AN ORDER CHARACTERIZATION OF COMMUTATIVITY FOR \( C^* \)-ALGEBRAS

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Abstract. In this paper, we investigate the problem of when a \( C^* \)-algebra is commutative through operator-monotonic increasing functions. The principal result is that the function \( e^{t} \), \( t \in [0, \infty) \), is operator-monotonic increasing on a \( C^* \)-algebra \( A \) if and only if \( A \) is commutative. Therefore, \( C^* \)-algebra \( A \) is commutative if and only if \( e^{x+y} = e^{x}e^{y} \) in \( A^+ \mathbb{C} \) for all positive elements \( x, y \) in \( A \).

1. Introduction

Let \( A \) be a \( C^* \)-algebra. The sets of self-adjoint and positive elements of \( A \) are denoted by \( A_H \) and \( A_+ \), respectively. \( A^+ \mathbb{C} \) is the \( C^* \)-algebra obtained from \( A \) by adjoining an identity to \( A \). For \( x \in A \), the spectrum of \( x \) in \( A \) is denoted by \( \sigma(x) \). If \( A \) is commutative, \( \Omega \) denotes the spectral space of \( A \) which is locally compact in the weak * topology and \( C^0_0(\Omega) \) denotes the set of all continuous functions on \( \Omega \) vanishing at infinity. For other notations, we will follow [9].

For the commutativity of \( C^* \)-algebras, a lot of results have been obtained from various points of view, for example, the nilpotent and ideal characterizations given by I. Kaplansky (see [2] and [7]), the numerical characterizations given by M. J. Crabb, J. Duncan and C. M. McGregor (see [1]) and J. Duncan and P. J. Taylor (see [3]), the order characterizations given by T. Ogasawara (see [11]), S. Sherman (see [12]), M. J. Crabb, J. Duncan and C. M. McGregor (see [1]) and M. Fukamiya, M. Misonou and Z. Takeda (see [5]), *-representation characterizations given by C. F. Skau (see [13]) and S. Wright (see [14]) and spectral characterizations given by R. Nakamoto (see [14]) and Y. Kato (see [8]).

From [6], we know that some continuous real-valued functions defined on a subset \( S \) of the real field \( \mathbb{R} \) are operator-monotonic increasing on \( S \) and others are not. Therefore, we introduce the following concept.

Definition 1. Let \( A \) be a \( C^* \)-algebra and \( f \) a continuous real-valued function defined on a subset \( S \) of the real field \( \mathbb{R} \). We say that the function \( f \) is operator-monotonic increasing on \( A \) associated with \( S \) if \( f(x) \leq f(y) \) in \( A^+ \mathbb{C} \) whenever \( x \) and \( y \) are self-adjoint elements of \( A \), \( x \leq y \) and \( \sigma(x) \cup \sigma(y) \subseteq S \). We denote by...
$M_A(S)$ the set of operator-monotonic increasing functions on $A$ associated with the subset $S$ of the real field $R$.

The exponential function plays an important role in the study of $C^*$-algebras (see [3], [9], etc.). In this paper, we will give another order characterization of commutativity for $C^*$-algebras through operator-monotonic increasing functions.

2. Lemmas

In this section, we derive the basic properties of operator-monotonic increasing functions on a $C^*$-algebra associated with a subset of the real field $R$.

**Lemma 1.** Let $A$ be a $C^*$-algebra and $S$ a subset of the real field $R$.

(i) If $f \in M_A(S)$, then $f$ is a monotonic increasing function on $S$.

(ii) $M_A(S)$ is a convex cone.

(iii) If $f$ belongs to $M_A(S)$ and takes values in some subset $T$ of the real field $R$ and $g$ belongs to $M_A(T)$, then the composition $(g \circ f)(t) = g(f(t))$ is also in $M_A(S)$.

**Proof.** Direct verification according to Definition 1.

**Corollary 1.** Let $A$ be a $C^*$-algebra. Then the function $f(t) = t^2 + at + b, a, b \in R$ and $t \in [-\frac{a}{2}, \infty)$, is operator-monotonic increasing on $A$ if and only if $A$ is commutative.

