C*-ALGEBRAS ASSOCIATED WITH BRANCHED COVERINGS

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Abstract. In this note we analyze the C*-algebra associated with a branched covering both as a groupoid C*-algebra and as a Cuntz-Pimsner algebra. We determine conditions when the algebra is simple and purely infinite. We indicate how to compute the K-theory of several examples, including one related to rational maps on the Riemann sphere.

1. Introduction

Given a branched covering \( \sigma : X \to X \) of a locally compact space \( X \), we define its C*-algebra to be the C*-algebra of the \( r \)-discrete groupoid \( \Gamma \) associated by Renault to the corresponding partially defined local homeomorphism \( T \). More precisely,

\[ \Gamma = \Gamma(X, \sigma) = \{(x, m - n, y) \mid m, n \in \mathbb{N}, x \in \text{dom}(T^m), y \in \text{dom}(T^n), T^m x = T^n y\}, \]

where \( T \) is the restriction of \( \sigma \) to the nonsingular set \( X \setminus S = U \). Here \( \text{dom}(T^k) \) is the domain of \( T^k \).

It turns out that this C*-algebra is isomorphic to an augmented Cuntz-Pimsner algebra associated with a C*-correspondence \((A, E)\), where \( A = C_0(X) \), \( E = \tilde{C}_c(U) \), viewed as an \( A \)-Hilbert module via the right multiplication

\[ ((\xi f))(x) = \xi(x)f(\sigma(x)), \xi \in E, f \in A, x \in U, \]

and the inner product

\[ \langle \xi, \eta \rangle(x) = \sum_{\sigma(y) = x} \overline{\xi(y)} \eta(y). \]

The left multiplication is given by the map

\[ \varphi : A \to L(E), (\varphi(f)\xi)(x) = f(x)\xi(x). \]

If the singular set is empty, so that \( U = X \), we recover previous results (see \[7\], \[2\]).

We consider several examples of C*-algebras arising from branched coverings, and indicate how to compute their K-theory. These examples include some of the algebras considered by R. Exel in \[9\], in the case of a partial homeomorphism.

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2. Branched coverings and groupoids

We collect some facts here about branched coverings and some examples for future reference.

**Definition 2.1.** Let $X, X'$ be locally compact, second countable Hausdorff spaces and let $S \subset X$, $S' \subset X'$ be closed subsets such that $U = X \setminus S$ and $V = X' \setminus S'$ are dense in $X$ and $X'$, respectively. A continuous surjective map $\sigma : X \to X'$ is said to be a branched covering with branch sets $S$ (upstairs) and $S'$ (downstairs) if:

1. the components of preimages of open sets of $X'$ are a basis for the topology of $X$ (in particular $\sigma$ is an open map),
2. $\sigma(S) = S'$, $\sigma(U) = V$, and
3. $\sigma|_U$ is a local homeomorphism.

We mention that in the original definition given by Fox (see [10]), $X \setminus S$ and $X' \setminus S'$ are supposed to be connected and $S$ and $S'$ are supposed to be of codimension 2 for topological reasons. We make no such restrictions here; in particular, we allow disconnected $X \setminus S$ and $X' \setminus S'$.

**Examples 2.2.**

a) (Folding the interval) Let $X = X' = [0, 1]$, and let the map

$$\sigma(t) = \begin{cases} 
2t, & 0 \leq t \leq 1/2, \\
2 - 2t, & 1/2 \leq t \leq 1.
\end{cases}$$

Then $S = \{1/2\}$, $S' = \{1\}$.

b) Let $X = X' = \mathbb{D}^2$ (the closed unit disc) and $\sigma(z) = z^p$ ($p \geq 2$). Then $S = S' = \{0\}$.

c) Let $X = S^k$ be the $k$-sphere, $X' = \mathbb{D}^k$ be the $k$-disc, and let $\sigma$ be the projection onto the equatorial plane. Then $S = S' = S^{k-1}$ (identified with the equator).

d) Let $G$ be a finite group acting nonfreely on a compact manifold $X$, let $X' = X/G$ be the corresponding orbifold, and let $\sigma : X \to X'$ be the quotient map. Then $\sigma$ is a branched covering with $S$ = the set of points $s$ with nontrivial isotropy group $G_s$. The previous examples are just particular cases of this, but note that not every branched covering is associated with a group action.

