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CARDINAL ELEMENTARY EXTENSIONS

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Abstract. We determine the consistency strength of some model theoretic extension properties for cardinals.

Let M, N be two nonempty sets. We say that N *extends* M iff for every structure $\mathfrak{M} = (M, \varepsilon, A_1, \ldots, A_n)$ there are $B_1, \ldots, B_n$ such that $\mathfrak{N} = (N, \varepsilon, B_1, \ldots, B_n)$ is an elementary extension of $\mathfrak{M}$. In this note we make some observations on this property for cardinals. More precisely we shall determine the consistency strength of the two properties "$\varkappa^+$ extends $\varkappa$" and "$\varkappa^{++}$ extends $\varkappa$". First note that if some $\tau > \varkappa$ extends $\varkappa$, then $\varkappa$ is weakly Π^1_1-indescribable.

Now let us look at the property "$\varkappa^{++}$ extends $\varkappa$" which is easy to analyze. Its consistency strength over ZFC is simply given by "for some regular $\tau > \varkappa$ V_τ extends $V_\varkappa$". Let us call a $\varkappa$ with the latter property *sublime* (with witness τ). Clearly, a witness τ for the sublimity of $\varkappa$ is inaccessible. So if we force with $\mathrm{Col}(\varkappa^+, < \tau)$ we get a model where $\varkappa^{++}$ extends $\varkappa$. This gives one direction of our previous claim. For the other one just assume that some regular $\tau > \varkappa$ extends $\varkappa$. We shall show that $\varkappa$ is sublime in L with witness τ. To get this we only need to know that τ extends $\varkappa$ in L. For then L_τ extends $L_\varkappa$ which yields that $\varkappa$ is sublime in L. To see that τ extends $\varkappa$ in L we need a small but familiar argument. Since we shall need it once more we make it a lemma.

Lemma 1. *Assume that $\tau > \varkappa$ extends $\varkappa$ and $\mathrm{cf}(\tau) > \omega$. Let M be a transitive set with $\varkappa \in M$ and $|M| = \varkappa$. Then there is an elementary embedding $j\colon M \to N$ with N transitive, $j \restriction \varkappa = \mathrm{id} \restriction \varkappa$ and $j(\varkappa) = \tau$.*

Proof. Let π be an isomorphism of (M, ε) onto $(\varkappa, E)$. Build a structure $\mathfrak{A} = (\varkappa, \varepsilon, E, A)$ such that A witnesses that $E|\alpha$ is well founded for every $\alpha < \varkappa$. Now choose some elementary extension $\mathfrak{B} = (\tau, \varepsilon, \bar{E}, B)$ of $\mathfrak{A}$. Then $\bar{E}|\alpha$ is well founded for every $\alpha < \tau$, so $\bar{E}$ is well founded since $\mathrm{cf}(\tau) > \omega$. Also $\bar{E}$ is extensional. So let $\sigma\colon (\tau, \bar{E}) \xrightarrow{\sim} (N, \varepsilon)$ where N is transitive. Finally set $j = \sigma \circ \pi$. □

Now let us return to the situation before the lemma. We have some regular $\tau > \varkappa$ which extends $\varkappa$ and want to show that τ extends $\varkappa$ in L. Let $\mathfrak{A} = (\varkappa, \varepsilon, A_1, \ldots, A_n)$ be a structure in L. Choose some $\alpha < \varkappa^+$ such that $\mathfrak{A} \in L_\alpha$. By the lemma let $j\colon L_\alpha \to N$ be an elementary embedding with N transitive, $j \restriction \varkappa = \mathrm{id} \restriction \varkappa$, $j(\varkappa) = \tau$. Let $\mathfrak{B} = j(\mathfrak{A})$. Because $N = L_\gamma$ for some γ we have that $\mathfrak{B} \in L$. However, $\mathfrak{B}$ is an elementary extension of $\mathfrak{A}$ with universe τ.

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So we have shown

Theorem 1. *The following theories are equiconsistent*:
(1) $ZFC +$ "$\exists\varkappa\ \varkappa^{++}$ *extends* $\varkappa$",
(2) $ZFC +$ "$\exists\varkappa\ \varkappa$ *is sublime*".

