COMPLETE ORTHOGONAL DECOMPOSITION
HOMOMORPHISMS BETWEEN MATRIX ORDERED
HILBERT SPACES

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ABSTRACT. The purpose of this paper is to show that a complete order homo-
morphism and a complete orthogonal decomposition homomorphism between
the non-commutative $L^2$-spaces induce respectively an isomorphism and a $\ast$-
isomorphism between the associated reduced von Neumann algebras.

1. Introduction

In [C] Connes studied an order isomorphism on a Hilbert space and introduced an
orientable homogeneous selfdual cone to construct a von Neumann algebra. On the
other hand, Schmitt and Wittstock [SW] introduced a matrix ordered Hilbert space
to handle a non-commutative order and characterized it using the face property of
the family of selfdual cones. From the point of view of the complete positivity of
the maps on a matrix ordered Hilbert space, we showed in [M2] the relationship
between an order isomorphism or an orthogonal decomposition isomorphism de-
fined by Yamamuro [Y] and an isomorphism of a von Neumann algebra. In the
present article we shall generalize their results to the case where a complete order
homomorphism is not necessarily bijective.

We shall use the notation as introduced in [SW] with respect to the matrix
ordered standard forms.

Let $M_n$ and $M_{n,m}$ be respectively a set of all $n \times n$ and $n \times m$ matrices over
$\mathbb{C}$. For a Hilbert space $H$ and $n \in \mathbb{N}$, put $H_n = H \otimes M_n$. Let $(H, H^+_n, n \in \mathbb{N})$, where $H^+_n$ denotes a selfdual cone in $H_n$, be a matrix ordered Hilbert space, and
let $(\hat{H}_n, \hat{H}^+_n, n \in \mathbb{N})$ be another one. Let $h$ be a bounded linear map of $H$ into $\hat{H}$.
A bijective linear map $h$ is called an order isomorphism if $hH^+_n = \hat{H}^+_n$. We call
$h$ a complete order isomorphism if $h_n H^+_n = \hat{H}^+_n$ for every $n \in \mathbb{N}$. We call $h$ an
o.d. (orthogonal decomposition) homomorphism if $h$ is 1-positive and $(h\xi, h\eta) = 0$
whenever $\xi, \eta \in H^+$ and $(\xi, \eta) = 0$. If $h_n$ is an o.d. homomorphism for every $n \in \mathbb{N}$,
we call $h$ a complete o.d. homomorphism. A bijective map $h$ is called a complete
o.d. isomorphism if both $h$ and $h^{-1}$ are complete o.d. homomorphisms.

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From now on, let \((M, H, H_n^+, n \in \mathbb{N})\) and \((\hat{M}, \hat{H}, \hat{H}_n^+, n \in \mathbb{N})\) be matrix ordered standard forms of von Neumann algebras. Here we use the notation

\[
\text{Ad}(h) : x \in M \mapsto hxh^{-1} \in B(\hat{H})
\]

for the invertible map \(h : H \to \hat{H}\).

Throughout this paper, we assume a Hilbert space to be separable.

2. Results

The main results are as follows:

**Theorem A.** Let \((M, H, H_n^+, n \in \mathbb{N})\) and \((\hat{M}, \hat{H}, \hat{H}_n^+, n \in \mathbb{N})\) be matrix ordered standard forms. Suppose that \(h\) is a complete order homomorphism of \(H\) into \(\hat{H}\) with support projection \(e\) and range projection \(f\). Put \(N = M \cap \{e\}'\) and \(\hat{N} = \hat{M} \cap \{f\}'\). If \(e\) is completely positive and \(hH^+\) is a selfdual cone in the closed range space of \(h\), then we obtain the following properties:

1. \(f\) is completely positive.
2. \((eM|_{eH}, eH, e_nH_n^+, n \in \mathbb{N})\) and \((f\hat{M}|_{f\hat{H}}, f\hat{H}, f_n\hat{H}_n^+, n \in \mathbb{N})\) are matrix ordered standard forms.
3. \(h|_{eH}\) is a complete order isomorphism of \(eH\) onto \(f\hat{H}\), and \(\text{Ad}(h|_{eH})\) is an isomorphism of \(eMe\) onto \(f\hat{M}f\).

