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LOCATING SUBSETS OF A HILBERT SPACE

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ABSTRACT. This paper deals with locatedness of convex subsets in inner product and Hilbert spaces which plays a crucial role in the constructive validity of many important theorems of analysis.

1. INTRODUCTION

In Bishop's constructive mathematics, the framework of this paper, locatedness of subsets (especially convex subsets) of normed spaces plays a crucial role in the validity of many important theorems of analysis such as the Hahn-Banach and separation theorems [1, Chapter 7.4], [7], the open and unopen mapping theorems [5], and the existence theorems of Minkowski functionals [9]. (Recall that subset C of a normed space X is *located* if

$$\rho(x, C) := \inf\{\|x - y\| : y \in C\}$$

exists for each $x \in X$.)

Richman [10] extended the definition of weakly totally boundedness, which was first defined in [8] for separable Hilbert spaces, to inner product spaces which are not necessarily separable as follows: a subset C of an inner product space X is *weakly totally bounded* if for each $x \in X$, $\{\langle x, y\rangle : y \in C\}$ is a totally bounded subset of $\mathbb{C}$. Then he showed that

- an inhabited, bounded, convex, balanced subset of an inner product space is located if and only if it is weakly totally bounded, and
- a bounded operator on a Hilbert space has an adjoint if and only if its image of the unit ball is located.

In this paper, we show that we can remove balancedness in the first result, and partly remove boundedness in the second. Although classically an operator on a Hilbert space having an adjoint is bounded, it seems the most we can say constructively is that the operator is sequentially continuous, a weaker property than boundedness (see [2, Theorem 4] and [3, Corollary 2]), and hence our results have many applications (see, for example [4]). We assume the reader has access to [1, Chapters 4 and 7] or [6, Chapter 2] which provide the basic material on metric, normed and Hilbert spaces upon which our work is founded.

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2. Main results

Lemma 1. *Let C be a bounded, located convex subset of an inner product space X whose diameter is bounded by $M > 0$. Then for each $y \in C$ and each $\epsilon > 0$, either*

1. $\mathrm{Re}\langle x, y\rangle + \epsilon^2/(4M^2) < \mathrm{Re}\langle x, y'\rangle$ *for some* $y' \in C$, *or*
2. $\mathrm{Re}\langle x, z\rangle < \mathrm{Re}\langle x, y\rangle + \epsilon$ *for all* $z \in C$.

Proof. We may assume that $\epsilon < M^2$. Since $\rho(x + y, C)$ exists, either

$$\|x + y - y'\|^2 < \|x\|^2 - \epsilon^2/(2M^2) \text{ for some } y' \in C$$

or

$$\|x\|^2 - \epsilon^2/M^2 < \|x + y - z\|^2 \text{ for all } z \in C.$$

In the former case, we have

$$\|x\|^2 + 2\,\mathrm{Re}\langle x, y - y'\rangle + \|y - y'\|^2 < \|x\|^2 - \frac{\epsilon^2}{2M^2},$$

and hence

$$\mathrm{Re}\langle x, y\rangle + \frac{\epsilon^2}{4M^2} \le \mathrm{Re}\langle x, y\rangle + \frac{\|y - y'\|^2}{2} + \frac{\epsilon^2}{4M^2} < \mathrm{Re}\langle x, y'\rangle.$$

In the latter case, letting $\lambda := \epsilon/M^2$, as $0 < \lambda < 1$, we have for each $z \in C$

$$\begin{aligned} \|x\|^2 - \frac{\epsilon^2}{M^2} &< \|x + y - ((1-\lambda)y + \lambda z)\|^2 \\ &= \|x + \lambda(y - z)\|^2 \\ &= \|x\|^2 + 2\lambda\,\mathrm{Re}\langle x, y - z\rangle + \lambda^2\|y - z\|^2 \\ &\le \|x\|^2 + 2\lambda\,\mathrm{Re}\langle x, y - z\rangle + \lambda^2 M^2, \end{aligned}$$

and hence

$$\mathrm{Re}\langle x, z\rangle < \mathrm{Re}\langle x, y\rangle + \frac{\epsilon^2}{2\lambda M^2} + \frac{\lambda M^2}{2} = \mathrm{Re}\langle x, y\rangle + \epsilon.$$

□

Theorem 2. *Let C be an inhabited, bounded, located convex subset of an inner product space X. Then* $\sup\{\mathrm{Re}\langle x, y\rangle : y \in C\}$ *exists for each* $x \in X$.