We are left with the vague task of defending our implicit claim that sublimity is a natural large cardinal property. We can only offer the following result which at least locates the strength of sublimity.

Proposition 1. (a) *Assume $V = L$. Then every sublime cardinal is subtle. Hence there are many subtle cardinals below any sublime one.*
(b) *Let $\varkappa$ be weakly ineffable. Then some $\alpha > \varkappa$ is sublime.*

Proof. (a) Let $\varkappa$ be sublime with witness τ. Assume that $\varkappa$ is not subtle, so there is a sequence $\vec{S} = \langle S_\alpha | \alpha \in C\rangle$, C club in $\varkappa$, $S_\alpha \subseteq \alpha$, such that $S_\alpha \neq S_\beta \cap \alpha$ for all $\alpha, \beta \in C$, $\alpha < \beta$. Let $\vec{S}$ be the $<_L$-least such that $\vec{S} \in L_\gamma$ where $\gamma < \varkappa^+$. By the lemma let $j\colon L_\gamma \to L_{\bar\gamma}$ be an elementary embedding with $j \restriction \varkappa = \mathrm{id} \restriction \varkappa$ and $j(\varkappa) = \tau$. Let $j(\vec{S}) = \langle \bar{S}_\alpha | \alpha \in D\rangle$. Clearly, $j(\bar{S}) \restriction \varkappa = \vec{S}$ and D is closed in τ, so $\varkappa \in D$ and $\vec{S}_\varkappa \subseteq \varkappa$. So let $\vec{S}_\varkappa \in L_\delta$ and $\vec{S} \in L_\delta$ where $\delta < \varkappa^+$ and choose an elementary embedding $k\colon L_\delta \to L_{\bar\delta}$ with $k \restriction \varkappa = \mathrm{id} \restriction \varkappa$ and $k(\varkappa) = \tau$. Now we see that $j(\vec{S}) = k(\vec{S})$ since they are both equal to the $<_L$-least sequence which shows that τ is not subtle. But then $k(\bar{S}_\varkappa) \cap \varkappa = \bar{S}_\varkappa$ yields that $\bar{S}_\varkappa \cap \alpha = S_\alpha$ for many $\alpha \in C$ which is a contradiction. So $\varkappa$ must be subtle. By reflection many $\alpha < \varkappa$ are subtle.

(b) We shall show that some $\alpha < \varkappa$ is sublime with witness $\varkappa$. Assume not. Let $I = \{\alpha < \varkappa | \alpha \text{ is inaccessible}\}$. For $\alpha \in I$ choose $S_\alpha \subseteq V_\alpha$ such that $(V_\alpha, \varepsilon, S_\alpha)$ has no elementary extension of the form $(V_\varkappa, \varepsilon, S)$. Let $C_\alpha = \{\beta < \alpha | (V_\beta, \varepsilon, S_\alpha \cap V_\beta) \prec (V_\alpha, \varepsilon, S_\alpha)\}$. So C_α is club in α. Since $\varkappa$ is weakly ineffable there are $S \subseteq V_\varkappa$ and $C \subseteq \varkappa$ such that $X = \{\alpha \in I | S_\alpha = S \cap V_\alpha \text{ and } C_\alpha = C \cap \alpha\}$ is unbounded in $\varkappa$. However, then it is easy to see that $(V_\alpha, \varepsilon, S \cap V_\alpha) \prec (V_\varkappa, \varepsilon, S)$ for any $\alpha \in C$ and $X \subseteq C$. So we have $(V_\alpha, \varepsilon, S_\alpha) \prec (V_\varkappa, \varepsilon, S)$ for any $\alpha \in X$ which is a contradiction. ☐

We now turn to our analysis of the property "$\varkappa^+$ extends $\varkappa$". This is much stronger, for it implies that $0^\#$ exists. To see this observe that if $\varkappa^+$ extends $\varkappa$, then $\varkappa$ is weakly Π^1_1-indescribable and $(\varkappa^+)^L < \varkappa^+$ because $L_\varkappa \prec L_{\varkappa^+}$. This is known to imply the existence of $O^\#$. We shall later give a proof for this. On the other hand it is well known that any cardinal $\lambda > \varkappa$ extends $\varkappa$ if $\varkappa$ is Ramsey. A special proof of this will be useful for us. For this we need the following concept.

