THE DIRICHLET-JORDAN TEST
AND MULTIDIMENSIONAL EXTENSIONS

MICHAEL TAYLOR

(Communicated by Christopher D. Sogge)

Abstract. If $\mathcal{F}$ is a foliation of an open set $\Omega \subset \mathbb{R}^n$ by smooth $(n-1)$-dimensional surfaces, we define a class of functions $B(\Omega, \mathcal{F})$, supported in $\Omega$, that are, roughly speaking, smooth along $\mathcal{F}$ and of bounded variation transverse to $\mathcal{F}$. We investigate geometrical conditions on $\mathcal{F}$ that imply results on pointwise Fourier inversion for these functions. We also note similar results for functions on spheres, on compact 2-dimensional manifolds, and on the 3-dimensional torus. These results are multidimensional analogues of the classical Dirichlet-Jordan test of pointwise convergence of Fourier series in one variable.

Suppose $f \in L^1(\mathbb{R}^n)$, with Fourier transform

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int f(x)e^{-ix\cdot\xi} \, dx.$$  

We set

$$S_Rf(x) = (2\pi)^{-n/2} \int_{|\xi| \leq R} \hat{f}(\xi)e^{ix\cdot\xi} \, d\xi.$$  

When $n = 1$, the Dirichlet-Jordan test for pointwise convergence of $S_Rf(x)$ as $R \to \infty$ states that, if $f$ has bounded variation, then for each $x \in \mathbb{R}$,

$$\lim_{R \to \infty} S_Rf(x) = \frac{1}{2} \lim_{\varepsilon \to 0} [f(x + \varepsilon) + f(x - \varepsilon)].$$  

This can be established as follows. Pick a function $h(t)$, equal to 0 for $0 < t < 1$ for $0 < t \leq 1$, smooth on $(0, \infty)$, and equal to 0 for $t \geq 2$. Set $h(0) = 1/2$. By Riemann’s localization principle there is no loss of generality in assuming $f$ has compact support. If $f$ has bounded variation, its distributional derivative $f' = \mu$ is a (signed) measure, and we have

$$f(x) = \int h(x - y) \, d\mu(y) + g(x),$$

with $g \in C_0^\infty(\mathbb{R})$. If $f(x)$ is adjusted to equal the right side of (3) at each point of discontinuity, then (3) holds for all $x \in \mathbb{R}$. Then we have

$$S_Rf(x) = \int S_Rh(x - y) \, d\mu(y) + S_Rg(x).$$
Obviously \( S_{Rg}(x) \rightarrow g(x) \) for all \( x \). The Dirichlet-Jordan result can then be proven using the following two properties of \( S_{Rh} \):

\[
S_{Rh}(x) \rightarrow h(x), \quad \text{for every } x \in \mathbb{R}
\]

(including \( x = 0 \)), and, for some \( C < \infty \), independent of \( x, R \),

\[
|S_{Rh}(x)| \leq C.
\]

To establish \( \text{(6)} \), one can appeal to the Dini test, or use localization and smoothness for \( x \neq 0 \), plus a symmetrization argument to cover the case \( x = 0 \). The bound \( \text{(7)} \) is a consequence of the analysis of the Gibbs phenomenon for \( S_{Rh} \). From this, the Dirichlet-Jordan result can be deduced via Lebesgue’s dominated convergence theorem. Let us state an abstract version of this last segment of the argument.

Let \((Y, \mathcal{B})\) be a set with sigma algebra, let \( \mu \) be a finite signed measure on \( \mathcal{B} \), and let \( X \) be a set. Let \( h_R : X \times Y \rightarrow \mathbb{C} \) be given, for each \( R \in (0, \infty) \). Assume that \( h_R(x, \cdot) \) is \( \mathcal{B} \)-measurable, for each \( x \in X, R \in (0, \infty) \), that

\[
|h_R(x, y)| \leq C, \quad \forall x \in X, y \in Y, R \in (0, \infty),
\]

and that

\[
\lim_{R \rightarrow \infty} h_R(x, y) = h(x, y), \quad \forall x \in X, y \in Y.
\]

Then

\[
\lim_{R \rightarrow \infty} \int_Y h_R(x, y) \, d\mu(y) = \int_Y h(x, y) \, d\mu(y), \quad \forall x \in X.
\]

As mentioned, this is simply a consequence of the dominated convergence theorem. The role played by \( X \) here is, in essence, trivial, except for the fact that it arises in nontrivial contexts.

