alpha A\|_* \|f\|_{p,\omega}. \tag{2.10}$$

On the other hand, if $|x|<1$ and $|x-y|\le 1$, then it is easy to see that

$$|e^{iP(x,y)} - e^{iP_0(x,y)}| \le C \sum_{|\gamma|=k,|\beta|=l} |a_{\gamma,\beta}||x-y| = C|x-y|.$$

Denote $f_0(y) = f(y)\chi_{\{|y|\le 2\}}$, then $T_{0,2}^A f(x) = T_{0,2}^A f_0(x)$ if $|x|<1$. Hence,

$$|T_{0,2}^A f(x)| \le C \int_{|x-y|<1} \left| \frac{\Omega(x-y)}{|x-y|^{n+m-2}} R_m(A;x,y) f_0(y) \right| dy \le C M_{\Omega,A} f_0(x).$$

When ω satisfies one of the conditions (a), (b) and (c), by Lemma 1 we have

$$\int_{|x|<1} |T_{0,2}^A f(x)|^p \omega(x)dx \le C\Big(\sum_{|\alpha|=m-1} \|D^\alpha A\|_* \Big)^p \int_{|y|<2} |f(y)|^p \omega(y)dy.$$

Using the same argument as in [11, p. 189], we may obtain

$$\int_{|x-h|<1} |T_{0,2}^A f(x)|^p \omega(x)dx \le C\Big(\sum_{|\alpha|=m-1} \|D^\alpha A\|_* \Big)^p \int_{|y-h|<2} |f(y)|^p \omega(y)dy.$$

Integrating the above inequality with respect to h, we have

$$\|T_{0,2}^A f\|_{p,\omega} \le C \sum_{|\alpha|=m-1} \|D^\alpha A\|_* \|f\|_{p,\omega}. \tag{2.11}$$

Thus, (2.9) follows from (2.10) and (2.11).

Now we turn our attention to the operator T_j^A, $j \ge 1$. Obviously,

$$|T_j^A f(x)| \le \int_{2^{j-1}<|x-y|\le 2^j} \frac{|\Omega(x-y)|}{|x-y|^{n+m-1}} |R_m(A;x,y)||f(y)|\,dy \le C M_{\Omega,A} f(x),$$

where C is independent of j. If ω satisfies one of the conditions (a), (b) and (c), then by (2.5) there is an $\varepsilon > 0$ such that $\omega^{1+\varepsilon}$ satisfies the same condition, too. Hence, it follows from Lemma 1 that

$$\|T_j^A f\|_{p,\omega^{1+\varepsilon}} \le C \sum_{|\alpha|=m-1} \|D^\alpha A\|_* \|f\|_{p,\omega^{1+\varepsilon}}, \tag{2.12}$$

where C is independent of j. On the other hand, we know (see (2.7) in [1, p. 171]) that there are $C, \delta > 0$ depending only on the total degree of $P(x,y)$ such that for all $j \ge 1$

$$\|T_j^A f\|_2 \le C 2^{-\delta j} \sum_{|\alpha|=m-1} \|D^\alpha A\|_* \|f\|_2. \tag{2.13}$$

Hence, using the Stein-Weiss interpolation theorem with change of measures [13] between (2.12) and (2.13), we may obtain

$$\|T_j^A f\|_{p,\omega} \le C2^{-\delta\theta j} \sum_{|\alpha|=m-1} \|D^\alpha A\|_* \|f\|_{p,\omega} \tag{2.14}$$

with $0 < \theta < 1$. Thus, summing (2.14) over all $j \ge 1$ and combining the estimate (2.9) for T_0^A, we have

$$\|T^A f\|_{p,\omega} \le C \sum_{|\alpha|=m-1} \|D^\alpha A\|_* \|f\|_{p,\omega},$$

where C depends only on n, p, q, A_p constant of ω and the total degree $\deg P$ of $P(x,y)$. Therefore, we finish the proof that (ii) implies (i).

The proof that (i) implies (ii). Suppose that $P(x,y)$ has property $\mathcal{P}$. Decompose

$$\begin{aligned} T^A f(x) &= \int_{|x-y|<1} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_m(A;x,y) f(y)\,dy \\ &\quad + \int_{|x-y|\ge 1} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_m(A;x,y) f(y)\,dy \\ &:= T_0^A f(x) + T_\infty^A f(x). \end{aligned}$$