Definition. Let $\mathfrak{A} = (X, \varepsilon, A_1, \dots, A_n)$ be a structure where X is a transitive set and $\varkappa = On \cap X$. An $\mathfrak{A}$-*mouse* is a structure $M = (J^{B,U}_\beta, \varepsilon, B, U)$ such that $\beta > \varkappa$, $\mathfrak{A} \in M$, U is a normal measure on $\varkappa$ in M and M is iterable with respect to U.

Now let $\varkappa$ be Ramsey and let $\mathfrak{A} = (\varkappa, \varepsilon, A_1, \dots, A_n)$ be a structure. Then an $\mathfrak{A}$-mouse M exists. This is part of the folklore. We may assume that $|M| = \varkappa$, so given any cardinal $\lambda > \varkappa$ we can iterate M λ times to get an embedding $\pi\colon M \to N$ which is Σ_1-preserving. However, then $\pi(\mathfrak{A})$ is an elementary extension of $\mathfrak{A}$ with universe (λ, ε). This shows that λ extends $\varkappa$. Moreover, this argument motivates the following concept.

Definition. $\varkappa$ is a *Keisler* cardinal iff every structure $\mathfrak{A} = (\varkappa, \varepsilon, A_1, \dots, A_n)$ has an elementary extension $\mathfrak{B} = (\tau, \varepsilon, B_1, \dots, B_n)$ for which a $\mathfrak{B}$-mouse exists.

An easy condensation argument shows that we can always assume that τ has cardinality $\varkappa$ and a $\mathfrak{B}$-mouse of cardinality $\varkappa$ exists.

Theorem 2. *The following theories are equiconsistent*:

(1) ZFC + "$\exists\varkappa\ \varkappa^+$ *extends* $\varkappa$",
(2) ZFC + "$\exists\varkappa\ \varkappa$ *is Keisler*".

Proof. (2)→(1) Here we actually have that if $\varkappa$ is Keisler, then any cardinal $\lambda > \varkappa$ extends $\varkappa$. Given $\mathfrak{A}$ choose a $\mathfrak{B}$-mouse M of cardinality $\varkappa$ with $\mathfrak{A} \prec \mathfrak{B}$ and iterate M λ-times to find an elementary extension of $\mathfrak{B}$ (hence $\mathfrak{A}$) with universe (λ, ε).

(1)→(2) Let $\varkappa^+$ extend $\varkappa$. Since a measurable cardinal is Keisler we may assume that there is no inner model with a measurable cardinal. We shall show that $\varkappa$ is Keisler in the Dodd-Jensen core model K (see [1]). Using our assumption $\neg L^\mu$ we get that $(\varkappa^+)^K = \varkappa^+$. To see this set $\tau = (\varkappa^+)^K$. If we had that $\tau < \varkappa^+$, then the lemma would give us an elementary embedding of K_τ into a transitive set N. Since $\varkappa \geq \omega_2$ results in [1] would give us L^μ. So we have $\tau = \varkappa^+$. Now let $\mathfrak{A} = (\varkappa, \varepsilon, A_1, \dots, A_n)$ be a structure in K. Since τ extends $\varkappa$ there are a transitive set in $\bar{K}$ and $B_1, \dots, B_n$ such that $On \cap \bar{K} = \tau$ and $(K_\varkappa, \varepsilon, A_1, \dots, A_n) \prec (\bar{K}, \varepsilon, B_1, \dots, B_n)$. Set $\mathfrak{B} = (\tau, \varepsilon, B_1, \dots, B_n)$. Hence $\mathfrak{A} \prec \mathfrak{B}$. Since $\varkappa$ cannot be the largest cardinal in $\bar{K}$ we must have that $\bar{K} \neq \bar{K}_\tau$. In fact there is a mouse $M \in K_\tau$ such that $M \notin \bar{K}$. Let N be the τ-th iterate of M. Standard arguments give that N is a $\mathfrak{B}$-mouse. □

It can be shown that every Keisler cardinal is inaccessible. We now turn to the question of how large a Keisler cardinal is. We shall show that there is one below any ω_1-Erdös cardinal. However, since the concept of a Keisler cardinal looks rather exotic we want to locate its position in the hierarchy of large cardinals quite precisely. For this we have to recall some definitions from [2].