e) Let $X$ be the unit circle, identified with the one-point compactification of $\mathbb{R}$. Consider $\sigma : \mathbb{R} \to X$ the usual map which wraps $\mathbb{R}$ around the circle. Here $\sigma$ is not defined everywhere, but the groupoid construction will make sense.

f) Let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial map of degree $k \geq 2$, and define $p(\infty) = \infty$. We get a map $p : \mathbb{S}^2 \to \mathbb{S}^2$, which is a branched covering with

$$S' = \{w \in \mathbb{S}^2 \mid \text{the equation } p(z) = w \text{ has multiple roots}\},$$

$$S = p^{-1}(S').$$

In a similar way, each rational map $q : \mathbb{S}^2 \to \mathbb{S}^2$ gives a branched covering.

g) If $\sigma : U \to V$ is a homeomorphism between two open subsets of $X$, then we are in the situation considered by Exel in [9], by taking $A = C_0(X)$, $I = C_0(V)$, $J = C_0(U)$.

While one can build $C^*$-correspondences from any branched covering $\sigma : X \to X'$, in this note we shall restrict our attention to spaces covering themselves in a branched fashion.

For a locally compact space $X$ and a local homeomorphism $T$ from an open subset $\text{dom}(T)$ of $X$ onto an open subset $\text{ran}(T)$ of $X$, Renault denotes by $\text{Germ}(X, T)$
the groupoid of germs of \( G(X, T) \), which in turn is the full pseudogroup generated by restrictions \( T|_Y \), where \( Y \) is an open subset of \( X \) on which \( T \) is injective. He proves that the groupoid of germs coincides with the semidirect product groupoid
\[
\Gamma(X, T) = \{(x, m-n, y) \mid m, n \in \mathbb{N}, x \in \text{dom}(T^m), y \in \text{dom}(T^n), T^m x = T^n y\},
\]
if \((X, T)\) is essentially free in the sense that for all \( m, n \), there is no nonempty open set on which \( T^m \) and \( T^n \) agree.

The topology on \( \Gamma \) is generated by the open sets
\[
(Y; m, n; Z) = \{(y, m-n, z) \mid (y, z) \in Y \times Z, T^m(y) = T^n(z)\},
\]
where \( Y \) and \( Z \) are open subsets on which \( T^m \) and \( T^n \) are injective, respectively. Since the range and the source maps are local homeomorphisms, \( \Gamma \) becomes a locally compact (Hausdorff) \( r \)-discrete groupoid, and so we may consider its \( C^* \)-algebra.

Denote by \( c : \Gamma(X, T) \to \mathbb{Z} \) the cocycle defined by \( c(x, k, y) = k \), i.e., \( c \) is the so-called position cocycle. Then \( c^{-1}(0) \) is an equivalence relation \( R(X, T) \), which is the increasing union of sub-relations
\[
R_N = \{(x, y) \in X \times X \mid \exists n \leq N \text{ with } x, y \in \text{dom}(T^n), T^n x = T^n y\}.
\]

Notice that \( R_0 \) is the diagonal of \( X \times X \), and each \( R_N \) is \( r \)-discrete, because the range and source maps are local homeomorphisms.

3. \( C^* \)-algebras defined by a branched covering

**Definition 3.1.** Given a branched covering \( \sigma : X \to X \) with branch set \( S \), consider the local homeomorphism \( T : X \setminus S \to X \setminus \sigma(S) \), and assume that \((X, T)\) is essentially free. We define \( C^*(X, \sigma) \) to be \( C^*(\Gamma(X, T)) \).

Let \( A = C_0(X) \), and let \( E = C_c(U) \), where \( U = X \setminus S \), with the structure of a Hilbert \( A \)-module given by the formulae
\[
(\xi f)(x) = \xi(x) f(\sigma(x)), \quad \xi \in E, f \in A, x \in U,
\]
\[
\langle \xi, \eta \rangle(y) := \sum_{\sigma(x) = y} \overline{\xi(x) \eta(x)}, \quad x \in \sigma(U), \quad \xi, \eta \in E.
\]

In other words, the inner product is given by \( \langle \xi, \eta \rangle = P(\xi \eta) \), where \( P \) is the extension of \( P : C_c(U) \to C_c(\sigma(U)) \),
\[
(P\xi)(y) = \sum_{\sigma(x) = y} \xi(x).
\]

Note that the inner products generate the ideal \( C_0(\sigma(U)) \) in \( A \). The left module structure on \( E \) is defined by the equation
\[
\varphi : A \to L(E), \quad (\varphi(f)\xi)(x) = f(x)\xi(x) \quad f \in A, \xi \in E.
\]

It is straightforward to verify that \( \varphi(f) \) is in \( L(E) \) with adjoint \( \varphi(\hat{f}) \), \( f \in A \). It is also straightforward to verify that \( \varphi \) is injective. Thus, we may form the augmented Cuntz-Pimsner algebra \( \hat{O}_E \).

