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THE MIXED HODGE STRUCTURE ON THE FUNDAMENTAL GROUP OF A PUNCTURED RIEMANN SURFACE

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ABSTRACT. Given a compact Riemann surface $\bar{X}$ of genus g and distinct points p and q on $\bar{X}$, we consider the non-compact Riemann surface $X := \bar{X} \setminus \{q\}$ with basepoint $p \in X$. The extension of mixed Hodge structures associated to the first two steps of $\pi_1(X,p)$ is studied. We show that it determines the element $(2g\,q - 2\,p - K)$ in $\mathrm{Pic}^0(\bar{X})$, where K represents the canonical divisor of $\bar{X}$ as well as the corresponding extension associated to $\pi_1(\bar{X},p)$. Finally, we deduce a pointed Torelli theorem for punctured Riemann surfaces.

INTRODUCTION

Let q be a point in a compact Riemann surface $\bar{X}$ of genus g. In this paper we want to study the complement of q in $\bar{X}$, i.e. $X := \bar{X} \setminus \{q\}$ with a basepoint $p \in X$. We refer to this situation as a *punctured Riemann surface X with puncture q and basepoint p.*

For the fundamental group $\pi_1(\bar{X},p)$ of the compact Riemann surface $(\bar{X},p)$, Hain and Pulte ([Hai87b], [Pul88]) proved that the extension of mixed Hodge structures associated to the quotient of its group ring by W_{-3} determines the base point (see Theorem 2.1). From this result and from the classical Torelli theorem they derived a *pointed Torelli theorem* (see Theorem 2.3) as a corollary.

For the fundamental group $\pi_1(\bar{X} \setminus \{q\}, p)$ of the punctured Riemann surface $(\bar{X} \setminus \{q\}, p)$, the corresponding extension of mixed Hodge structures w_{pq}, i.e. the extension associated to the quotient of its group ring by W_{-3}, is one dimension bigger than in the compact case. We show that it determines the element

$$(2g\,q - 2\,p - K) \text{ in } \mathrm{Pic}^0(\bar{X}),$$

where K represents the canonical divisor, and the corresponding extension associated to $\pi_1(\bar{X},q)$ (see Theorem 1.2). This may have implications on possible normal functions on the moduli space of complex projective curves (cf. [HL97], 7.4).

Finally, we prove that this extension w_{pq} determines both, the basepoint p and the puncture q. This, together with the pointed Torelli theorem of Hain and Pulte yields a *punctured pointed Torelli theorem* (see Theorem 2.8).

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1. Extensions and the theta divisor

For the definition of iterated integrals and of the mixed Hodge structure (MHS) on the fundamental group we refer to the introductory article [Hai87b].

Let $\bar{X}$ be a compact Riemann surface of genus g and let q be a point on $\bar{X}$. We consider the pointed space (X,p), where $X := \bar{X} \setminus \{q\}$ and p is a basepoint on X. Denote by $J \subset \mathbb{Z}\pi_1(X,p)$ and $\bar{J} \subset \mathbb{Z}\pi_1(\bar{X},p)$ the augmentation ideals in the group rings of the respective fundamental groups. Note that $J/J^2 = H_1(X)$ resp. $\bar{J}/\bar{J}^2 = H_1(\bar{X})$, and since we remove only a single point from $\bar{X}$, we have that $X \hookrightarrow \bar{X}$ induces an isomorphism of pure Hodge structures between $H_1(X)$ and $H_1(\bar{X})$, both of weight -1. This allows us to identify these two Hodge structures. Similarly, we identify the weight 1 Hodge structures $H^1(X)$ and $H^1(\bar{X})$. We will write just H_1 and H^1. The *MHS on the fundamental group* $\pi_1(X,p)$ resp. $\pi_1(\bar{X},p)$ consists by definition of MHS's on the integral lattices J/J^{s+1} resp. $\bar{J}/\bar{J}^{s+1}$ for $s \geq 2$.