Definition. (a) Let $f : [A]^{<\omega} \to On$, $A \subseteq On$. Assume that X is an infinite homogeneous set for f. Then set $\mathrm{typ}_f(X) = \langle \delta_n | n < \omega \rangle$, where $f''[X]^n = \{\delta_n\}$.

(b) Let $f : [A]^{<\omega} \to On$, $A \subseteq On$. A sequence $\langle X_\alpha | \alpha < \tau \rangle$ is called homogeneous for f iff every X_α is homogeneous for f and $\mathrm{typ}_f(X_\alpha) = \mathrm{typ}_f(X_\beta)$ for all $\alpha, \beta < \tau$.

(c) $\varkappa$ is $< \tau$-*Erdös* iff for every regressive $f : [C]^{<\omega} \to \varkappa$, C club in $\varkappa$, there is a homogeneous sequence $\langle X_\alpha | \alpha < \tau \rangle$ for f such that $\mathrm{otp}(X_\alpha) \geq \max\{\omega, \alpha\}$.

(d) $\varkappa$ is nearly $< \tau$-*Erdös* iff for every pair of functions f, g with $f\colon [C]^{<\omega} \to \lambda$ and $g\colon [C]^{<\omega} \to \varkappa$ regressive, $C \subseteq \varkappa$ club, $\lambda < \varkappa$, there is a sequence $\langle X_\alpha | \alpha < \tau \rangle$, $\mathrm{otp}(X_\alpha) \geq \max\{\omega, \alpha\}$, such that $\langle X_\alpha | \alpha < \tau \rangle$ is homogeneous for f and every set X_α for $\alpha < \tau$ is homogeneous for g.

Clearly, $< \omega_1$-Erdös cardinals and nearly $< \omega_1$-Erdös cardinals are very close to each other. So the following result is quite precise. We assume that the reader knows the standard techniques of translating the combinatorial definitions above into their model theoretic counterparts.

Proposition 2. (a) *Every Keisler cardinal is nearly* $< \omega_1$*-Erdös. Hence there are nearly* $< \omega_1$*-Erdös cardinals below any Keisler cardinal.*

(b) *Let* $\varkappa$ *be* $< \omega_1$*-Erdös. Then some* $\tau < \varkappa$ *is a Keisler cardinal.*

Proof. (a) Let $\varkappa$ be a Keisler cardinal. Let f, g be a pair of functions with $f\colon [C]^{<\omega} \to \lambda$ and $g\colon [C]^{<\omega} \to \varkappa$ regressive where $C \subseteq \varkappa$ is club and $\lambda < \varkappa$. Consider the structure $\mathfrak{A} = (V_\varkappa, \varepsilon, f, g)$. Since $\varkappa$ is Keisler there is some elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ for which a $\mathfrak{B}$-mouse M exists. Now let $\pi\colon M \to N$ be the ω_1-th iteration map and let $\pi(\mathfrak{B}) = (Z, \varepsilon, \bar{f}, \bar{g})$. Let I consist of the first ω_1 iteration points of M. Then I is homogeneous for $\bar{f}$ and $\bar{g}$. Now set $a = \text{typ}_{\bar{f}}(I)$. Then $a \in V_\varkappa$ since $\text{rng}(\bar{f}) \subseteq \lambda$. Now let $\omega \leq \xi < \omega_1$ and fix a bijection $h_\xi\colon \omega \to \xi$. Now there is a canonical tree $T = T(h_\xi, a, \bar{f}, \bar{g})$ of height ω such that any infinite branch of T gives a set X such that $\text{otp}(X) = \xi$, X is homogeneous for $\bar{f}$ and $\bar{g}$ and $\text{typ}_{\bar{f}}(X) = a$. Since $\pi(\mathfrak{B})$ is a model of ZFC (and $\text{cf}(On \cap Z) > \omega$) there is some $X \in Z$ such that $\text{otp}(X) = \xi$, X is homogeneous for $\bar{f}$ and $\bar{g}$ and $\text{typ}_{\bar{f}}(X) = a$. Now use that $\mathfrak{A}$ is an elementary submodel of $\pi(\mathfrak{B})$. This shows that $\varkappa$ is nearly $< \omega_1$-Erdös. Since $\varkappa$ is also weakly compact this property reflects.