**Theorem 3.2.** The \( C^* \)-algebras \( \hat{O}_E \) and \( C^*(\Gamma(X, T)) \) are isomorphic.
The proof of the theorem will be given in several steps, using the notion of Fell bundle.

We recall briefly the Pimsner construction from [13]. A $C^*$-correspondence is a pair $(E, A)$, where $E$ is a (right) Hilbert module over a $C^*$-algebra $A$, and where $A$ acts to the left on $E$ via a $^*$-homomorphism $\varphi : A \to L(E)$, from $A$ to the bounded adjointable module maps on $E$. We shall always assume that our map $\varphi$ is injective. The module $E$ is not necessarily full, in the sense that the span of the inner products $\langle E, E \rangle$ may be a proper ideal of $A$. Given a $C^*$-correspondence $(E, A)$, Pimsner constructs a $C^*$-algebra $O_E$, which generalizes both the crossed products by $\mathbb{Z}$ and the Cuntz-Krieger algebras. The $C^*$-algebra generated by $O_E$ and $A$ is denoted by $\hat{O}_E$, and is called the augmented Cuntz-Pimsner algebra of the correspondence. The algebra $O_E$ is a quotient of the generalized Toeplitz algebra $T_E$ generated by the creation operators $T_\xi$, $\xi \in E$ on the Fock space

$$E_+ = \bigoplus_{n=0}^{\infty} E^{\otimes n}.$$  

Here $E^{\otimes 0} = A$, and for $n \geq 1$, $E^{\otimes n}$ denotes the $n$-th tensor power of $E$, balanced via the map $\varphi$. The creation operators $T_\xi$, $\xi \in E$, are defined by the formulae

$$T_\xi a = \xi a, \quad \text{for } a \in A,$$

and

$$T_\xi (\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n, \quad \text{for } \xi_1 \otimes \cdots \otimes \xi_n \in E^{\otimes n}.$$

A $C^*$-correspondence should be viewed as a generalization of an endomorphism of a $C^*$-algebra. Just as an endomorphism of a $C^*$-algebra can be “extended” to an automorphism of a larger algebra, so a general $C^*$-correspondence can be “extended” to an “invertible” correspondence. We will not go into much detail here except to indicate how this leads to another presentation of $O_E$.

Pimsner considers a new pair $(E_\infty, F_E)$, where $F_E$ is the $C^*$-algebra generated by all the compact operators $K(E^{\otimes n}), \ n \geq 0$, in $\lim L(E^{\otimes n})$, and $E_\infty = E \otimes F_E$. The advantage is that $E_\infty$ becomes an $F_E$-$F_E$ bimodule, such that the dual or adjoint module $E_\infty^*$ is also an $F_E$-$F_E$ bimodule. More accurately, $E_\infty$ has two $F_E$-valued inner products with respect to which $E_\infty$ satisfies all the axioms of an imprimitivity bimodule over $F_E$, except possibly the one asserting that the left inner product is full. The $C^*$-algebra $O_E$ is represented on the two-sided Fock space

$$E_\infty = \bigoplus_{n \in \mathbb{Z}} E^{\otimes n},$$

where for $n < 0$, $E^{\otimes n}_\infty$ means $(E^{\otimes n}_\infty)^{\otimes -n}$. It is isomorphic to the $C^*$-algebra generated by the multiplication operators $M_\xi \in L(E_\infty)$, where for $\xi \in E_\infty$, $M_\xi \varphi = \xi \otimes \varphi$.

Given a branched covering $\sigma : X \to X$, we first want to identify the $C^*$-algebra $F_E$. Recall that $T : U \to \sigma(U)$ is the local homeomorphism associated to $\sigma$.