Multidimensional analogues of functions for which \( \text{(6)}-\text{(7)} \) hold arise as follows. Let \( \Sigma \) be a smooth \((n - 1)\)-dimensional surface in \( \mathbb{R}^n \). Let \( \mathcal{C}_1(\Sigma) \) denote the set of caustic points of order \( \geq 1 \), in the terminology used in \( \text{§10 of [PT]} \). (This follows Definition 5.2.3 of [Dm], in the case where \( \Lambda \) is the Lagrangian flow-out of the unit normal bundle of \( \Sigma \).) Let \( \mathcal{O}_\Sigma \) be an open neighborhood of \( \mathcal{C}_1(\Sigma) \). Let \( h(x) \) be a piecewise smooth function, with compact support, with simple jump across \( \Sigma \). For \( x \in \Sigma \), set \( h(x) \) equal to the mean value of its limits from each side. The fact that

\[
S_{Rh}(x) \rightarrow h(x), \quad \forall x \in \mathbb{R}^n \setminus \mathcal{O}_\Sigma,
\]

follows from Proposition 26 in \( \text{§10 of [PT]} \) (the result for \( x \in \Sigma \) holding by the analysis in \( \text{§11} \)). The fact that, for any compact \( K \subset \mathbb{R}^n \setminus \mathcal{O}_\Sigma \),

\[
|S_{Rh}(x)| \leq C_K, \quad \forall R \in (0, \infty), x \in K,
\]

follows from the analysis of the Gibbs phenomenon in \( \text{§11 of [PT]} \) (cf. also [CV]). We note that \( \mathcal{C}_1(\Sigma) \) is empty when \( n = 2 \). Also, when \( n = 3 \), \( \mathcal{C}_1(\Sigma) \) is empty if \( \Sigma \) is real analytic and not part of a sphere (as noted by [K]).

Now suppose we have a foliation of an open set \( \Omega \subset \mathbb{R}^n \) by such surfaces. More precisely, suppose we have smooth functions \( u_1, \ldots, u_n-1, v \) on \( \Omega \), producing a diffeomorphism

\[
(u_1, \ldots, u_{n-1}, v) : \Omega \rightarrow Q \subset \mathbb{R}^n,
\]

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
where \( Q \) is the open cube \(( -\pi, \pi ) \times \cdots \times ( -\pi, \pi )\). We consider the family of surfaces \( \Sigma_c = \{ x \in \Omega : v(x) = c \} \). Assume that \( \mathcal{O} \) is an open neighborhood of the union of the sets \( C_1(\Sigma_c) \). Fix \( \varphi \in C_0^\infty(\Omega) \). Let \( h_t : \Omega \to \mathbb{R} \) be given by

\[
h_t(x) = \begin{cases} 
1 & \text{if } v(x) > t, \\
\frac{1}{2} & \text{if } v(x) = t, \\
0 & \text{if } v(x) < t.
\end{cases}
\]

Let \( K \) be any compact set in \( \mathbb{R}^n \setminus \mathcal{O} \). Then, for each \( g \in C^\infty(\Omega) \), we have

\[
|S_R(\varphi h_t)(x)| \leq C_K(g), \quad \forall R \in (0, \infty), x \in K, t \in I = (-\pi, \pi).
\]

Hence, if we set \( \Phi(g)(R, x, t) = S_R(\varphi h_t)(x) \), we have

\[
\Phi : C^\infty(\Omega) \to L^\infty((0, \infty) \times K \times (-\pi, \pi)).
\]

Now, if we compose this with the inclusion \( \iota : L^\infty((0, \infty) \times K \times (-\pi, \pi)) \to L^\infty_{\text{loc}}((0, \infty) \times K \times (-\pi, \pi)) \), it is easy to see that the map \( \iota \circ \Phi : C^\infty(\Omega) \to L^\infty_{\text{loc}}((0, \infty) \times K \times (-\pi, \pi)) \) is continuous. It follows that the map \( \Phi \) in (15) has closed graph. Hence, we can apply the closed graph theorem and deduce that

\[
\sup_{x \in K, t \in I, R \in (0, \infty)} |S_R(\varphi h_t)(x)| \leq C_K \| g \|_{H^\ell(\Omega)},
\]

for some finite \( \ell \). This estimate can also be demonstrated by a recollection of what makes geometrical optics constructions work, up to any given finite order, and its implementation for the analysis of the Gibbs phenomenon in [PT]. (It would be of interest to study the optimal value of \( \ell \), but we will not pursue this here. We will stipulate that \( \ell > n/2 \).)

