PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 4, Pages 1005–1014
S 0002-9939(00)05641-0
Article electronically published on October 16, 2000

SUBSEMIVARIETIES OF Q-ALGEBRAS

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(Communicated by Dale Alspach)

ABSTRACT. A variety is a class of Banach algebras V, for which there exists a family of laws $\{\|P\| \leq K_p\}_P$ such that V is precisely the class of all Banach algebras A which satisfies all of the laws (i.e. for all P, $\|P\|_A \leq K_p$). We say that V is an H-variety if all of the laws are homogeneous. A semivariety is a class of Banach algebras W, for which there exists a family of homogeneous laws $\{\|P\| \leq K_P\}_P$ such that W is precisely the class of all Banach algebras A, for which there exists $K > 0$ such that for all homogeneous polynomials P, $\|P\|_A \leq K^i \cdot K_P$, where $i = \deg(P)$. However, there is no variety between the variety of all IQ-algebras and the variety of all IR-algebras, which can be defined by homogeneous laws alone. So the theory of semivarieties and the theory of varieties differ significantly. In this paper we shall construct uncountable chains and antichains of semivarieties which are not varieties.

1. Introduction

Let A be a Banach algebra and $P(X_1, \ldots, X_n)$ a polynomial (in several non-commuting variables without constant). We define

$$\|P\|_A = \sup\{\|P(x_1, \ldots, x_n)\| \colon x_i \in A, \|x_i\| \leq 1 \ (1 \leq i \leq n)\}.$$

By a law we mean a formal expression $\|P\| \leq K$ where $K \in R$ and P is a polynomial and we say that A satisfies the above law if $\|P\|_A \leq K$. We say that a law is homogeneous if P is homogeneous. A variety is a class of Banach algebras (real or complex) for which there exists a family of laws such that all of its members satisfy all of the laws, or equivalently, a variety is a class of Banach algebras which is closed under taking closed subalgebras, quotient algebras by closed ideals, products, and images under isometric isomorphisms (see [1]).

A semivariety is a class of Banach algebras for which there exists a family of laws $\{\|P\| \leq K_P\}$ such that it is precisely the class of all Banach algebras A for which there are K, $a > 0$ such that for all P, $\|P\|_{A,a} \leq K \cdot K_P$, where

$$\|P\|_{A,a} = \sup\{\|P(x_1, \ldots, x_n)\| \colon x_i \in A, \|x_i\| \leq a \ (1 \leq i \leq n)\}.$$

Every semivariety is an H-semivariety (can be obtained by homogeneous laws), but not every variety is an H-variety ([1] and [2]) which is a significant difference between them.

Let V be a variety and let $\{L_a\}_{a \in I}$ be the family of all families of laws which determines V. We define

$$|P|_V = \inf\{K \colon \exists a \in I; (\|P\| \leq K) \in L_a\}.$$

Received by the editors December 11, 1998 and, in revised form, June 10, 1999.
1991 *Mathematics Subject Classification.* Primary 46H99; Secondary 06B20.

The family $\{|P|_V\}_P$ is a family of laws which determines V. By means of the mapping $P \to |P|_V$ we can compare the elements of the lattice of all varieties and the elements of all semivarieties.

1.1. Lemma. *Let V and W be two varieties. Then $V \subseteq W$ if and only if for all polynomials P, $|P|_V \leq |P|_W$.*

Proof. See [2].

Each semivariety can be generated by an H-variety by means of the operation "$\wedge$" (if V is a class of Banach algebras, $\hat{V}$ will be the class of all Banach algebras isomorphic to members of V (see [1])).

1.2. Lemma. *Let A and B be two semivarieties generated by the H-varieties, respectively, V and W. Then $A \subseteq B$ if and only if there exists $K > 0$, such that for all homogeneous polynomials P of degree i,*

$$|P|_V \leq K^i |P|_W.$$

Proof. See [3].