Note that $E \otimes E$ is a quotient of $C_c(U) \otimes C_c(U)$, where we identify $f \otimes \eta$ with $\xi \otimes \varphi(f) \eta$ for any $\xi, \eta \in E$ and any $f \in A$. Therefore $E \otimes E$ can be identified, as a vector space, with the completion of the compactly supported continuous functions on the set $U \cap \sigma(U)$. In a similar way, $E^{\otimes n}$ is identified (as a vector space) with $C_c(U \cap \sigma(U) \cap \cdots \cap \sigma^n(U))$. The structure of a Hilbert $A$-module on $E^{\otimes n}$ is given

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by the equations
\[(\xi f)(x) = \xi(x)f(\sigma^n(x))\]
and
\[\langle \xi, \eta \rangle_n = P_n(\xi \eta).\]
Here \(P_n : C_c(U \cap \sigma(U) \cap \ldots \cap \sigma^{n-1}(U)) \to C_c(\sigma^n(U))\) is given by
\[P_n(\xi)(y) = \sum_{\sigma^n(x) = y} \xi(x).\]

**Proposition 3.3.** The C*-algebra \(K(E)\) is isomorphic to \(C^*(R(T))\), where
\[R(T) = \{(x, y) \in U \times U \mid T(x) = T(y)\}\]
is the equivalence relation associated with \(T = \sigma|_U\).
More generally, \(K(E^\otimes n) \simeq C^*(R(T^n))\), where
\[R(T^n) = \{(x, y) \in \text{dom}(T^n) \times \text{dom}(T^n) \mid T^n(x) = T^n(y)\}\]
is the equivalence relation associated with \(T^n\).

**Proof.** It is known that \(K(E) = E \otimes E^*\), the tensor product balanced over \(A\), where \(E^*\) is the adjoint of \(E\). Since \(\xi f \otimes \eta^* = \xi \otimes f\eta^*\), it follows that, as a set, \(K(E) = C_c(R(T))\). The multiplication of compact operators is exactly the convolution product on \(C_c(R(T))\); therefore, as C*-algebras, \(K(E) = C^*(R(T))\).

In the same way, using the fact that \(K(E^\otimes n) = (E^\otimes n) \otimes (E^\otimes n)^*\), we get \(K(E^\otimes n) = C^*(R(T^n))\).

If we take
\[R_N := \bigcup_{n=0}^{N} R(T^n),\]
where \(R_0\) is the diagonal of \(X\), then the natural inclusion \(C^*(R_N) \to C^*(R_{N+1})\) is induced by the map
\[L(E^\otimes N) \to L(E^\otimes N+1), \quad F \mapsto F \otimes I.\]

**Corollary 3.4.** We have \(\mathcal{F}_E = \bigcup_{n=0}^{\infty} C^*(R_N)\). In particular, \(\mathcal{F}_E\) is isomorphic to \(C^*(R(X,T))\); where \(R(X,T)\) is defined at the end of the previous section.

In order to establish an isomorphism between \(C^*(\Gamma)\) and \(\mathcal{O}_E\) in our setting, we follow [8] and use the notion of a Fell bundle. We show that both \(C^*(\Gamma)\) and \(\mathcal{O}_E\) are isomorphic to the C*-algebra associated to isomorphic Fell bundles over \(Z\). This point of view was suggested by Abadie, Eilers and Exel in [1]. We recall the definition of a Fell bundle and of the associated C*-algebra (cf. [13] for a more general situation).

**Definition 3.5.** Consider a Banach bundle \(p : \mathcal{B} \to \mathbb{Z}\). A multiplication on \(\mathcal{B}\) is a continuous map \(\mathcal{B} \times \mathcal{B} \to \mathcal{B}\) \(((b_1, b_2) \to b_1 b_2)\) which satisfies the conditions:

a) \(p(b_1 b_2) = p(b_1) + p(b_2), b_1, b_2 \in \mathcal{B}\),
b) \(B_k \times B_l \to B_{k+l}\) is bilinear,
c) \((b_1 b_2)b_3 = b_1(b_2 b_3)\),
d) \(||b_1 b_2|| \leq ||b_1|| ||b_2||\).
An *involution* is a continuous map \( B \to B, b \mapsto b^* \), which satisfies:

- \( p(b^*) = -p(b), b \in B \),
- \( B_k \to B_{-k} \) is conjugate linear \( \forall k \in \mathbb{Z} \),
- \( b^{**} = b \).

The bundle \( B \) together with these maps is said to be a *Fell bundle* if in addition:

- \( (b_1 b_2)^* = b_2^* b_1^* \),
- \( \|b^* b\| = \|b\|^2 \), and
- \( b^* b \geq 0 \ \forall b \in B \).