**Theorem B.** With \((M, H, H_n^+, n \in \mathbb{N})\) and \((\hat{M}, \hat{H}, \hat{H}_n^+, n \in \mathbb{N})\) as before, let \(h\) be a completely positive o.d. homomorphism of \(H\) into \(\hat{H}\) with support projection \(e\) and range projection \(f\). If \(h\) has the closed range, then we obtain the following properties:

1. \(e\) belongs to \(M \cap M'\).
2. \(f\) is completely positive.
3. \((f\hat{M}|_{f\hat{H}}, f\hat{H}, f_n\hat{H}_n^+, n \in \mathbb{N})\) is a matrix ordered standard form.
4. \(h|_{eH}\) is a complete o.d. isomorphism of \(eH\) onto \(f\hat{H}\), and \(\text{Ad}(h|_{eH})|_{Me}\) is a *-isomorphism of \(Me\) onto \(f\hat{M}f\).

We need some lemmata to prove Theorem A.

**Lemma 1.** Let \((M, H, J, H^+)\) be a standard form, and let \(\hat{H}\) be a Hilbert space with a selfdual cone \(\hat{H}^+\). Suppose that \(h\) is a linear bijection of \(H\) onto \(\hat{H}\) such that \(hH^+ = \hat{H}^+\). Then, for the polar decomposition \(h = u|h|\) of \(h\), \(u\) is a 1-positive isometry of \(H\) onto \(\hat{H}\), and there exists a positive invertible operator \(k\) in \(M\) such that \(|h| = kJkJ\).

**Proof.** Since for every \(\xi \in \hat{H}^+\), \((h^*\xi, \eta) = (\xi, h\eta) \geq 0\) holds for all \(\eta \in H^+\), it follows from the selfduality of \(H^+\) that \(h^*H^+ \subset H^+\). Hence \(h^*hH^+ \subset H^+\). Since \((h^{-1})^* = (h^*)^{-1}, h^*hH^+ = H^+\). By [3] Theorem 3.3] there exists a positive invertible operator \(k\) in \(M\) such that \(h^*h = k^2Jk^2J\). Note that we may assume \(H^+ = P_{\xi_0}\) with a cyclic and separating vector \(\xi_0 \in H^+\) by the unicity of the standard form. Then \(|h| = kJHk^{-1}k^{-1}JH = hH^+ = \hat{H}^+\).

This completes the proof. \(\square\)
Lemma 2. With \((M, H, H_n^+, n \in \mathbb{N})\) a matrix ordered standard form and \((\hat{H}, \hat{H}_n^+, n \in \mathbb{N})\) a matrix ordered Hilbert space, let \(h\) be an order isomorphism of \(H\) onto \(\hat{H}\). If \(h\) is completely positive, then \(h\) is a complete order isomorphism. In addition, there exists a von Neumann algebra \(M\) such that \((M, H, H_n^+, n \in \mathbb{N})\) is a matrix ordered standard form, and \(\text{Ad}(h)\) is an isomorphism of \(M\) onto \(M\).

Proof. Let \(h = u|h|\) be the polar decomposition of \(h\). By Lemma 1, \(|h|\) can be written as \(|h| = kJ_{H^+} + kJ_{H^+}\) for some positive invertible operator \(k \in M\). Hence

\[
|h_n|H_n^+ = (k \otimes 1_n)J_{H_{n}^+}(k \otimes 1_n)J_{H_{n}^+}H_n^+ = H_n^+.
\]

Then \(u_nH_n^+ = h_nH_n^+ \subset \hat{H}_n^+\). Since \(u\) is unitary, \(u\) is a complete order isomorphism of \(H\) onto \(\hat{H}\). Thus we see that \(h\) is a complete order isomorphism and \(u\) is a complete o.d. isomorphism. By [M2], Proposition 2.6, Theorem 2.7] we obtain the desired result.

Lemma 3. With \((M, H, H_n^+, n \in \mathbb{N})\) a matrix ordered standard form, let \(e\) be a completely positive projection on \(H\). Then there exists a von Neumann algebra \(A\) such that \((A, eH, e_nH_n^+, n \in \mathbb{N})\) is a matrix ordered standard form. In addition, if \(N = M \cap \{e\}'\), then

\[
A = eM|eH = N|eH.
\]

Proof. Put \(J = J_{H^+}, K = eH, K_n = e_nH_n, K_n^+ = eH^+, K_n^+ = e_nH_n^+\) for every \(n \in \mathbb{N}\). There exists by [M1, Lemma 1] a von Neumann algebra \(A\) such that \((A, K, K_n^+)\) is a matrix ordered standard form. The inclusion \(eM|_K \subset A\) follows from the first part of the proof of [M1, Lemma 2]. We prove that \(A \subset N|_K\). Note that in a standard form \((M, H, J, H^+)\) the map \(q \mapsto qJ_qJH^+\) is an order isomorphism of the set of all projections in \(M\) onto the set of all closed faces in \(H^+\) (see [C, Theorem 4.2 c])). If \(p\) is a projection in \(A\), then \(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}J_{K_2^+}^+ \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}J_{K_2^+}\), which shall be denoted by \(F\), is a closed face in \(K_2^+\), and

\[
P_F = \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}J_{K_2^+}^+ \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}J_{K_2^+}.
\]