This definition of the MHS's is possible because of Chen's π_1-De Rham-Theorem, telling us that integration of iterated integrals yields isomorphisms

$$\begin{aligned} H^0\bar{B}_s\big(E^\bullet(\bar{X}\log q),\ p\big) &\xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}(J/J^{s+1},\ \mathbb{C}) =: \big(J/J^{s+1}\big)^*_{\mathbb{C}} \qquad \text{resp.} \\ H^0\bar{B}_s\big(E^\bullet(\bar{X}),\ p\big) &\xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}(\bar{J}/\bar{J}^{s+1},\ \mathbb{C}) =: \big(\bar{J}/\bar{J}^{s+1}\big)^*_{\mathbb{C}}. \end{aligned} \tag{1.1}$$

Here $E^\bullet(\bar{X}\log q)$ denotes the differential graded algebra (dga) of C^∞-forms on $X = \bar{X} \setminus \{q\}$ with logarithmic singularities at q and $E^\bullet(\bar{X})$ denotes the dga of smooth complex valued forms on $\bar{X}$. The objects on the left of (1.1) are the complex vector spaces of iterated integrals of length $\leq s$, which are homotopy functionals — considered as functions on the fundamental group. These vector spaces can be described purely algebraically in terms of the augmented dga's $E^\bullet(\bar{X}\log q)$ and $E^\bullet(\bar{X})$. This is part of a general construction, *the reduced bar construction*, whence the elaborate notation (cf. [Che76] or [Hai87a]). Here we identify these different descriptions deliberately.

In the two cases under consideration, the weight filtration $W_\bullet$ is already given on the lattices J/J^{s+1} resp. $\bar{J}/\bar{J}^{s+1}$ by the J-adic filtration, i.e.

$$W_{-l}\ J/J^{s+1} = J^l/J^{s+1} \text{ resp. } W_{-l}\ \bar{J}/\bar{J}^{s+1} = \bar{J}^l/\bar{J}^{s+1} \text{ for } 0 < l \leq s+1.$$

For $l = 1$ and $s = 1$ we recover the pure Hodge structure on homology, i.e. $W_{-1}J/J^2 = J/J^2 = H_1 = \bar{J}/\bar{J}^2 = W_{-1}\bar{J}/\bar{J}^2$. The weights -1 and -2 give rise to two extensions of MHSs w_{pq} and w_p, related by the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & J^2/J^3 \arrow[r] \arrow[d] & J/J^3 \arrow[r] \arrow[d] & J/J^2 \arrow[r] \arrow[d, "="] & 0;\ w_{pq} \\ 0 \arrow[r] & \bar{J}^2/\bar{J}^3 \arrow[r] & \bar{J}/\bar{J}^3 \arrow[r] & \bar{J}/\bar{J}^2 \arrow[r] & 0;\ w_p. \end{tikzcd} \tag{1.2}$$

The multiplication in the group rings defines surjective maps $J/J^2 \otimes J/J^2 \to J^2/J^3$ and $\bar{J}/\bar{J}^2 \otimes \bar{J}/\bar{J}^2 \to \bar{J}^2/\bar{J}^3$ whose dual morphisms are inclusions $(J^2/J^3)^* \hookrightarrow H^1 \otimes H^1$ and $(\bar{J}^2/\bar{J}^3)^* \hookrightarrow H^1 \otimes H^1$. It is well-known (cf. [Hai87b]) that in both cases, the image of the above inclusions coincides with the kernel of the cup-product. Hence the inclusions give isomorphisms $(J^2/J^3)^* \cong H^1 \otimes H^1$ and $(\bar{J}^2/\bar{J}^3)^* \cong K$, where $K := \ker\{\cup : H^1(\bar{X}) \otimes H^1(\bar{X}) \to H^2(\bar{X})\}$. As $\cup$ is a morphism of Hodge structures, K inherits a pure Hodge structure of weight 2 from $H^1 \otimes H^1$. Dualizing the diagram (1.2) yields extensions of MHS's m_{pq} and m_p — dual to w_{pq} and w_p