Note that \( B_0 \) is a \( C^* \)-algebra. Denote by \( C_c(B) \) the collection of compactly supported continuous sections. Of course, in our setting, because \( \mathbb{Z} \) has the discrete topology, continuous, compactly supported sections are really elements of the algebraic direct sum \( \sum B_k \). Given \( \xi, \eta \in C_c(B) \), define the multiplication and involution by means of the formulae

\[
(\xi \cdot \eta)(k) = \sum_l \xi(k-l)\eta(l),
\]

\[
\xi^*(k) = \xi(-k)^*.
\]

Then \( C_c(B) \) becomes a \( * \)-algebra. Let \( P : C_c(B) \to B_0 \) be the restriction map \( \xi \mapsto \xi(0) \). With the inner product \( \langle \xi, \eta \rangle = P(\xi^* \eta) \), \( C_c(B) \) becomes a pre-Hilbert \( B_0 \)-module. For \( \xi \in C_c(B) \), put \( \|\xi\|_2 := \|\langle \xi, \xi \rangle\|^{1/2} \), and denote the completion of \( C_c(B) \) with this norm by \( L^2(B) \). Notice that

\[
L^2(B) = \bigoplus_{k \in \mathbb{Z}} B_k
\]

as Hilbert \( B_0 \)-modules. We have an embedding \( C_c(B) \to L(L^2(B)) \) given by left multiplication. Denote by \( C^*(B) \) the completion of \( C_c(B) \) with respect to the operator norm. The map \( P : C_c(B) \to B_0 \), \( \xi \mapsto \xi(0) \) extends to a conditional expectation \( P : C^*(B) \to B_0 \).

**Proof of Theorem 3.2.** To the pair \( (E_\infty, \mathcal{F}_E) \), we can associate the Fell bundle \( B \), where \( B_n := E_\infty^\otimes n, n \in \mathbb{Z} \). The multiplication is given by the tensor product, where we identify \( E_\infty^* \otimes E_\infty \) with \( \mathcal{F}_E \) and \( E_\infty \otimes E_\infty^* \) with the ideal \( \mathcal{F}_E^1 \) of \( \mathcal{F}_E \) generated by the (images of) \( K(E_\infty) \), \( n \geq 1 \), in \( \lim inf L(E_\infty^\otimes n) \). (See [18].) The involution is obvious. Then

\[
L^2(B) = E_\infty = \bigoplus_{n \in \mathbb{Z}} E_\infty^\otimes n.
\]

Since \( E_\infty \) is generated by \( \mathcal{F}_E \) and \( E_\infty \), it follows that the \( C^* \)-algebra generated by the operators \( M_\xi \) is isomorphic to \( C^*(B) \). Hence, \( \mathcal{O}_E \simeq C^*(B) \).

For the groupoid \( \Gamma = \Gamma(X, T) \) and \( l \in \mathbb{Z} \), take

\[
\Gamma_l := \{(x, k, y) \in \Gamma \mid k = l\} = \{(x, y) \in X \times X \mid x_n = y_{n+l} \text{ for large } n\},
\]

and \( D_l = C_c(\Gamma_{-l}) \) (closure in \( C^*(\Gamma) \)). Then it is easy to see that \( C^*(\Gamma) \) is isomorphic to \( C^*(D) \). However, \( D_l = C^*(R(X, T)) \simeq \mathcal{F}_E = B_0 \), \( D_1 = C_c(\Gamma_{-1}) \simeq E_\infty = B_1 \), etc. Therefore the Fell bundles \( B \) and \( D \) are isomorphic.

That concludes the proof of the theorem.

Renault [20] has nice criteria for this \( C^* \)-algebra to be simple and purely infinite. Applying them here, we obtain the following two propositions.
Proposition 3.6. Let \( \sigma : X \to X \) be an essentially free branched covering. Assume that for every nonempty open set \( D \subset X \) and every \( x \in X \), there exist \( m, n \in \mathbb{N} \) such that \( x \in \text{dom}(T^n) \) and \( T^m x \in T^m D \). Then \( C^*(X, \sigma) \) is simple.

Proposition 3.7. Assume that for every nonempty open set \( D \subset X \) there exists an open set \( D' \subset D \) and \( m, n \in \mathbb{N} \) such that \( T^m(D) \) is strictly contained in \( T^m(D') \). Then \( C^*(X, \sigma) \) is purely infinite.