Here \(P_F\) denotes the projection of \(K_2^+\) onto the (closed) linear span of \(F\) in \(K_2\). There then exists a projection \(P = \begin{bmatrix} a & b \\ b & c \end{bmatrix}\) in \(M_2(M)\) such that \(P|_F = PJ_2PJ_2\), where \(P|_F\) denotes the projection of \(H_2\) onto the closed linear span of the face \((F)\) generated by \(F\) in \(H_2^+\). It follows from [H] Lemma II.1.7] that \(P_F\Xi = e_2P|_F\Xi = P|_F\Xi\) for all \(\Xi \in K_2\). By setting \(\Xi = \begin{bmatrix} 0 & \xi \\ 0 & 0 \end{bmatrix}\) we have

\[
\begin{bmatrix} 0 & \xi \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} bJbJ\xi & bJcJ\xi \\ cJbJ\xi & cJcJ\xi \end{bmatrix}
\]

for all \(\xi \in K\). It follows from [SW Corollary 3.3] that \(b\xi = 0\) for all \(\xi \in K\). Using both equalities \(c = cJcJ\xi\) and \(b^*b + c^2 = c\) since \(P\) is a projection, we have

\[
c\xi = c^2JcJ\xi = (c - b^*b)JcJ\xi = cJcJ\xi = \xi.
\]

Moreover, by setting \(\Xi = \begin{bmatrix} 0 & \xi \\ 0 & 0 \end{bmatrix}\) we have

\[
p\xi = eaJcJ\xi
\]

for all \(\xi \in K\). Therefore, when \(\xi\) is an element of \(K^+\), \(p\xi = eaJcJ\xi = ea\xi\). Hence, \(p\xi = ea\xi\) for all \(\xi \in K\) because \(K\) is spanned by \(K^+\). Since \(e_2P|_F = P|_F\) for \(e_2\),
ea = ae. Consequently, we obtain
\[ N|_K \subset eM|_K \subset A \subset N|_K. \]
Therefore, we obtain the desired equalities. \( \square \)

**Remark.** In the above lemma, if we assume that \( (eH, e_nH_n^+, n \in \mathbb{N}) \) is a matrix ordered Hilbert space having the conditions for a matrix ordered standard form and \( e \) is 2-positive instead of the complete positivity, then Lemma 3 holds. Namely, if there exists a matrix ordered standard form \( (A, K, K^+_n, n \in \mathbb{N}) \) and \( eH^+ = K^+, e^2H^+_2 = K^+_2 \), then \( A = eM|_K = N|_K. \)

**Proof of Theorem A.** Put \( K = eH, K^+ = eH^+, K^+_n = e_nH^+_n, \hat{K} = hH, \hat{K}^+ = hH^+ \) and \( \hat{K}^+_n = h_nH^+_n \). Let \( h = u|h| \) and \( h_0 = u_0|h_0| \) be the polar decompositions of \( h \) and \( h_0 = h|_K \), respectively. Since \( e \) is a completely positive projection, it follows from Lemma 3 that \( (eM|_K, K, K^+_n) \) is a matrix ordered standard form. By assumption \( h_0 \) is an order isomorphism of \( K \) onto \( \hat{K} \). Hence by Lemma 2 \( |h_0| \) is a complete order isomorphism on \( K \). Therefore, \( |h_0|nK^+_n \) is a selfdual cone in \( K_n \), and so is \( \hat{K}^+_n \) in \( K_n \) because of the complete positivity of \( h \). Since \( f_n \) is the support projection of \( h_n \), it follows that \( \hat{K}^+_n \subset f_n\hat{H}^+_n. \) If \( \xi \in \hat{K}^+_n, \eta \in \hat{H}^+_n \), then \( (\xi, f_n\eta) = (\xi, \eta) \geq 0. \) Hence \( \hat{K}^+_n \subset (f_n\hat{H}^+_n)' \) (in \( \hat{K}_n \)). Therefore, \( \hat{K}^+_n = \hat{K}^+_n' \supset (f_n\hat{H}^+_n)' \supset f_n\hat{H}^+_n \) (in \( \hat{K}_n \)). Hence \( \hat{K}^+_n = f_n\hat{H}^+_n \), which means that \( f \) is completely positive. Therefore, by Lemma 3 \( (fM|_{\hat{K}_n}, \hat{K}^+_n) \) is a matrix ordered standard form and \( \text{Ad}(h_0) \) is an isomorphism of \( eM|_K \) onto \( fM|_{\hat{K}}. \) \( \square \)

Now, we examine the properties of o.d. homomorphisms between two ordered Hilbert spaces.