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1 & \longrightarrow & \left(J/J^3\right)^* & \longrightarrow & H^1 \otimes H^1 & \longrightarrow & 0;\ m_{pq} \\ & & =\big\uparrow & & \big\uparrow & & \big\uparrow & & \\ 0 & \longrightarrow & H^1 & \longrightarrow & \left(\bar{J}/\bar{J}^3\right)^* & \longrightarrow & K & \longrightarrow & 0;\ m_p. \end{array}$$

Since the exact sequence

$$0 \to K_{\mathbb{Z}} \to H^1_{\mathbb{Z}} \otimes H^1_{\mathbb{Z}} \xrightarrow{\cup} H^2(\bar{X}, \mathbb{Z}) \to 0 \tag{1.3}$$

of Hodge structures of weight 2 splits over $\mathbb{Q}$ but not over $\mathbb{Z}$, let us first clarify the nature of the embedding $K \hookrightarrow H^1 \otimes H^1$. Identify $H^2(\bar{X}, \mathbb{Z})$ with $\mathbb{Z}$. There is a bilinear form

$$b : \left(H^1_{\mathbb{Z}} \otimes H^1_{\mathbb{Z}}\right) \times \left(H^1_{\mathbb{Z}} \otimes H^1_{\mathbb{Z}}\right) \longrightarrow \mathbb{Z},$$

given by $b\left((x_1 \otimes x_2), (y_1 \otimes y_2)\right) := (x_1 \cup y_2) \cdot (y_1 \cup x_2)$, which has mixed signature and is nondegenerate. Consider the rank 1 sublattice $Q_{\mathbb{Z}}$ of $H^1_{\mathbb{Z}} \otimes H^1_{\mathbb{Z}}$ orthogonal to the kernel of the cup-product $K_{\mathbb{Z}} \subset H^1_{\mathbb{Z}} \otimes H^1_{\mathbb{Z}}$ with respect to b. The submodule $Q_{\mathbb{Z}}$ projects to $2g\, H^2(\bar{X}, \mathbb{Z})$ under $\cup$.

$Q_{\mathbb{Z}}$ is generated by one element $\mathfrak{X}$ in $H^1_{\mathbb{Z}} \otimes H^1_{\mathbb{Z}}$, which is invariant under complex conjugation. Hence $H^1_{\mathbb{Z}} \otimes H^1_{\mathbb{Z}}$ induces on $Q_{\mathbb{Z}}$ a $\mathbb{Z}$-HS, isomorphic to $H^2(\bar{X}, \mathbb{Z})$ or $\mathbb{Z}(-1)$. Since (1.3) splits over the rationals we have $K_{\mathbb{Q}} \oplus Q_{\mathbb{Q}} = H^1_{\mathbb{Q}} \otimes H^1_{\mathbb{Q}}$.

Note that m_p, the restriction of m_{pq} to K, is the extension associated to $\pi_1(\bar{X}, p)$, which is used in the pointed Torelli theorem of Hain and Pulte (Theorem 2.3).

Definition 1.1. Define $k_{pq} = [0 \to H^1 \to E_{pq} \to Q \to 0] \in \mathrm{Ext}_{\mathrm{MHS}}\left(Q; H^1\right)$ to be the restriction of the extension m_{pq} to an extension of Q by H^1.

Let $\Psi : \mathrm{Ext}_{\mathrm{MHS}}\left(Q; H^1\right) \xrightarrow{\cong} \mathrm{Pic}^0(\bar{X})$ be the natural isomorphism (see [Car80]). Then the main theorem of this paper is

Theorem 1.2. *In* $\mathrm{Pic}^0(\bar{X})$ *we have* $\Psi(k_{pq}) = (2g\, q - 2\, p - K)$.