1.3. Lemma. *If a variety V is determined by the homogeneous laws*

$$\{\|P\| \leq K_P\}_{p \in L},$$

where L is a class of homogeneous polynomials, then

$$\widehat{V} = \{A \colon \exists K > 0; \|P\|_A \leq K^i \cdot K_P \quad \text{for all } P \in L\},$$

where $i = \deg(P)$.

Proof. See [3].

2. Generators of semivarieties

2.1. Definition. A variety is called a *polynomial identity variety* if its elements satisfy a class of algebraic identities.

2.2. Lemma. *Each polynomial identity variety is a semivariety.*

Proof. See [3].

The following theorem, with Lemma 2.2, shows that the lattice of all semivarieties is small compared with the lattice of all varieties.

2.3. Theorem. *Let D be a semivariety. Let N_2 be the variety of all nilpotent Banach algebras of class 2 (i.e. Banach algebras in which all products vanish). Then:*

(i) *if $D = N_2$, then N_2 is the only variety which generates D;*
(ii) *if $D \neq N_2$, then there exist uncountably many H-varieties each of which generates D.*

Proof. (i) is straightforward.

(ii) Given $0 < a < 1$, let $D \neq N_2$. There is an H-variety V such that $D = \widehat{V}$. Let A be a Banach algebra such that $V = V(A)$. Now let A_a be the algebra derived from the Banach algebra A by changing the product of A to "$\circ$" defined as follows:

$$x \circ y = a(xy) \qquad (x, y \in A).$$

Let $V_a = H(A_a)$, for all $1 > a > 0$, where $H(A_a)$ is the H-variety determined by the homogeneous laws $\|P\|_{A_a} \geq \|P\|$. It is easy to see that for all homogeneous P and all $1 > a > 0$,

$$|P|_{V_a} = \|P\|_{V(A_a)} = \|P\|_{A_a}.$$

But for all homogeneous P of degree i, $\|P\|_{A_a} = a^{i-1}\|P\|_A$. So for all homogeneous P and all $1 > a > 0$,

$$|P|_{V_a} = a^{i-1}\|P\|_A.$$

Therefore, if $1 > a > 0$, $1 > b > 0$ and $a \neq b$, then $V_a \neq V_b$. Now we shall prove that $D = \widehat{V}_a$ for all $1 > a > 0$. Since $V_a \subseteq V, \widehat{V}_a \subseteq D$. Now let $B \in D$. By Lemma 1.3, there is $K > 0$ such that

$$\|P\|_B \leq K^i\|P\|_A = K^i a^{1-i}\|P\|_{A_a} \leq (k/a^2)^i\|P\|_{A_a} = (k/a^2)^i|P|_{V_a}$$

for all homogeneous P of degree i. So, $B \in \widehat{V}_a$; hence $D \subseteq \widehat{V}_a$, which completes the proof of the theorem.

3. Chains and antichains of subsemivarieties of Q-algebras

We saw that most of the well-known semivarieties are varieties, and each semivariety has uncountably many generators; however, we shall try to build uncountable chains and antichains of subsemivarieties of Q-algebras which are not varieties. (A Q-algebra is a Banach algebra which is bicontinuously isomorphic with the quotient of a uniform algebra by a closed ideal, and the class of all Q-algebras is a semivariety (see [7]).)

3.1. Definition. Let $I = (0, 1)$. We say that the family $\{a_i\}_{i\in I}$ of real decreasing sequences is increasing if for all $i, j \in I$ and all $n \in N$, $j \geq i$ implies

$$a_j(n) \geq a_i(n).$$

3.2. Theorem. *There exists an uncountable chain of semivarieties which are not varieties.*

Proof. Let $I = (0, 1)$ and let $\{a_i\}_{i\in I}$ be an increasing family of sequences such that there exists $C > 1$, so that for all $i \in I$,