Now, if \( \mu \) is a finite (signed) measure on \( I \) we can say that, for each \( g \in H^\ell(\Omega) \),

\[
f(x) = \int_I g(x) \varphi(x) h_t(x) \, d\mu(t) \Rightarrow S_R f(x) \to f(x), \quad \forall x \in \mathbb{R}^n \setminus \mathcal{O}.
\]

This class of synthesized functions is somewhat constrained, but it will serve as a starting point for an analysis of a much more natural class of functions, which we will now introduce.

Let \( \Omega \subset \mathbb{R}^n \) be open and let \( \mathcal{F} = \{ \Sigma_c : c \in I \} \) be a foliation of \( \Omega \) by smooth \((n-1)\)-dimensional surfaces. Let \( \mathcal{M}(\Omega) \) denote the space of finite (signed) Borel measures on \( \Omega \). We say

\[
f \in \mathcal{B}(\Omega, \mathcal{F})
\]

if \( f \) is a compactly supported element of \( L^\infty(\Omega) \) with the property that

\[
X_1 \cdots X_k f \in \mathcal{M}(\Omega),
\]

for any \( k \), and any smooth vector fields \( X_1, \ldots, X_k \) on \( \Omega \), provided that at most one of them is not tangent to \( \mathcal{F} \). One would have the same class of functions if one insisted the one exceptional vector field be \( X_1 \) (or that it be \( X_k \)). The following is our main result.

**Theorem 1.** Given \( f \in \mathcal{B}(\Omega, \mathcal{F}) \), there exists a Borel measurable \( \tilde{f} \), equal to \( f \) a.e., such that, as \( R \to \infty \),

\[
S_R f(x) \to \tilde{f}(x), \quad \forall x \in \mathbb{R}^n \setminus \mathcal{O},
\]