1.1. **Riemann's constant.** Let $u : \mathrm{Pic}^0(\bar{X}) \to \mathrm{Jac}(\bar{X})$ be the Abel-Jacobi map and define the divisor

$$W_{p,g-1} := \left\{ \sum_{j=1}^{g-1} u(q_j - p) \;\middle|\; \sum_{j=1}^{g-1} q_j \in \bar{X}^{(g-1)} \right\}.$$

Denote the *theta divisor* on $\mathrm{Jac}(\bar{X})$ by Θ and *Riemann's constant* by $\kappa_p \in \mathrm{Jac}(\bar{X})$, such that Riemann's classical theorem[1] reads: $\Theta = W_{p,g-1} + \kappa_p$.

Using the Riemann-Roch theorem one can prove that Riemann's constant κ_p is related to the canonical divisor by the fact that for any divisor K of a holomorphic 1-form holds $u\big((2g-2)p - K\big) = 2\kappa_p$ and that the canonical divisor is characterized by this equation (for a proof we refer to [GH78], p. 340). Theorem 1.2 is then a consequence of the following theorem, whose proof will be given in the sequel.

Theorem 1.3. *In the Jacobian* $\mathrm{Jac}(\bar{X})$ *we have* $u\big(\Psi(k_{pq}) + 2g(p - q)\big) = 2\kappa_p$.

[1] Proofs of this theorem can be found in [Rie92] (VI, 22., pp. 132-136; XI, pp. 213-224) or [Lan02]. For proofs in modern language we refer to [Lew64], [Mum83] (Theorem 3.1, pp. 149-151) or to [GH78]. In the theory of θ-functions it is more convenient to define $\varkappa_p$ to be an element of $\mathbb{C}^g$ like in [Rie92], [Lan02], [Lew64], [Fay73] (here Riemann's constant is defined as $-\varkappa_p$) and [Mum83].

The rest of this section is devoted to the proof of Theorem 1.3. First we interpret the right-hand side of the equation by means of an expression for κ_p in terms of iterated integrals, as it was already known to Riemann.

To present this formula we need some more notation. Denote by $\gamma_1, \dots, \gamma_{2g}$ and δ (representing a small loop around q) a system of elements in $\pi_1(X,p)$ having the property, that the fundamental group $\pi_1(X,p)$ is the quotient of the free group $F\langle\gamma_1, \dots, \gamma_{2g}, \delta\rangle$ generated by the γ_i and δ subject to the commutator relation

$$[\gamma_1,\ \gamma_{g+1}]\cdots[\gamma_g,\ \gamma_{2g}] = \delta. \tag{1.4}$$

Let $dz_1, \dots, dz_g$ be a basis of holomorphic 1-forms on $\bar{X}$, such that $\int_{\gamma_\nu} dz_i = \delta_{i\nu}$, i.e. the period matrix can be written $\Omega = (\omega_{i\mu})_{\substack{i=1,\dots,g \\ \mu=1,\dots,2g}} = (\Omega_1, \Omega_2) = (I, Z)$. By Riemann's bilinear relations, Z is a symmetric $g \times g$-matrix with positive definite imaginary part. Having made these choices we may represent the Jacobian of $\bar{X}$ as $\mathrm{Jac}(\bar{X}) := \mathbb{C}^g/\Omega\mathbb{Z}^{2g}$. The following expresssion of κ_p in terms of iterated integrals

$$\kappa_p = \left[-\sum_{\nu=1}^{g} \int_{\gamma_\nu} dz_i dz_\nu + \frac{1}{2}\int_{\gamma_{g+i}} dz_i\right]_{i=1,\dots,g} \in \mathrm{Jac}(\bar{X}) \tag{1.5}$$

was known to Bernhard Riemann in 1865 (see [Rie92], p. 213, or [Fay73]).