$$(*) \qquad \lim_{n\to\infty}[a_i(1)\cdots a_i(n)/(a_i(1))^n]^{1/n^c} = i.$$

For each $i \in I$, let $w_i(n) = a_i(1)\cdots a_i(n)$, and let $L^1_{a_i}$ is the weighted sequence algebra. Suppose for all $i \in I$, V_i is the H-variety determined by the laws

$$\|X_1\cdots X_n\| \leq a_i(1)\cdots a_i(n)/(a_i(1))^n.$$

Since

$$\|X_1\cdots X_n\|_{L^1_{a_i}} = a_i(1)\cdots a_i(n)/(a_i(1))^n$$

(see [2], Theorem 4.5), it follows that

$$|X_1\cdots X_n|_{V_i} = a_i(1)\cdots a_i(n)/(a_i(1))^n.$$

Now consider the family $\{\widehat{V}_i\}_{i\in I}$ of semivarieties. First we shall prove that all elements of $\{\widehat{V}_i\}_{i\in I}$ are different. Let $i, j \in I$ and $\widehat{V}_i = \widehat{V}_j$. Then, there exist $K > 0$ and $L > 0$ such that for all homogeneous polynomials P of degree r,

$$|P|_{V_i} \leq K^r|P|_{V_j} \quad \text{and} \quad |P|_{V_j} \leq L^r|P|_{V_i}.$$

Therefore for all $n \geq 1$,

$$a_i(1) \cdots a_i(n)/(a_i(1))^n \leq K^n a_j(1) \cdots a_j(n)/(a_j(1))^n$$

and

$$a_j(1) \cdots a_j(n)/(a_j(1))^n \leq L^n a_i(1) \cdots a_i(n)/(a_i(1))^n.$$

Hence

$$[a_j(1)/a_i(1)]^n [a_i(1) \cdots a_i(n)/a_j(1) \cdots a_j(n)] \leq K^n$$

and

$$[a_i(1)/a_j(1)]^n [a_j(1) \cdots a_j(n)/a_i(1) \cdots a_i(n)] \leq L^n.$$

So by $(*)$, we have

$$i/j \leq 1 \quad \text{and} \quad j/i \leq 1.$$

Thus, $i = j$, so the elements of $\{\widehat{V}_i\}$ are different.

Now we shall prove that, for each $i \in I$, $\widehat{V}_i$ is not a variety. Let $i \in I$ and $m \geq 1$. Let for all $n \geq 1$,

$$a_i^m(n) = \begin{cases} 1 & \text{if } m \geq n, \\ a_i(1) \cdots a_i(n)/(a_i(1))^n & \text{otherwise.} \end{cases}$$

Let $a_i^m = (a_i^m(n))_{n=1}^\infty$. Then a_i^m is a decreasing positive sequence of real numbers. Let

$$A_i^m = L^1_{a_i^m}.$$

Then for all $n \geq 1$,

$$\|X_1 \cdots X_n\|_{A_i^m} = a_i^m(1) \cdots a_i^m(n)/(a_i^m(1))^n.$$

However, $a_i^m(1) = 1$, so for all $n \geq 1$,

$$\|X_1 \cdots X_n\|_{A_i^m} = a_i^m(2) \cdots a_i^m(n).$$

It is clear that if $m \geq n$, then

$$(**) \qquad \|X_1 \cdots X_n\|_{A_i^m} = 1.$$

Now let $m < n$. Then we have

$$(***) \qquad \|X_1 \cdots X_n\|_{A_i^m} = a_i^m(2) \cdots a_i^m(n) \leq a_i^m(n) = a_i(1) \cdots a_i(n)/(a_i(1))^n.$$

Now let

$$L = [a_i(1) \cdots a_i(m)/(a_i(1))^m]^{-1/m}.$$

Next we shall prove that for all $n \geq 1$,

$$\|X_1 \cdots X_n\|_{A_i^m} \leq L^n a_i(1) \cdots a_i(n)/(a_i(1))^n.$$

If $m \geq n$, then we have

$$[a_i(1) \cdots a_i(m)/(a_i(1))^m]^n \leq [a_i(1) \cdots a_i(n)/(a_i(1))^n]^m.$$