(b) Let $<^*$ be a well ordering of $V_\varkappa$. Assume that no $\tau < \varkappa$ is a Keisler cardinal. For notational reasons let us assume that for every $\tau < \varkappa$ there is an $A_\tau \subseteq \tau$ such that $(\tau, \varepsilon, A_\tau)$ shows that τ is not Keisler. Let A_τ be the $<^*$-least such. We shall derive a contradiction. Consider the structure $\mathfrak{M} = (V_\varkappa, \varepsilon, <^*)$. Since $\varkappa$ is $< \omega_1$-Erdös there is a sequence $\langle I_\alpha | \alpha < \omega_1 \rangle$ such that $\text{otp}(I_\alpha) = \omega(1+\alpha)$, I_α is a remarkable set of indiscernibles for $\mathfrak{M}$, and setting $\delta_\alpha = \sup(\text{Hull}_{\mathfrak{M}}(I_\alpha) \cap \min I_\alpha)$ we have that $\delta_\alpha = \delta_\beta$ for all $\alpha, \beta < \omega_1$ and I_α, I_β have the same indiscernibility type in $\mathfrak{M}$ with parameters less than δ, the common value of all the δ_α. For $\alpha < \omega_1$ let $\pi_\alpha : \text{Hull}_{\mathfrak{M}}(I_\alpha \cup \delta) \to \mathfrak{N}_\alpha$ be the transitive collapse and let $\bar{I}_\alpha = \pi_\alpha'' I_\alpha$. Then we have $\mathfrak{N}_\alpha \prec \mathfrak{N}_\beta$ for $\alpha \leq \beta < \omega_1$. So let $\mathfrak{N}$ be the union of this chain and set $\bar{I} = \bigcup_{\alpha<\omega} \bar{I}_\alpha$. Then $\bar{I}$ is a club set of indiscernibles for $\mathfrak{N}$, $\delta = \min \bar{I}$ and $\bar{I}$ generates $\mathfrak{N}$ with parameters less than δ. Let $\eta = (\delta^+)^{\mathfrak{N}}$ and set $H = H_\eta^{\mathfrak{N}}$. By well-known results the uncountable club generating set of indiscernibles $\bar{I}$ gives us some U such that $M = (H, \varepsilon, U)$ thinks that U is a normal measure on δ and such that M is amenable and iterable with respect to U. Now let γ be the first element of I_0 and set $B = \pi_0(A_\gamma) \subseteq \delta$. Set $\mathfrak{B} = (\delta, \varepsilon, B)$. Clearly, we have $(\gamma, \varepsilon, A_\gamma) \prec (\rho, \varepsilon, A_\rho)$ for any $\rho \in I_0$. Hence there is some $\tau < \gamma$ such that $(\tau, \varepsilon, A_\tau) \prec (\gamma, \varepsilon, A_\gamma)$. So using π_0 we find some $\tau < \delta$ such that $(\tau, \varepsilon, A_\tau) \prec \mathfrak{B}$. However, this contradicts the definition of A_τ since M is essentially a $\mathfrak{B}$-mouse. Formally, we can replace M by $(J_{\delta+1}^{B,U}, \varepsilon, B, U \cap J_{\delta+1}^{B,U})$. □

Let us conclude with some remarks on other extension properties. Obviously, the proof of Theorem 1 essentially treats all the cases "τ extends $\varkappa$" where $\tau > \varkappa$ is regular, $\tau \neq \varkappa^+$ and τ is not the successor of a singular cardinal. On the other hand successors of singular cardinals behave like $\varkappa^+$. Just use the covering lemma. For extension properties of the form "some or some specific singular cardinal extends $\varkappa$" one has to introduce versions of sublimity with singular witnesses. These are actually slightly weaker.

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