**Proposition 4** (cf. [DY, (2.1)]). Let \( (M, H, J, H^+) \) be a standard form, and let \( \hat{H} \) be a Hilbert space with a selfdual cone \( \hat{H}^+. \) Then \( h \) is an o.d. homomorphism of \( H \) into \( \hat{H} \) if and only if \( hH^+ \subset \hat{H}^+ \) and \( |h| \in M \cap M'. \)

One can give the similar proof to that of Dang-Yamanufo.

**Proposition 5** (cf. [DY, (3.1)]). With \( (M, H, J, H^+) \) and \( \hat{H}, \hat{H}^+ \) as before, if \( h \) is a bijective o.d. homomorphism of \( H \) to \( \hat{H} \), then \( h \) is an o.d. isomorphism.

**Proof.** Let \( h = u|h| \) be the polar decomposition of \( h. \) Using the argument in the proof of Proposition 4, we see that \( h \) is an order isomorphism. By Lemma 1, \( |h| \) can be written as \( |h| = kJk \) for some positive invertible operator \( k \in M. \) Since \( u = hk^{-1}Jk^{-1}J, \) it follows that \( uH^+ \subset \hat{H}^+. \) Hence \( u \) is an o.d. homomorphism, and so \( u \) is an o.d. isomorphism. Hence \( |h| \) is an o.d. homomorphism. By [Y] (3.4), \( k \) belongs to \( M \cap M'. \) This means that \( |h|^{-1} \) is an o.d. homomorphism. Therefore, \( h \) is an o.d. isomorphism. This completes the proof. \( \square \)

**Lemma 6.** With \( (M, H, H_n^+, n \in \mathbb{N}) \) a matrix ordered standard form and \( (\hat{H}, \hat{H}_n^+, n \in \mathbb{N}) \) a matrix ordered Hilbert space, let \( h \) be a completely positive o.d. homomorphism of \( H \) into \( \hat{H}. \) Then \( h_nH_n^+ \) is a selfdual subcone of \( \hat{H}_n^+ \) and \( (h\hat{H}^+, h_nH_n^+, n \in \mathbb{N}) \) is a matrix ordered Hilbert space, and \( h \) is a complete o.d. homomorphism.

**Proof.** Let \( h = u|h| \) be the polar decomposition of \( h. \) Using Proposition 4, we see that \( |h| \) belongs to \( M \cap M'. \) Hence \( |h_n| \) belongs to \( M_n \cap M'_n. \) This implies that \( h_n \) is an o.d. homomorphism, i.e., \( h \) is a complete o.d. homomorphism. To complete
the proof, it suffices to show that $h_nH^+_n$ is selfdual; hence $|h_n|H^+_n$ is selfdual. Recall that for every $n \in \mathbb{N}$ the selfdual cone $H^+_n$ is generated by the elements $[x_iJ_H + x_jJ_H + \xi^j]_{i,j=1}^n, x_1, \ldots, x_n \in M, \xi \in H^+$. If we set $h = |h| + \varepsilon 1, \varepsilon > 0$, then for such elements $x_i, \xi$, we get

$$\lim_{\varepsilon \to 0} |h_n| [h\varepsilon^{-\frac{1}{2}} x_iJ_H + h\varepsilon^{-\frac{1}{2}} x_jJ_H + \xi^j]_{i,j=1}^n = \lim_{\varepsilon \to 0} [|h|^{-\frac{1}{2}} x_iJ_H + x_jJ_H + \xi^j]_{i,j=1}^n$$

where $e$ denotes the support projection of $|h|$ and it belongs to the center of $M$. This implies that $|h_n|H^+_n$ is dense in the selfdual cone $e_nH^+_n$.

**Proof of Theorem B.** We apply Proposition 4, Proposition 5, Lemma 6 and [M2, Theorem 2.7].

Applying Lemma 3 and Theorem B, we obtain the following corollary:

**Corollary 7.** For matrix ordered standard forms $(M, H, H^+_n, n \in \mathbb{N})$ and $(\hat{M}, \hat{H}, \hat{H}^+_n, n \in \mathbb{N})$, suppose that $u$ is a completely positive partial isometry with initial projection $e$ and final projection $f$. Put $\rho(x) = \bar{x}u^*$ for all $x \in eMe$. Then $(eM|eH, e_nH^+_n, n \in \mathbb{N})$ and $(f\hat{M}|f\hat{H}, f_n\hat{H}^+_n, n \in \mathbb{N})$ are matrix ordered standard forms, and $\rho$ is a *-isomorphism of $eMe$ onto $fMf$.

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