So by $(**)$, we have

$$\begin{aligned} \|X_1 \cdots X_n\|_{A_i^m} &= 1 \leq [a_i(1) \cdots a_i(m)/(a_i(1))^m]^{-n/m} \cdot [a_i(1) \cdots a_i(n)/(a_i(1))^n] \\ &= L^n a_i(1) \cdots a_i(n)/(a_i(1))^n. \end{aligned}$$

Now let $m < n$. Since $L \geq 1$, by $(***)$ we have

$$\begin{aligned}\|X_1 \cdots X_n\|_{A_i^m} &\leq a_i(1) \cdots a_i(n)/(a_i(1))^n \\ &\leq L^n a_i(1) \cdots a_i(n)/(a_i(1))^n,\end{aligned}$$

so that for all $i \in I$ and all $m \geq 1$,

$$A_i^m \in \widehat{V}_i.$$

Now we shall prove that for all $i \in I$,

$$\prod_{m=1}^{\infty} A_i^m \notin \widehat{V}_i.$$

Since for all $n \geq 1$

$$\|X_1 \cdots X_n\|_{\prod A_i^m} = \sup_{m \geq 1} \|X_1 \cdots X_n\|_{A_i^m}$$

(for a proof see [3]), so by $(**)$ we have

$$\|X_1 \cdots X_n\|_{\prod A_i^m} = 1.$$

Now let $\prod_{m=1}^{\infty} A_i^m \in \widehat{V}_i$. So there exists $M > 0$ such that for all $n \geq 1$,

$$M^n a_i(1) \cdots a_i(n)/(a_i(1))^n \geq 1.$$

So by $(*)$, $i \geq 1$, which is a contradiction, so that for all $i \in I$, $\prod_{m=1}^{\infty} A_i^m \notin \widehat{V}_i$. Therefore for all $i \in I$, $\widehat{V}_i$ is not a variety.

The family $\{\widehat{V}_i\}$ is also a chain, because if $i, j \in I$ and $j > i$, we take $L = a_j(1)/a_i(1)$. Then for all $n \geq 1$,

$$|X_1 \cdots X_n|_{V_i} \leq L^n |X_1 \cdots X_n|_{V_j},$$

so $\widehat{V}_i \subseteq \widehat{V}_j$.

It remains to show that there is an increasing family $\{a_i\}_{i \in I}$ of sequences which also satisfies $(*)$. For example, let $I = (0, 1)$ and for each $i \in I$, let

$$a_i = \{i^{2k-2}\}_{k=1}^{\infty}.$$

Then $\{a_i\}$ is an increasing family of sequences. Let $C = 2$. We have

$$\lim_{n \to \infty} [a_i(1) \cdots a_i(n)/(a_i(1))^n]^{1/n^2} = \lim_{n \to \infty} (i \cdot i^2 \cdots i^{n-1})^{2/n^2} = i.$$

3.3. *Remark.* In Theorem 3.2, for each $i \in I$, V_i is a non-polynomial identity variety (see Lemma 2.2), so there exists an uncountable tower of varieties whose elements are non-polynomial identity varieties.