1.2. **Extension data.** According to [Car80] we need two things for the computation of the extension data $k_{pq} \in \mathrm{Ext}_{\mathrm{MHS}}(Q, H^1)$: a Hodge filtration preserving section $s_F : (Q, F^\bullet) \to (E_{pq}, F^\bullet)$ and an integral retraction $r_\mathbb{Z} : (E_{pq})_\mathbb{Z} \to H^1_\mathbb{Z}$.

Let $dx_1, \dots, dx_{2g}$ be the real harmonic 1-forms such that $\int_{\gamma_j} dx_i = \delta_{ij}$. Then a generator $\mathfrak{X}$ of $Q_\mathbb{Z}$ is given by $\mathfrak{X} := \sum_{\nu=1}^{g} ([dx_\nu] \otimes [dx_{g+\nu}] - [dx_{g+\nu}] \otimes [dx_\nu])$. Riemann's first bilinear relation tells us that $\mathfrak{X} \in F^1(H^1_\mathbb{C} \otimes H^1_\mathbb{C})$. To be more precise, we can write

$$\sum\nolimits_{\nu=1}^{g} (dx_\nu \otimes dx_{g+\nu} - dx_{g+\nu} \otimes dx_\nu) = \sum\nolimits_{j,k=1}^{g} (a_{jk} dz_j \otimes d\bar{z}_k + \bar{a}_{jk} d\bar{z}_j \otimes dz_k)$$

with $A = (a_{jk})_{jk} = \left(\bar{\Omega}_2\Omega_1^t - \bar{\Omega}_1\Omega_2^t\right)^{-1} = (\bar{Z} - Z)^{-1}$. Observe: $A^t = -\bar{A}$.

Define $\wedge\mathfrak{X} := \sum_{\nu=1}^{g} (dx_\nu \wedge dx_{g+\nu} - dx_{g+\nu} \wedge dx_\nu) \in F^1E^2(\bar{X})$. The strictness of the differential with respect to the Hodge filtration on $E^\bullet(\bar{X}\log q)$ implies that there is a $\mu_q \in F^1E^1(\bar{X}\log q)$ such that $\wedge\mathfrak{X} + d\mu_q = 0$. This condition implies that the iterated integral $\int \mathfrak{X} + \mu_q := \int \sum_{\nu=1}^{g} (dx_\nu dx_{g+\nu} - dx_{g+\nu} dx_\nu) + \mu_q$ is a homotopy functional.

A Hodge fitration preserving section s_F is then defined by $s_F(\mathfrak{X}) = \int \mathfrak{X} + \mu_q$ and an integral retraction $r_\mathbb{Z}$ is given by the map, which sends an iterated integral $\int I$ of length ≤ 2 with values in $\mathbb{Z}$ to $r_\mathbb{Z}(\int I) := \sum_{j=1}^{2g} (\int_{\gamma_j} I)[dx_j]$. Again a standard computation shows

$$\boxed{u \circ \Psi(k_{pq}) = \left(\sum_{\nu=1}^{g} (\int_{\gamma_\nu} dz_i \int_{\gamma_{g+\nu}} \mathfrak{X} + \mu_q - \int_{\gamma_{g+\nu}} dz_i \int_{\gamma_\nu} \mathfrak{X} + \mu_q)\right)_{i=1,\dots,g} \in \mathrm{Jac}\,\bar{X}.}$$

1.3. **A higher reciprocity law.** Generally for functions $F, G : \pi_1(X,p) \to \mathbb{C}$ we introduce $\Pi(F; G) := \sum_{\nu=1}^{g} \left(F(\gamma_\nu)G(\gamma_{g+\nu}) - F(\gamma_{g+\nu})G(\gamma_\nu)\right)$. For instance, Riemann's classical period relation reads $\Pi(\int dz_i; \int dz_j) = 0$. With this notation we can state a higher reciprocity law.