The \( C^* \)-algebra of the equivalence relation \( R(X, T) \) plays the role of the AF-algebra. It is an inductive limit of unital noncontinuous trace algebras, similar to the dimension drop algebras considered by Elliott et al.

4. K-theory Computations and Examples

We use the six-term exact sequence obtained by Pimsner to compute the K-theory of the \( C^* \)-algebras associated to a branched covering.

Theorem 4.1. Let \( \sigma : X \to X \) be a branched covering of a locally compact space \( X \) with singular sets \( S \) and \( S' \). Let \( U = X \setminus S \). We have an exact sequence:

\[
\begin{array}{c}
K_0(C_0(U \cap \sigma(U))) \rightarrow K_0(C_0(X)) \rightarrow K_0(C^*(X, \sigma)) \\
\rightarrow K_1(C^*(X, \sigma)) \rightarrow K_1(C_0(U \cap \sigma(U)))
\end{array}
\]

Proof. Theorem 4.9 of [18] for the augmented \( C^* \)-algebra \( \mathcal{O}_E \) yields the following six-term exact sequence:

\[
\begin{array}{c}
K_0(I) \\
\delta \uparrow
\end{array} \quad \longrightarrow \quad \begin{array}{c}
\otimes_{(i_f - [E])} K_0(C_0(X)) \\
\otimes_{(i_f - [E])} K_0(C^*(X, \sigma)) \\
\downarrow \delta
\end{array} \quad \begin{array}{c}
i \downarrow
\end{array} \quad \begin{array}{c}
K_0(C_0(X)) \\
K_0(C^*(X, \sigma)) \\
K_1(I)
\end{array}
\]

where \( I \) is the ideal \( I = \varphi^{-1}(K(E)) \cap C_0(\sigma(U)) \) in \( A = C_0(X) \), and where, recall, \( C^*(X, \sigma) \simeq \mathcal{O}_E \). The maps \( \delta \) are the boundary maps corresponding to the Toeplitz extension for \( E \) described in [18]. The map \( i \) is the map on K-theory induced by the inclusion of \( C_0(X) \) in \( C^*(X, \sigma) \), and the two instances of \( \otimes_{(i_f - [E])} \) are the maps induced by Kasparov multiplication by \( i_f - [E] \in KK(I, C_0(X)) \). (Here, \( K_* (C_0(X)) \) is identified with \( KK_*(\mathbb{C}, C_0(X)) \), \( K_* (\mathcal{O}_E) \) is identified with \( KK_*(\mathbb{C}, \mathcal{O}_E) \), and \( K_* (I) \) is identified with \( KK_*(\mathbb{C}, I) \).) However, \( \varphi^{-1}(K(E)) = C_0(U); \) therefore \( I = C_0(U \cap \sigma(U)) \).

Example 4.2. Consider the folding of the interval \( \sigma : [0, 1] \to [0, 1] \). Then

\[
C^*(R_0) = C([0, 1]), C^*(R_1) = \{ f : [0, 1] \to \mathbb{M}_2 \mid f(\frac{1}{2}) \in \mathbb{C} \otimes I_2 \},
\]

\[
C^*(R_2) = \{ f : [0, 1] \to \mathbb{M}_4 \mid f(0), f(\frac{1}{2}), f(\frac{3}{4}), f(1) \in \mathbb{M}_2 \otimes I_2, f(\frac{1}{2}) \in \mathbb{C} \otimes I_4 \},
\]

etc.

Since \( (0, 1], \sigma \) is essentially free, we may consider \( C^*([0, 1], \sigma) \) defined above. Note that this \( C^* \)-algebra is not simple, since the orbit of 0 is \( [0, 1] \) which is not dense. To compute its K-theory, note that \( \sigma(U) = [0, 1/2) \cup (1/2, 1], \sigma(U) = [0, 1) \); therefore \( I = C_0([0, 1/2) \cup (1/2, 1]) \). The exact sequence becomes

\[
\begin{array}{c}
0 \rightarrow \mathbb{Z} \rightarrow K_0 \\
\uparrow \\
K_1 \rightarrow 0 \rightarrow \mathbb{Z}
\end{array}
\]

hence \( C^*([0, 1], \sigma) \) has \( K_0 = \mathbb{Z} \oplus \mathbb{Z} \) and \( K_1 = 0 \).
Example 4.3. Let
\[ q(z) = \frac{(z^2 + 1)^2}{4z(z^2 - 1)}, \quad z \in S^2 = \mathbb{C} \cup \{ \infty \}. \]