Proof. Given $\epsilon > 0$ and $x \in X$, we will construct $y \in C$ such that $\mathrm{Re}\langle x, z\rangle < \mathrm{Re}\langle x, y\rangle + \epsilon$ for all $z \in C$. Let $M > 0$ be a bound of diameter of C and choose $y_0 \in C$. Construct an increasing binary sequence $\{\lambda_n\}_{n=0}^{\infty}$ with $\lambda_0 = 0$ and a sequence $\{y_n\}_{n=0}^{\infty}$ in C such that

$$\begin{aligned} \lambda_n = 0 &\Rightarrow \mathrm{Re}\langle x, y_{n-1}\rangle + \epsilon^2/(4M^2) < \mathrm{Re}\langle x, y_n\rangle, \\ \lambda_n = 1 &\Rightarrow \mathrm{Re}\langle x, z\rangle < \mathrm{Re}\langle x, y_{n-1}\rangle + \epsilon. \end{aligned}$$

Assume that we have constructed $\lambda_0, \ldots, \lambda_n$ and $y_0, \ldots, y_n$. If $\lambda_n = 1$, set

$$\lambda_{n+1} := 1 \quad \text{and} \quad y_{n+1} := y_n.$$

Otherwise, applying Lemma 1, if $\mathrm{Re}\langle x, y_n\rangle + \epsilon^2/(4M^2) < \mathrm{Re}\langle x, y'\rangle$ for some $y' \in C$, set $\lambda_{n+1} := 1$ and $y_{n+1} := y'$; or else if $\mathrm{Re}\langle x, z\rangle < \mathrm{Re}\langle x, y_n\rangle + \epsilon$ for all $z \in C$, set $\lambda_{n+1} := 1$ and $y_{n+1} := y_n$.

Choose a positive integer N such that $\|x\| \cdot M < \mathrm{Re}\langle x, y_0\rangle + N \cdot \epsilon^2/(4M^2)$. If $\lambda_N = 0$, then

$$\|x\| \cdot M < \mathrm{Re}\langle x, y_0\rangle + N \cdot \frac{\epsilon^2}{4M^2} < \mathrm{Re}\langle x, y_N\rangle \leq \|x\| \cdot M,$$

a contradiction. Therefore $\lambda_N = 1$. □

Lemma 3. *Let C be a convex subset of an inner product space X such that for every $y \in X$, $\sup\{\mathrm{Re}\langle y, z\rangle : z \in C\}$ exists, and let $x \in X$. Then for each $y \in C$, each positive integer n and each real number ϵ with $0 < \epsilon < 1$, one of the following holds:*

1. $\|x - y'\|^2 < \|x - y\|^2 - \epsilon^4/(16n)$ *for some* $y' \in C$*;*
2. $n/4 < \|x - y\|^2 + \|x - z'\|^2$ *for some* $z' \in C$*;*
3. $\|x - y\| < \|x - z\| + \epsilon$ *for all* $z \in C$*.*

Proof. If $\|x - y\| < \epsilon$, then $\|x - y\| < \|x - z\| + \epsilon$ for all $z \in C$. Hence we may assume that $\epsilon/2 < \|x - y\|$. Let $\delta := \epsilon^2/2$. Then since $\sup\{\mathrm{Re}\langle x - y, z\rangle : z \in C\}$ exists, either

$$\mathrm{Re}\langle x - y, z - y\rangle < \delta \text{ for all } z \in C$$

or

$$\delta/2 < \mathrm{Re}\langle x - y, z_0 - y\rangle \text{ for some } z_0 \in C.$$

In the former case, we have

$$\begin{aligned} \|x - y\|^2 &= \mathrm{Re}\langle x - y, x - y\rangle = \mathrm{Re}\langle x - y, x - z + z - y\rangle \\ &= \mathrm{Re}\langle x - y, x - z\rangle + \mathrm{Re}\langle x - y, z - y\rangle < \|x - y\|\|x - z\| + \delta \end{aligned}$$

for all $z \in C$, and therefore

$$\|x - y\| < \|x - z\| + \frac{\delta}{\|x - y\|} < \|x - z\| + \epsilon$$

for all $z \in C$. In the latter case, we divide it into three subcases: if $\|z_0 - y\|^2 < (3/2) \cdot \mathrm{Re}\langle x - y, z_0 - y\rangle$, setting $y' := z_0$, we have

$$\begin{aligned} \|x - y'\|^2 &= \|x - y - (z_0 - y)\|^2 = \|x - y\|^2 - 2\,\mathrm{Re}\langle x - y, z_0 - y\rangle + \|z_0 - y\|^2 \\ &< \|x - y\|^2 - \frac{\mathrm{Re}\langle x - y, z_0 - y\rangle}{2} < \|x - y\|^2 - \frac{\delta}{4} = \|x - y\|^2 - \frac{\epsilon^2}{8} \\ &< \|x - y\|^2 - \frac{\epsilon^2}{8} \cdot \frac{\epsilon^2}{2n} = \|x - y\|^2 - \frac{\epsilon^4}{16n}; \end{aligned}$$