3.4. Theorem. *There exists an uncountable antichain of semivarieties which are not varieties.*

Proof. Let $i > 1$. Let for each $n \geq 1$

$$C_n^i = \begin{cases} n2^n & \text{if there is } m \geq 1 \text{ such that } (2m+1)^i > n \geq (2m)^i, \\ -n2^n & \text{if there is } m \geq 1 \text{ such that } (2m)^i > n \geq (2m-1)^i. \end{cases}$$

Let $i > 1, j > 1$ and $j > i$. We have

$$\lim_{m \to \infty} [(2m)^i - (2m-1)^i] = \lim_{m \to \infty} [(2m+1)^{j/i} - (2m)^{j/i}] = \infty,$$

so there are $N_1 > 0$ and $N_2 > 0$ such that

$$m' \geq N_1 \Rightarrow (2m')^i - (2m'-1)^i > 4$$

and

$$m \geq N_2 \Rightarrow (2m+1)^{j/i} - (2m)^{j/i} > 4.$$

Let $N = \max(N_1, N_2)$. Let $n > N$; then there is N_3 such that

$$m'' \geq N_3 \Rightarrow (2m'')^j > n.$$

Let $m_0 = \max(N, N_3)$. Since $m_0 \geq N_3$,

$$(2m_0)^j > n. \tag{1}$$

Since $m_0 \geq N$, $m_0 \geq N_2$. Hence

$$(2m_0+1)^{j/i} - (2m_0)^{j/i} > 4,$$

so there is m'_0 such that

$$(2m_0+1)^{j/i} > 2m'_0 > 2m'_0 - 1 > (2m_0)^{j/i}$$

or

$$(2m_0+1)^j > (2m'_0)^i > (2m'_0-1)^i > (2m_0)^j. \tag{2}$$

But we have

$$m'_0 > [(2m_0)^{j/i} + 1]/2 \geq m_0 \geq N \geq N_1,$$

so there is n_0 such that

$$(2m'_0)^i > n_0 > n_0 - 1 > (2m'_0-1)^i. \tag{3}$$

Therefore, by (1), (2) and (3), there exist m_0, m'_0 and n_0 such that

$$(2m_0+1)^j > (2m'_0)^i > n_0 > n_0 - 1 > (2m'_0-1)^i > (2m_0)^j > n. \tag{4}$$

We have

$$\begin{aligned}
\sum_{k=1}^{n_0}(C_k^j - C_k^i) &= \sum_{k=1}^{n_0-2}(C_k^j - C_k^i) + C_{n_0-1}^j + C_{n_0}^j - C_{n_0-1}^i - C_{n_0}^i \\
&\geq -\sum_{k=1}^{n_0-2} 2k2^k + C_{n_0-1}^j + C_{n_0}^j - C_{n_0-1}^i - C_{n_0}^i \\
&\geq -(n_0-1)(2^{n_0}-4) + C_{n_0-1}^j + C_{n_0}^j - C_{n_0-1}^i - C_{n_0}^i.
\end{aligned}$$

By (4), we have

$$\begin{aligned}
C_{n_0-1}^j &= (n_0-1)2^{n_0-1}, \quad C_{n_0}^j = n_0 2^{n_0}, \\
C_{n_0-1}^i &= -(n_0-1)2^{n_0-1}, \quad C_{n_0}^i = -n_0 2^{n_0},
\end{aligned}$$

so that

$$\begin{aligned}
\sum_{k=1}^{n_0}(C_k^j - C_k^i) &\geq -(n_0-1)(2^{n_0}-4) + (n_0-1)2^{n_0-1} + n_0 2^{n_0} \\
&\quad + (n_0-1)2^{n_0-1} + n_0 2^{n_0} = n_0 2^{n_0+1} + 4(n_0-1).
\end{aligned}$$

Thus

$$1/n_0 \sum_{k=1}^{n_0}(C_k^j - C_k^i) \geq 2^{n_0+1} + 4(n_0-1)/n_0 \geq 2^{n_0+1} \geq 2^{n+1},$$

so that for all $i > 1, j > 1$ and $j > i$,

$$\sup_{n>1} 1/n \sum_{k=1}^{n} (C_k^j - C_k^i) = \infty. \tag{5}$$

Now let $i > 1$, $j > 1$ and $j > i$. There are $M_1 > 0$ and $M_2 > 0$ such that