Theorem 1.4. *For any holomorphic 1-form ω on $\bar{X}$ we have modulo periods of ω*

$$\sum_{\nu=1}^{g}\left\{\int_{\gamma_\nu}\omega\int_{\gamma_{g+\nu}}\mathfrak{X}+\mu_q-\int_{\gamma_{g+\nu}}\omega\int_{\gamma_\nu}\mathfrak{X}+\mu_q\right\}\equiv 2g\int_p^q\omega$$
$$+\sum_{j,k=1}^{g}a_{jk}\Big\{\Pi(\int\omega;\int dz_j\int d\bar{z}_k)+2\,\Pi(\int\omega\int d\bar{z}_k;\int dz_j)$$
$$-2\,\Pi(\int\omega dz_j;\int d\bar{z}_k)\Big\}.$$

1.3.1. *Observation.* The proof of the *higher reciprocity law* in Theorem 1.4 and also later the proof of the *higher period relation* of Theorem 1.6 are direct generalizations of Riemann's bilinear relations as they are proved in [Che77] or [Gun69]. We use the following procedure.

Let $c_i := (\gamma_i - 1)$ and $d := (\delta - 1)$ denote the elements in J corresponding to γ_i and δ in $\pi_1(X,p)$. If we interpret relation (1.4) in $\mathbb{Z}\pi_1(X,p)$ modulo J^4, we obtain

$$\begin{aligned}\textstyle\sum_{\nu=1}^{g}\{&c_\nu c_{g+\nu}-c_{g+\nu}c_\nu+(c_{g+\nu}c_\nu c_{g+\nu}-c_\nu c_{g+\nu}c_\nu)\\&-(c_\nu c_{g+\nu}c_{g+\nu}-c_{g+\nu}c_\nu c_\nu)\}\equiv d \mod J^4.\end{aligned}\tag{1.6}$$

When the linear extension of a homotopy functional $F:\pi_1(X,p)\to\mathbb{C}$ to $\mathbb{Z}\pi_1(X,p)$ satisfies $F(J^4)=0$, then it has to respect relation (1.6). For instance iterated integrals of length ≤ 3, which are homotopy functionals, are examples of such F.

Remark 1.5. Also in [PY96] the above described procedure is employed to derive *higher period relations for iterated integrals*. In subsection 1.4 we will apply the method to one specific iterated integral. Since the homotopy functionals in [PY96] do not take the polarization or likewise a puncture into account, there are no *higher reciprocity laws for iterated integrals*.

Proof of Theorem 1.4. Use the fact that for any closed path α,

$$\int_\alpha d\bar{z}_j dz_k+\int_\alpha dz_k d\bar{z}_j=\int_\alpha d\bar{z}_j\int_\alpha dz_k$$

to prove that the left-hand side of the equation in Theorem 1.4 equals

$$\Pi(\int\omega;\int\sum_{j,k=1}^{g}2a_{jk}dz_j d\bar{z}_k+\mu_q)-\Pi(\int\omega;\sum_{j,k=1}^{g}a_{jk}\int dz_j\int d\bar{z}_k).$$

Note that $\int I := \int\sum_{j,k=1}^{g}2a_{jk}\omega dz_j d\bar{z}_k+\omega\mu_q$ is a homotopy functional, so its values on both sides of (1.6) coincide. Recall that for 1-forms φ, ψ, χ and closed paths α, β with $a=(\alpha-1)$, $b=(\beta-1)$ and $ab=(\alpha\beta-\alpha-\beta+1)$ holds: $\int_{ab}\varphi\psi\chi=\int_a\varphi\int_b\psi\chi+\int_a\varphi\psi\int_b\chi$. Using this rule, a direct computation shows that the value of $\int I$ on the left-hand side of relation (1.6) takes the value:

$$\Pi(\int\omega;\int I)+\sum_{j,k=1}^{g}2a_{jk}\Big\{\Pi(\int\omega dz_j;\int d\bar{z}_k)$$
$$-\Pi(\int\omega\int d\bar{z}_k;\int dz_j)-\Pi(\int\omega;\int dz_j\int d\bar{z}_k)\Big\}.$$