if $n/2 < \|z_0 - y\|^2$, setting $z' := z_0$, we have

$$\begin{aligned} \frac{n}{2} &< \|z' - y\|^2 = \|(z' - x) + (x - y)\|^2 \\ &= 2\|x - z'\|^2 + 2\|x - y\|^2 - \|z' + y - 2x\|^2 \\ &\leq 2\|x - z'\|^2 + 2\|x - y\|^2; \end{aligned}$$

or if $\mathrm{Re}\langle x - y, z_0 - y\rangle < \|z_0 - y\|^2$ and $\|z_0 - y\|^2 < n$, setting

$$y' := y + \frac{\mathrm{Re}\langle x - y, z_0 - y\rangle}{\|z_0 - y\|^2}(z_0 - y)$$

which is in C as $0 < \mathrm{Re}\langle x - y, z_0 - y\rangle / \|z_0 - y\|^2 < 1$, we have

$$\begin{aligned}
\|x - y'\|^2 &= \left\| x - y - \frac{\mathrm{Re}\langle x - y, z_0 - y\rangle}{\|z_0 - y\|^2}(z_0 - y)\right\|^2 \\
&= \|x - y\|^2 - \frac{2(\mathrm{Re}\langle x - y, z_0 - y\rangle)^2}{\|z_0 - y\|^2} + \frac{(\mathrm{Re}\langle x - y, z_0 - y\rangle)^2}{\|z_0 - y\|^2} \\
&= \|x - y\|^2 - \frac{(\mathrm{Re}\langle x - y, z_0 - y\rangle)^2}{\|z_0 - y\|^2} < \|x - y\|^2 - \frac{\delta^2}{4n} \\
&= \|x - y\|^2 - \frac{\epsilon^4}{16n}.
\end{aligned}$$

□

Theorem 4. *Let C be an inhabited, bounded, convex subset of an inner product space X such that* $\sup\{\mathrm{Re}\langle x, y\rangle : y \in C\}$ *exists for each $x \in X$. Then C is located.*

Proof. Given ϵ with $0 < \epsilon < 1$, and $x \in X$, we will construct $y \in C$ such that $\|x - y\| < \|x - z\| + \epsilon$ for all $z \in C$. Let M be a positive integer such that $\|x - z\|^2 + \|x - z'\|^2 < M/4$ for all $z, z' \in C$, and choose $y_0 \in C$. Construct an increasing binary sequence $\{\lambda_n\}_{n=0}^{\infty}$ with $\lambda_0 = 0$ and a sequence $\{y_n\}_{n=0}^{\infty}$ in C such that

$$\begin{aligned}
\lambda_n = 0 \quad &\Rightarrow \quad \|x - y_n\|^2 < \|x - y_{n-1}\|^2 - \epsilon^4/(16M), \\
\lambda_n = 1 \quad &\Rightarrow \quad \|x - y_n\| < \|x - z\| + \epsilon \text{ for all } z \in C.
\end{aligned}$$

Assume that we have constructed $\lambda_0, \ldots, \lambda_n$ and $y_0, \ldots, y_n$. If $\lambda_n = 1$, set

$$\lambda_{n+1} := 1 \quad \text{and} \quad y_{n+1} := y_n.$$

Otherwise, applying Lemma 3, if there exists $y' \in C$ such that $\|x - y'\|^2 < \|x - y_n\|^2 - \epsilon^4/(16M)$, set $\lambda_{n+1} := 0$ and $y_{n+1} := y'$; if $\|x - y_n\| < \|x - z\| + \epsilon$ for all $z \in C$, then set $\lambda_{n+1} := 1$ and $y_{n+1} := y_n$; or else $M/4 < \|x - y_n\|^2 + \|x - z'\|^2 < M/4$ for some $z' \in C$, a contradiction.

Choose a positive integer N such that $\|x - y_0\|^2 < N \cdot \epsilon^4/(16M)$. If $\lambda_N = 0$, then

$$0 \le \|x - y_n\|^2 < \|x - y_0\|^2 - N \cdot \frac{\epsilon^4}{16M} < 0,$$

a contradiction. Therefore $\lambda_N = 1$. □

Corollary 5. *Let C be an inhabited, bounded, convex subset of an inner product space X. Then C is located if and only if* $\sup\{\mathrm{Re}\langle x, y\rangle : y \in C\}$ *exists for each $x \in X$.*

Although Theorem 7 of [10] shows that boundedness is necessary in Theorem 2 and hence Corollary 5, we can remove boundedness from Theorem 4 for Hilbert spaces.