**Proof.** Let

$$g(t) = t^2, t \in [0, \infty),$$

$$h(t) = t + \frac{a}{2}, t \in [-\frac{a}{2}, \infty),$$

and

$$k(t) = t + b - \frac{a^2}{4}, t \in [0, \infty).$$

Then

$$f(t) = (k \circ g \circ h)(t), t \in [-\frac{a}{2}, \infty),$$

and by Lemma 1(ii), the functions $h$ and $k$ are operator-monotonic increasing on $A$.

If $A$ is a commutative $C^*$-algebra, then the function $g$ is operator-monotonic increasing on $A$ by Definition 1. Therefore, the function $f$ is operator-monotonic increasing on $A$ according to Lemma 1(iii).

Conversely, if the function $f$ is operator-monotonic increasing on $A$, then, by Lemma 1(ii), the function $p(t) = (t + \frac{a}{2})^2, t \in [-\frac{a}{2}, \infty)$, is operator-monotonic increasing on $A$, and hence the function $q(t) = t^2, t \in [0, \infty)$, is operator-monotonic increasing on $A$. It follows that $A$ is a commutative $C^*$-algebra by T. Ogasawara's order characterization theorem (see [11]).

**Lemma 2.** Let $A$ be a $C^*$-algebra and $f(t), f_k(t), k = 1, 2, \ldots$, continuous real-valued functions defined on a subset $S$ of the real field $R$. If $f_k \in M_A(S), k = 1, 2, \ldots$, and $\{f_k\}$ converges to $f$ uniformly on compact subsets of $S$ when $k \to \infty$, then $f$ belongs to $M_A(S)$. 

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Proof. Take $x, y$ in $A_H$ such that

$$x \leq y$$

and

$$\sigma(x) \cup \sigma(y) \subseteq S.$$  

Let $\{e_{x}(\lambda)\}$ and $\{e_{y}(\lambda)\}$ be the spectral families of $x$ and $y$, respectively. By spectral theory, we have

$$x = \int_{\sigma(x)} \lambda d e_{x}(\lambda)$$

and

$$y = \int_{\sigma(y)} \lambda d e_{y}(\lambda).$$

Since $\{f_k\}$ converges to $f$ uniformly on compact subsets of $S$, we get

$$f(x) = \int_{\sigma(x)} f(\lambda) d e_{x}(\lambda)$$

$$= \lim_{k \to \infty} \int_{\sigma(x)} f_k(\lambda) d e_{x}(\lambda)$$

and

$$f(y) = \int_{\sigma(y)} f(\lambda) d e_{y}(\lambda)$$

$$= \lim_{k \to \infty} \int_{\sigma(y)} f_k(\lambda) d e_{y}(\lambda)$$

(see [7] Sec. 5.2]). The premise that $\{f_k : k = 1, 2, \ldots\} \subseteq M_A(S)$ implies that

$$f_k(x) \leq f_k(y), k = 1, 2, \ldots.$$

Therefore, $f(x) \leq f(y)$, and hence $f \in M_A(S)$. 

3. Main theorem and corollary

Now we come to the main results.

**Theorem 1.** Let $A$ be a $C^*$-algebra. Then the function $f(t) = e^t, t \in [0, \infty)$, is operator-monotonic increasing on $A$ if and only if $A$ is a commutative $C^*$-algebra.

**Proof.** Suppose $A$ is a commutative $C^*$-algebra. Then $A$ is isometrically * isomorphic to $C_0(\Omega)$ (see [9] p. 73]). Now it is clear that the function $f(t) = e^t, t \in [0, \infty)$, is operator-monotonic increasing on $A$.

Conversely, assume that $x$ and $y$ are elements in $A_+$ and $x \leq y$. Letting $0 < \epsilon < 1$, we have

$$x + \epsilon \leq y + \epsilon.$$
Because the function $t \mapsto -t^{-1}, t \in (0, \infty)$, is operator-monotonic increasing on $\mathcal{A}$ (see [4]), the function

$$g_\alpha(t) = \frac{1}{\alpha + 1} - \frac{1}{\alpha + t}, t \in (0, \infty),$$

where $\alpha \geq 0$, is operator-monotonic increasing on $\mathcal{A}$. For $c > 0$, let

$$B = \max\{\frac{1 - \epsilon}{e^{\epsilon} - 1} - \epsilon, 0\}.$$

Then, when $b > B$, we have

$$\left| \int_b^\infty g_\alpha(t)\,d\alpha \right| \leq \ln \frac{b + 1}{b + \epsilon} < c$$