Then \( q \) is a rational map of degree 4, which is a local homeomorphism, except at the points \( z \) such that the equation \( q(z) = w \) has double roots. We find
\[ U = S^2 \setminus \{ \pm i, \pm \sqrt{2} \pm 1 \}, \quad q(U) = S^2 \setminus \{ -1, 0, 1 \}. \]

It is known (Lattès 1918) that the Julia set of \( q \) is the whole Riemann sphere, and that every backward orbit \( \bigcup_{n \geq 0} q^{-n}(z) \) is dense. From Theorem 4.2.5 in [3], it follows that for any nonempty open set \( W \) of \( S^2 \), there is \( N \geq 0 \) such that \( q^N(W) = S^2 \). Notice that the forward orbits of all the points in the singular set \( \{ \pm i, \pm \sqrt{2} \pm 1 \} \) are finite. Denote by \( F \) the union of these orbits. Given any open set \( D \), we can find an open subset \( D' \subset D \) such that the closure \( \overline{D'} \) does not intersect the finite set \( F \). We can find a positive integer \( M \) such that \( q^M|_{D'} = S^2 \); hence \( \overline{D'} \) is strictly contained in \( q^M(D') \). In particular, \( (S^2, q) \) satisfies the hypotheses of Propositions 3.6 and 3.7. Hence \( C^*(S^2, q) \) is simple and purely infinite. We have
\[ I \simeq C_0(S^2 \setminus \{ \pm i, \pm \sqrt{2} \pm 1, 0, \pm 1 \}). \]

To compute the K-theory of \( I \), we use the more general short exact sequence
\[ 0 \to C_0(S^2 \setminus \{ p_1, p_2, ..., p_n \}) \to C(S^2) \xrightarrow{j} \mathbb{C}^n \to 0. \]

That gives
\[
\begin{array}{cccc}
K_0 & \to & \mathbb{Z}^2 & \xrightarrow{j_*} \mathbb{Z}^n \\
\uparrow & & \downarrow & \\
0 & \leftarrow & 0 & \leftarrow K_1
\end{array}
\]

Using the fact that \( j_* \) takes the Bott element into 0, it follows that
\[ \ker j_* \simeq \mathbb{Z}, \quad \text{coker} j_* \simeq \mathbb{Z}^{n-1}. \]

We obtain
\[ K_0(C_0(S^2 \setminus \{ p_1, p_2, ..., p_n \})) = \mathbb{Z}, \quad K_1(C_0(S^2 \setminus \{ p_1, p_2, ..., p_n \})) = \mathbb{Z}^{n-1}. \]

The exact sequence for our \( C^* \)-algebra becomes
\[
\begin{array}{cccc}
\mathbb{Z} & \to & \mathbb{Z}^2 & \to K_0(C^*(S^2, q)) \\
\uparrow & & & \downarrow \\
K_1(C^*(S^2, q)) & \leftarrow & 0 & \leftarrow \mathbb{Z}^8
\end{array}
\]

Note that for a general rational map \( q \) of degree \( \geq 2 \), the Julia set may not be the entire sphere. Its complement, the Fatou set, will provide an ideal in \( C^*(S^2, q) \) such that the quotient will be a simple \( C^* \)-algebra. For more details about the dynamics of a rational map, we refer to [3].
Example 4.4. Consider $S^1 = \mathbb{R} \cup \{\infty\}$, and let $\sigma : \mathbb{R} \to S^1, \sigma(t) = \exp(2\pi it)$. Then $I = C_0(\mathbb{R})$ with $K_0 = 0$ and $K_1 = \mathbb{Z}$; therefore we have the exact sequence

$$0 \to \mathbb{Z} \to K_0(C^*(S^1, \sigma)) \to K_1(C^*(S^1, \sigma)) \to \mathbb{Z}$$

Since the map $\mathbb{Z} \to \mathbb{Z}$ in the exact sequence is $id - id$, hence the zero map, it follows that

$$K_0(C^*(S^1, \sigma)) \simeq \mathbb{Z}^2, \quad K_1(C^*(S^1, \sigma)) \simeq \mathbb{Z}.$$  

Note that $C^*(S^1, \sigma)$ is simple, since every orbit is dense.

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