Theorem 6. *Let C be an inhabited, convex subset C of a Hilbert space H such that* $\sup\{\mathrm{Re}\langle x, y\rangle : y \in C\}$ *exists for each $x \in H$. Then C is located.*

Proof. Let $x \in H$, $0 < \epsilon < 1$, and choose $y_0 \in C$. If $\|x - y_0\| < \epsilon$, then $\|x - y_0\| < \|x - z\| + \epsilon$ for all $z \in C$. Hence we may assume that $0 < \|x - y_0\|$. Construct a ternary sequence $\{\lambda_n\}_{n=0}^{\infty}$ with $\lambda_0 = 0$ and a sequence $\{y_n\}_{n=0}^{\infty}$ in C such that for each $n \geq 1$, $\|x - y_n\| \leq \|x - y_{n-1}\|$ and

$$\begin{aligned} \lambda_n = -1 &\quad\Rightarrow\quad \|x - y_n\|^2 < \|x - y_{n-1}\|^2 - \epsilon^4/(16n), \\ \lambda_n = 0 &\quad\Rightarrow\quad n/4 < \|x - y_n\|^2 + \|x - z'\|^2 \text{ for some } z' \in C, \\ \lambda_n = 1 &\quad\Rightarrow\quad \|x - y_n\| < \|x - z\| + \epsilon \text{ for all } z \in C. \end{aligned}$$

Assume that we have constructed $\lambda_0, \dots, \lambda_n$ and $y_0, \dots, y_n$. If $\lambda_n = 1$, set

$$\lambda_{n+1} := 1 \quad \text{and} \quad y_{n+1} := y_n.$$

Otherwise, applying Lemma 3, if there exists $y' \in C$ such that $\|x - y'\|^2 < \|x - y_n\|^2 - \epsilon^4/(16n)$ set $\lambda_{n+1} := -1$ and $y_{n+1} := y'$; if $n/4 < \|x - y_n\|^2 + \|x - z'\|^2$ for some $z' \in C$, then, set $\lambda_{n+1} := 0$ and $y_{n+1} := y_n$; or if $\|x - y_n\| < \|x - z\| + \epsilon$ for all $z \in C$, set $\lambda_{n+1} := 1$ and $y_{n+1} := y_n$.

Let $\{N_k\}_{k=1}^{\infty}$ be a strictly increasing sequence of positive integers such that

$$\begin{aligned} k &< \frac{N_k}{4} - \|x - y_0\|^2, \\ \|x - y_0\|^2 &< \sum_{n=N_k+1}^{N_{k+1}} \frac{\epsilon^4}{16n}. \end{aligned}$$

Then for each k there exists n_k with $N_k < n_k \leq N_{k+1}$ such that $\lambda_{n_k} = 0$ or $\lambda_{n_k} = 1$: in fact, if $\lambda_n = -1$ for all n with $N_k < n \leq N_{k+1}$, then

$$0 \leq \|x - y_{N_{k+1}}\|^2 < \|x - y_{N_k}\|^2 - \sum_{n=N_k+1}^{N_{k+1}} \frac{\epsilon^4}{16n} \leq \|x - y_0\|^2 - \sum_{n=N_k+1}^{N_{k+1}} \frac{\epsilon^4}{16n} < 0,$$

a contradiction. Define a sequence $\{w_k\}_{k=1}^{\infty}$ as follows: if $\lambda_{n_k} = 0$, choose $z' \in C$ with $n_k/4 < \|x - y_{n_k}\|^2 + \|x - z'\|^2$ and set $w_k := x - z'$; if $\lambda_{n_k} = 1$, set $w_k := n_k(x - y_0)/\|x - y_0\|$. Then

$$\begin{aligned} \|w_k\|^2 &\geq \min\{n_k^2, n_k/4 - \|x - y_{n_k}\|^2\} \geq \frac{n_k}{4} - \|x - y_0\|^2 \\ &> \frac{N_k}{4} - \|x - y_0\|^2 > k, \end{aligned}$$

and hence $\|w_k\| \to \infty$ as $k \to \infty$. Applying the uniform boundedness theorem to the bounded linear functionals $\xi \mapsto \langle w_n, \xi \rangle$ on H, we can find $\xi_0 \in H$ such that the sequence $\{|\langle w_k, \xi_0 \rangle|\}_{k=1}^{\infty}$ is unbounded. Since C is weakly totally bounded, there exists a positive integer M such that $|\langle x - z, \xi_0 \rangle| < M$ for all $z \in C$. Choose K such that $M < |\langle w_K, \xi_0 \rangle|$. If $\lambda_{n_K} = 0$, then there exists $z' \in C$ such that $w_K = x - z'$, and hence

$$M < |\langle w_K, \xi_0 \rangle| = |\langle x - z', \xi_0 \rangle| < M,$$

a contradiction. Therefore $\lambda_{n_K} = 1$. □

Corollary 7. *The image of the unit ball under an operator on a Hilbert space having an adjoint is located.*

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