where \( \mathcal{O} \) is a neighborhood of the union of \( C_1(\Sigma_c), c \in I \).
To begin the proof, we note that $B(\Omega, \mathcal{F})$ is clearly a module over $C^\infty_0(\Omega)$. Hence, using a partition of unity, we can assume that $\Omega$ is as in (13), and $\Sigma_c = \{v = c\}$. Use the inverse of the diffeomorphism in (13) to pull $f$ back to a compactly supported element $g \in L^\infty(Q)$, with the property on $\omega = \partial g/\partial x_n$ that
\begin{equation}
\Delta^M_T \omega \in \mathcal{M}(Q), \quad M = 0, 1, 2, \ldots,
\end{equation}
where
\begin{equation}
\Delta_T = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{n-1}^2}.
\end{equation}
For the first $n-1$ factors of $(-\pi, \pi)$ in $Q$, throw in the endpoints and identify them, to regard $\omega$ as a compactly supported measure on $\mathbb{T}^{n-1} \times (-\pi, \pi)$. We have, for $\varphi$ continuous on $[-\pi, \pi]$,
\begin{equation}
|\langle \varphi(t)e^{-ik\cdot x'}, \Delta^M_T \omega \rangle| \leq C_M \|\varphi\|_{L^\infty},
\end{equation}
with $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{T}^{n-1}$, $k \in \mathbb{Z}^{n-1}$, so
\begin{equation}
|\langle \varphi(t)e^{-ik\cdot x'}, \omega \rangle| \leq C_M \langle k \rangle^{-M} \|\varphi\|_{L^\infty}.
\end{equation}
Hence, we have measures $\mu_k$ on $(-\pi, \pi)$, supported on $[-a, a]$ for some $a < \pi$, such that
\begin{equation}
\|\mu_k\|_{\mathcal{M}(I)} \leq C_M \langle k \rangle^{-2M}, \quad \omega = \sum_k e^{ik\cdot x'} \mu_k,
\end{equation}
where the norm denotes the total variation of $\mu_k$. Hence
\begin{equation}
g(x', y) = \int_{-\pi}^\pi \sum_k e^{ik\cdot x'} d\mu_k(t),
\end{equation}
so
\begin{equation}
f(x) = \sum_k e^{ik\cdot u(x)} \int_{-\pi}^{v(x)} d\mu_k(t)
\end{equation}
\begin{equation}
= \varphi(x) \sum_k e^{ik\cdot u(x)} \int_{-\pi}^{v(x)} d\mu_k(t)
\end{equation}
\begin{equation}
= \sum_k \varphi(x) g_k(x) \int_{-\pi}^{v(x)} d\mu_k(t),
\end{equation}
where we choose $\varphi \in C^\infty_0(\Omega)$ equal to 1 on the support of $f$, and set $g_k(x) = e^{ik\cdot u(x)}$, with $u(x) = (u_1(x), \ldots, u_{n-1}(x))$. The estimates done above imply convergence in sup-norm of the infinite series, to a function $\bar{f}(x)$ equal a.e. to $f(x)$. The analysis done above also shows that, for
\begin{equation}
f_k(x) = \varphi(x) g_k(x) \int_{-\pi}^{v(x)} d\mu_k(t),
\end{equation}
we have
\begin{equation}
S_R f_k(x) \rightarrow f_k(x), \quad x \in \mathbb{R}^n \setminus \mathcal{O},
\end{equation}
and, for each compact $K \subset \mathbb{R}^n \setminus \mathcal{O}$,
\begin{equation}
\sup_{R \in (0, \infty), x \in K} |S_R f_k(x)| \leq C_K \|g_k\|_{H^1(\Omega)} \|\mu_k\|_{\mathcal{M}(I)}.
\end{equation}
Now
\[ \| g_k \|_{H^\ell(\Omega)} \leq C(k)^\ell, \]
so, given \( N \), we can produce \( M = M(\ell, N) \) and apply \( \text{(26)} \) to obtain
\[ \sup_{R \in (0, \infty), x \in K} |S_R f_k(x)| \leq CKN(k)^{-N}. \]
Thus, from \( \text{(27)} \), we have, for \( x_2 \in K \),
\[ \lim_{R \to \infty} S_R f(x) = \sum_k f_k(x) = \hat{f}(x), \]
and the theorem is proven.

It is clear what sort of representative of the class of \( f \in B(\Omega, F) \) the function \( \hat{f}(x) \) is. If \( x_0 \in \Sigma_c \subset \Omega \), then \( \hat{f}(x_0) \) is the mean of the limit of \( \hat{f}(x) \) as \( x \to x_0 \) from within \( \{ \nu(x) > c \} \) and as \( x \to x_0 \) from within \( \{ \nu(x) < c \} \). In particular, for each \( x_0 \in \Omega \),
\[ \hat{f}(x_0) = \lim_{r \searrow 0} \frac{1}{V_n r^n} \int_{|y| < r} f(x_0 + y) \, dy, \]
where \( V_n \) is the volume of the unit ball in \( \mathbb{R}^n \).

There are other Riemannian manifolds \( M \) besides \( \mathbb{R}^n \) for which there are analogues of Theorem \( \text{(1)} \) with
\[ S_R f(x) = \chi_R(\sqrt{-\Delta}) f(x), \]
where \( \Delta \) is the Laplace-Beltrami operator on \( M \) and \( \chi_R(\lambda) \) is 1 for \( |\lambda| < R \), 0 for \( |\lambda| > R \), and 1/2 for \( |\lambda| = R \). One class of examples is the class of “strongly scattering manifolds,” in the terminology of \( \text{(PT)} \), \S 10. Using the “compactification” trick from \S 6 of \( \text{(PT)} \), we can extend Theorem \( \text{(1)} \) to the case where \( M \) is a sphere \( S^n \), or other compact rank-one symmetric space. Using results of \( \text{(BC)} \), we can extend Theorem \( \text{(1)} \) to compact 2-dimensional manifolds (and then \( \mathcal{O} \) is empty).

Using Theorem 5.4 of \( \text{(T)} \), we can extend Theorem \( \text{(1)} \) to the case \( M = \mathbb{T}^3 \), as long as all the leaves \( \Sigma_c \) of \( \mathcal{F} \) have nonzero Gauss curvature in \( \Omega \).

REFERENCES