for all $t \in [\epsilon, \|y\| + \epsilon]$. So $\int_0^\infty g_\alpha(t)\,d\alpha$ converges uniformly on $[\epsilon, \|y\| + \epsilon]$, and hence the function

$$s(t) = \ln t - \int_0^\infty g_\alpha(t)\,d\alpha$$

is continuous on $[\epsilon, \|y\| + \epsilon]$. Let $t_0 \in [\epsilon, \|y\| + \epsilon]$ such that

$$\left| \ln t_0 - \int_0^\infty g_\alpha(t_0)\,d\alpha \right| = \max_{t \in [\epsilon, \|y\| + \epsilon]} \left\{ \ln t - \int_0^\infty g_\alpha(t)\,d\alpha \right\}.$$

Then for $\delta > 0$, there are an integer $n$ and an equidistant division

$$0 = \alpha_0 < \alpha_1 < \ldots < \alpha_m = n$$

of the interval $[0, n]$ such that

$$\left| \ln t - \frac{n}{m} \sum_{k=1}^m g_{\alpha_k}(t) \right| < \delta$$

for all $t \in [\epsilon, \|y\| + \epsilon]$. So we have

$$\ln(x + \epsilon) \leq \frac{n}{m} \sum_{k=1}^m g_{\alpha_k}(x + \epsilon) + \delta$$

$$\leq \frac{n}{m} \sum_{k=1}^m g_{\alpha_k}(y + \epsilon) + \delta$$

$$\leq \ln(y + \epsilon) + 2\delta,$$

which shows that $\ln(x + \epsilon) \leq \ln(y + \epsilon)$. Choose real number $K$ such that $K \geq -2 \ln(t + \epsilon), t \in [0, \infty)$. Since the function $f(t) = e^t, t \in [0, \infty)$, is operator-monotonic increasing on $\mathcal{A}$, we get

$$e^{2\ln(x + \epsilon) + K} \leq e^{2\ln(y + \epsilon) + K},$$

and hence

$$(x + \epsilon)^2 \leq (y + \epsilon)^2$$

(see [7, Sec. 3.3]). Therefore, for $\epsilon \in (0, 1)$, the functions $f_\epsilon(t) = t^2 + 2\epsilon t + \epsilon^2, t \in (0, \infty)$, are operator-monotonic increasing on $\mathcal{A}$. It is clear that $\{g_\alpha\} = \{f_{\frac{\alpha}{\epsilon}}\}$ converges to $g(t) = t^2, t \in [0, \infty)$, uniformly on compact subsets of $[0, \infty)$. By Lemma 2, the function $f(t) = t^2, t \in [0, \infty)$, is operator-monotonic increasing on
A. So $\mathcal{A}$ is a commutative $C^*$-algebra by T. Ogasawara’s order characterization theorem.

**Corollary 2.** Let $\mathcal{A}$ be a $C^*$-algebra. Then $\mathcal{A}$ is commutative if and only if $e^{x+y} = e^x e^y$ in $\mathcal{A} + \mathcal{C}$ for all $x, y \in \mathcal{A}^+$.

**Proof.** If $\mathcal{A}$ is a commutative $C^*$-algebra, it is clear that $e^{x+y} = e^x e^y$ in $\mathcal{A} + \mathcal{C}$ for all $x, y \in \mathcal{A}^+$.

Conversely, assume that $x, y \in \mathcal{A}$, and $x \leq y$. Let $\mathcal{B}$ be the $C^*$-subalgebra generated by $(y - x)$. Then $\mathcal{B}$ is isometrically * isomorphic to $C_0^\infty(\sigma(y - x) \setminus \{0\})$ (see [9, p. 73]). Because $(y - x)(t) \geq 0, t \in \sigma(y - x) \setminus \{0\}, e^{y-x}(t) \geq 1$, that is, $e^{y-x} \geq 1$. So

$$e^x = e^x \cdot 1 \cdot e^y \leq e^y \cdot e^{y-x} \cdot e^y = e^y.$$  

Thus the function $f(t) = e^t, t \in [0, \infty)$, is operator-monotonic increasing on $\mathcal{A}$. By Theorem 1, $\mathcal{A}$ is a commutative $C^*$-algebra.

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