$$m' \geq M_1 \Rightarrow (2m'+1)^i - (2m')^i > 4$$

and

$$m \geq M_2 \Rightarrow (2m+2)^{j/i} - (2m+1)^{j/i} > 4.$$

Let $M = \max(M_1, M_2)$ and let $n > M$; then there is $M_3 > 0$ such that

$$m'' \geq M_3 \Rightarrow (2m''+1)^j > n.$$

Let $m_1 = \max(M_1, M_3)$. Since $m_1 \geq M_3$,

$$(2m_1+1)^j > n. \tag{6}$$

Also, since $m_1 \geq M$, $m_1 \geq M_2$. Hence

$$(2m_1+2)^{j/i} - (2m_1+1)^{j/i} \geq 4.$$

There is m_1' such that

$$(2m_1+2)^j > (2m_1'+1)^i > (2m_1')^i > (2m_1+1)^j. \tag{7}$$

But, we have

$$m_1' \geq [(2m_1+1)]^{j/i}/2 \geq m_1 \geq M_1,$$

so there is n_1 such that

$$(2m_1'+1)^i > n_1 > n_1 - 1 > (2m_1')^i. \tag{8}$$

Therefore, by (6), (7) and (8), there exist m_1, m_1' and n_1 such that

$$(2m_1+2)^j > (2m_1'+1)^i > n_1 > n_1 - 1 > (2m_1')^i > (2m_1+1)^j > n. \tag{9}$$

We have

$$\begin{aligned}\sum_{k=1}^{n_1}(C_k^j - C_k^i) &\leq \sum_{k=1}^{n_1} 2k2^k + C_{n_1-1}^j + C_{n_1}^j - C_{n_1-1}^i - C_{n_1}^i \\ &\leq (n_1-1)(2^{n_1}-4) + C_{n_1-1}^j + C_{n_1}^j - C_{n_1-1}^i - C_{n_1}^i.\end{aligned}$$

By (9), we have

$$\begin{aligned}C_{n_1-1}^j &= -(n_1-1)2^{n_1-1}, \quad C_{n_1}^j = -n_1 2^{n_1}, \\ C_{n_1-1}^i &= (n_1-1)2^{n_1-1}, \quad C_{n_1}^i = n_1 2^{n_1},\end{aligned}$$

so that

$$\begin{aligned}\sum_{k=1}^{n_1}(C_k^j - C_k^i) &\leq (n_1-1)(2^{n_1}-4) - (n_1-1)2^{n_1-1} - n_1 2^{n_1} - (n_1-1)2^{n_1-1} - n_1 2^{n_1} \\ &= -n_1 2^{n_1+1} - 4(n_1-1).\end{aligned}$$

Thus

$$1/n_1 \sum_{k=1}^{n_1}(C_k^j - C_k^i) \leq -2^{n_1+1} - [4(n_1-1)]/n_1 \leq -2^{n_1+1} \leq -2^{n+1},$$

so that, for all $i > 1$, $j > 1$ and $j > i$,

$$\inf_{n\geq 1} 1/n \sum_{k=1}^{n} (C_k^j - C_k^i) = -\infty. \tag{10}$$

By (5) and (10), we conclude that for all $i > 1$, $j > 1$ and $i \neq j$,

$$\sup_{n\geq 1} 1/n \sum_{k=1}^{n} (C_k^j - C_k^i) = +\infty. \tag{11}$$

Now let for all $i > 1$ and all $n > 1$ ($a_i(1) = 1$),

$$a_i(n) = \exp[-(n2^{2n} + C_n^i)].$$

Then for all $i > 1$, $\{a_i(n)\}_{n\geq 1}$ is a decreasing sequence of positive real numbers. Now let for all $i > 1$, V_i be the variety determined by the laws