By the process of the proof that (ii) implies (i), we see that T_∞^A is bounded on $L^p(\omega)$ if ω satisfies one of the conditions (a), (b) and (c) in the theorem. Therefore, if ω satisfies one of the conditions (a), (b) and (c), then T_0^A is also bounded on $L^p(\omega)$. Taking an $h \in \mathbb{R}^n$, for $|x-h| < 1$, we have $T_0^A f(x) = T_0^A[f(\cdot)\chi_{B(h,2)}(\cdot)](x)$, where $B(h,2) = \{y : |y-h| < 2\}$. The same argument as the proof of Lemma 6 tells us that

$$\begin{aligned} &\left(\int_{|x-h|\le 1} |T_0^A f(x)|^p \omega(x)dx\right)^{1/p} \\ &\qquad \le C \sum_{|\alpha|=m-1} \|D^\alpha A\|_* \left(\int_{|y-h|\le 2} |f(y)|^p \omega(y)dy\right)^{1/p}, \end{aligned} \tag{2.15}$$

where C is independent of h and f. Since $P(x,y)$ has property $\mathcal{P}$, we write

$$S^A f(x) = e^{iR_0(x,h)} p.v. \int_{|x-y|<1} e^{iP(x,y)} K_m(x,y) e^{-iP(x-h,y-h)} e^{-iR_1(y,h)} f(y)\,dy$$

for $h \in \mathbb{R}^n$, where $K_m(x,y) = \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_m(A;x,y)$. Expanding $e^{-iP(x-h,y-h)}$ into the Taylor series, we have

$$\begin{aligned} e^{-iP(x-h,y-h)} &= \sum_{k=0}^\infty \frac{(-i)^k}{k!} \Big[\sum_{\alpha,\beta} a_{\alpha\beta}(x-h)^\alpha (y-h)^\beta\Big]^k \\ &= \sum_{k=0}^\infty \frac{(-i)^k}{k!} \sum_{\mu,\nu} b_{\mu\nu}(x-h)^\mu (y-h)^\nu. \end{aligned}$$

Thus, we have

$$\begin{aligned}
&\Big(\int_{|x-h|\le 1}|S^A f(x)|^p\omega(x)dx\Big)^{1/p}\\
&\le \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{\mu,\nu}|b_{\mu\nu}|\Big(\int_{|x-h|\le 1}\Big|(x-h)^{\mu}\\
&\qquad\times\int_{|x-y|<1}e^{iP(x,y)}K_m(x,y)e^{-iR_1(y,h)}(y-h)^{\nu}f(y)\,dy\Big|^p\omega(x)dx\Big)^{1/p}\\
&\le \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{\mu,\nu}|b_{\mu\nu}||\xi^{\mu}|\Big(\int_{|x-h|<1}\big|T_0^A[e^{-iR_1(\cdot,h)}(\cdot-h)^{\nu}f(\cdot)](x)\big|^p\omega(x)dx\Big)^{1/p},
\end{aligned}$$

where $\xi=(1,1,\cdots,1)$. By (2.15), we have

$$\begin{aligned}
&\Big(\int_{|x-h|\le 1}|S^A f(x)|^p\omega(x)dx\Big)^{1/p}\\
&\le C\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{\mu,\nu}|b_{\mu\nu}||\xi^{\mu}|\Big(\int_{|y-h|<2}|f(y)(y-h)^{\nu}|^p\omega(y)dy\Big)^{1/p}\\
&\le C\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{\mu,\nu}|b_{\mu\nu}||\xi^{\mu}\cdot\eta^{\nu}|\Big(\int_{|y-h|<2}|f(y)|^p\omega(y)dy\Big)^{1/p}\\
&\le C\sum_{k=0}^{\infty}\frac{1}{k!}\big(\sum_{\alpha,\beta}|a_{\alpha\beta}||\xi^{\alpha}\cdot\eta^{\beta}|\big)^k\Big(\int_{|y-h|<2}|f(y)|^p\omega(y)dy\Big)^{1/p},
\end{aligned}$$

where $\eta=(2,2,\cdots,2)$. Note that

$$\sum_{k=0}^{\infty}\frac{1}{k!}\big(\sum_{\alpha,\beta}|a_{\alpha\beta}||\xi^{\alpha}\cdot\eta^{\beta}|\big)^k=\exp\big(\sum_{\alpha,\beta}|a_{\alpha\beta}||\xi^{\alpha}\cdot\eta^{\beta}|\big)\le C,$$

we have

$$\Big(\int_{|x-h|\le 1}|S^A f(x)|^p\omega(x)dx\Big)^{1/p}\le C\Big(\int_{|y-h|<2}|f(y)|^p\omega(y)dy\Big)^{1/p}.$$

Thus, (ii) follows from integrating the last inequality with respect to h and the proof of the theorem is complete.

Remark. Consider the multilinear oscillatory singular integral operator defined by

$$T_{A_1,A_2}f(x)=p.v.\int_{\mathbb{R}^n}e^{iP(x,y)}\frac{\Omega(x-y)}{|x-y|^{n+M-2}}\prod_{j=1}^{2}R_{m_j}(A_j;x,y)f(y)\,dy,$$

where $n,m_1,m_2\ge 2$, $M=m_1+m_2$ and $D^{\alpha}A_1,\ D^{\beta}A_2\in BMO$ for $|\alpha|=m_1-1$ and $|\beta|=m_2-1$, respectively. Repeating the arguments of the theorem and Lemma 1, we can obtain the same conclusions as the theorem and Lemma 1. We omit the details here.

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