$$\|X_1 \cdots X_n\| \leq a_i(1) \cdots a_i(n)/(a_i(1))^n \qquad (n \geq 1).$$

Since

$$\|X_1 \cdots X_n\|_{L^1_{a_i}} = a_i(1) \cdots a_i(n)/(a_i(1))^n,$$

it follows that for all $n \geq 1$ and all $i > 1$,

$$|X_1 \cdots X_n|_{V_i} = a_i(1) \cdots a_i(n)/(a_i(1))^n. \tag{12}$$

For all $i > 1$, $j > 1$ and $i \neq j$ we have

$$\begin{aligned}
&\sup_{n\geq 1}[a_i(1) \cdots a_i(n)/a_j(1) \cdots a_j(n)]^{1/n} \\
&\quad = \sup_{n\geq 1} \left\{ \exp\left[\sum_{k=1}^{n}(-k2^{2k} - C_k^i)\right] / \exp\left[\sum_{k=1}^{n}(-k2^{2k} - C_k^j)\right] \right\}^{1/n} \\
&\quad = \sup_{n\geq 1} \left[\exp 1/n \sum_{k=1}^{n} (C_k^j - C_k^i)\right] = \exp \sup_{n\geq 1} 1/n \sum_{k=1}^{n} (C_k^j - C_k^i).
\end{aligned}$$

So by (11), we have

$$\sup[a_i(1) \cdots a_i(n)/(a_j(1) \cdots a_j(n)]^{1/n} = +\infty. \tag{13}$$

Now we shall prove that $\{\widehat{V}_i\}_{i>1}$ is an uncountable antichain of semivarieties whose elements are not varieties. Let $i > 1$, $j > 1$ and $i \neq j$. Then by (13), for all $K > 0$ and all $L > 0$ there exist m and n such that

$$[(a_j(1) \cdots a_j(m))/(a_i(1) \cdots a_i(m))]^{1/m} > K$$

and

$$[(a_i(1) \cdots a_i(n))/(a_j(1) \cdots a_j(n))]^{1/n} > L.$$

But for all $i > 1$ and all $j > 1$, $a_i(1) = a_j(1)$, so for all $K > 0$ and all $L > 0$ there exist m and n such that

$$[a_j(1) \cdots a_j(m)]/(a_j(1))^m > K^m[a_i(1) \cdots a_i(m)]/(a_i(1))^m$$

and

$$[a_i(1) \cdots a_i(n)]/(a_i(1))^n > L^n[a_j(1) \cdots a_j(n)]/(a_j(1))^n.$$

Thus, by (12), for all $K > 0$ and all $L > 0$, there exist m and n such that

$$|X_1 \cdots X_m|_{V_j} > K^m|X_1 \cdots X_m|_{V_i}$$

and

$$|X_1 \cdots X_n|_{V_i} > L^n |X_1 \cdots X_n|_{V_j}.$$

Therefore, $\widehat{V}_i \not\subset \widehat{V}_j$ and $\widehat{V}_j \not\subset \widehat{V}_i$.

Finally, we shall show that each element of $\{\widehat{V}_i\}$ is not a variety. For each $i > 1$ and all $m \geq 1$, let

$$a_i^m(n) = \begin{cases} 1 & \text{if } m \geq n, \\ \exp\left[-\sum_{k=1}^n (k2^{2k} + C_k^i)\right] & \text{otherwise.} \end{cases} \qquad (n \geq 1)$$

Then $\{a_i^m(n)\}_{n\geq 1}$ is a decreasing sequence of positive real numbers. Let for all $i > 1$ and all $m \geq 1$, $A_i^m = L^1_{a_i^m}$. Then for all $i > 1$, $m \geq 1$ and $n \geq 1$,

$$\|X_1 \cdots X_n\|_{A_i^m} = [a_i^m(1) \cdots a_i^m(n)]/(a_i^m(1))^n.$$

So that if $m \geq n$, then

$$\|X_1 \cdots X_n\|_{A_i^m} = 1$$

and if $n > m$, then

$$\|X_1 \cdots X_n\|_{A_i^m} \leq a_i^m(n)/(a_i^m(1))^n = \exp\left[-\sum_{k=1}^n (k2^{2k} + C_k^i)\right].$$

Now let for all $m \geq 1$, $L_m = \exp(2^{2m+2} + 2^{m+1})$. Let $i > 1$, $m \geq 1$ and $n \geq 1$. If $m \geq n$, then

$$\begin{aligned} L_m^n &\left\{\exp\left[-\sum_{k=1}^n (k2^{2k} + C_k^i)\right] = \exp\left[n2^{2m+2} + n2^{m+1} - \sum_{k=1}^n (k2^{2k} + C_k^i)\right]\right\} \\ &\geq \exp\left[n2^{2m+2} + n2^{m+1} - \sum_{k=1}^n (k2^{2k} + k2^k)\right] \\ &\geq \exp\left[n2^{2m+2} + n2^{m+1} - n \cdot \sum_{k=1}^m (2^{2k} + 2^k)\right] \\ &= \exp[n2^{2m+2} + n2^{m+1} - n(2^{2m+2} - 4)/3 - n(2^{m+1} - 2)] \\ &= \exp[(2n/3)2^{2m+2} + (10/3)n] \geq 1 = \|X_1 \cdots X_n\|_{A_i^m}. \end{aligned}$$

Now let $n > m$. Since for all $m \geq 1$, $L_m \geq 1$,

$$\|X_1 \cdots X_n\|_{A_i^m} \leq \exp\left[-\sum_{k=1}^n (k2^{2k} + C_k^i)\right] \leq L_m^n \exp\left[-\sum_{k=1}^n (k2^{2k} + C_k^i)\right].$$

Thus, for all $i > 1$, $m \geq 1$ and $n \geq 1$

$$\begin{aligned} \|X_1 \cdots X_n\|_{A_i^m} \leq L_m^n \exp\left[-\sum_{k=1}^n (k2^{2k} + C_k^i)\right] &= L_m^n [a_i(1) \cdots a_i(n)]/(a_i(1))^n \\ &= L_m^n |X_1 \cdots X_n|_{V_i}, \end{aligned}$$

so that for all $i > 1$ and all $m \geq 1$, $A_i^m \in \widehat{V}_i$. We have $\|X_1 \cdots X_n\|_{\prod_m A_i^m} = \sup \|X_1 \cdots X_n\|_{A_1^m} = 1$. Let $L > 0$ be such that for all $n \geq 1$,

$$\|X_1 \cdots X_n\|_{\prod_m A_i^m} \leq L^n \exp\left[-\sum_{k=1}^{n}(k2^{2k} + C_k^i)\right].$$

Thus we have

$$L^n \exp\left[-\sum_{k=1}^{n}(k2^{2k} + C_k^i)\right] \leq L^n \exp\left[-(n2^{2n} + C_n^i)\right] \leq L^n \exp[-(n2^{2n} - n2^n)].$$

For all $n \geq 1$,

$$L \cdot \exp(2^n - 2^{2n}) \geq 1,$$

which is a contradiction, so for all $i > 1$, $\prod_m A_i^m \notin \widehat{V}_i$. Therefore, for all $i > 1$, $\widehat{V}_i$ is not a variety. Thus $\{\widehat{V}_i\}_{i>1}$ is an uncountable antichain of semivarieties which are not varieties.

3.5. *Remark.* In Theorems 3.2 and 3.4, all of the weighted sequence algebras are Q-algebras (see [9], Theorem 5.3), so by considering the intersections of the constructed semivarieties with the semivariety of all Q-algebras, we conclude that: There are uncountable chain and antichain of subsemivarieties of Q-algebras which are not varieties.

3.6. Corollary. *There are uncountable chain and antichain of subvarieties of IQ-algebras which are not semivarieties.*

Proof. Consider that, for all varieties U and W, $\widehat{U \cap W} = \widehat{U} \cap \widehat{W}$.

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