According to our observation 1.3.1 this has to be equal to the value of the homotopy functional $\int I$ applied to the right-hand side of (1.6). We compute this value as follows. From $\wedge\mathfrak{X} + d\mu_q = 0$ we can determine the shape of μ_q. Using Stokes' theorem, a standard argument shows that there is a simply connected holomorphic coordinate plot (U, z) on $\bar{X}$ containing q and all of a representing path for $\delta \in \pi_1(X, p)$ such that on U we may write $\mu_q = \frac{2g}{2\pi i} \frac{dz}{z} + \varphi$, where φ is a smooth (non-closed) 1-form in $E^1(U)$. Since this representative of δ is 0-homotopic in U, the homotopy functional $\sum_{j,k=1}^{g} 2a_{jk} \int \omega dz_j d\bar{z}_k + \omega\varphi$ vanishes on it. Consequently $\int_\delta I = \int_\delta \omega(\frac{2g}{2\pi i}\frac{dz}{z}) = 2g \int_p^q \omega$. Putting all ingredients together provides the proof. $\square$

1.4. **A higher period relation.** Recall that our period matrix Ω is of the form (I, Z) where Z is symmetric and has positive imaginary part. Like before set $A = (\bar{Z} - Z)^{-1}$. Define for $i = 1, \dots, g$ the $g \times g$-matrices

$$I_1^i := \left(\int_{c_\nu} dz_i dz_j \right)_{\nu,j} \quad \text{and} \quad I_2^i := \left(\int_{c_{g+\nu}} dz_i dz_j \right)_{\nu,j} \in \mathrm{Mat}(g \times g; \mathbb{C}).$$

Then we define the following two vectors with entries in $\mathrm{Mat}(g \times g; \mathbb{C})$:

$$I_1 = \begin{pmatrix} I_1^1 \\ \vdots \\ I_1^g \end{pmatrix}, \quad I_2 = \begin{pmatrix} I_2^1 \\ \vdots \\ I_2^g \end{pmatrix} \quad \in \mathrm{Mat}\big(g \times 1; \mathrm{Mat}(g \times g)\big).$$

For a matrix M, denote by $\operatorname{tr} M$ the *trace* of M and by $\operatorname{diag} M$ its *diagonal*. Define *the trace* of a vector of matrices to be the vector consisting of the traces of its components. The following theorem is the announced *higher period relation*.

Theorem 1.6. *With the above notation, we have*

$$\begin{aligned}&(2\operatorname{tr}(I_2 A) - 2\operatorname{tr}(I_1 A Z)) + (\operatorname{diag}(ZAZ) - Z\operatorname{diag}(AZ)) \\ &\quad + (\operatorname{diag}(ZA) - Z\operatorname{diag}(A)) + (\operatorname{diag}(AZ) - Z\operatorname{diag}(ZA)) \equiv 0 \mod (I, Z)\mathbb{Z}^{2g}.\end{aligned}$$

Proof. Apply the homotopy functional $\sum_{j,k=1}^{g} a_{jk} \int dz_j\, dz_i\, dz_k$ to (1.6). $\square$

We use this higher period relation to continue our computation of the extension k_{pq}. After Theorem 1.4 it makes sense to speak of k_{pp}; we have

$$\Psi(k_{pq}) = 2g(q - p) + \Psi(k_{pp}).$$

In the above introduced notation, Theorem 1.4 tells us that $u \circ \Psi(k_{pp}) \in \mathbb{C}^g / \Omega\mathbb{Z}^{2g}$ can be written as

$$\begin{aligned}u \circ \Psi(k_{pp}) &= \operatorname{diag}(ZA\bar{Z}) - Z\operatorname{diag}(A) \\ &\qquad + 2\operatorname{diag}(ZA) - 2Z\operatorname{diag}(A\bar{Z}) - 2\operatorname{tr}(I_1 A\bar{Z}) + 2\operatorname{tr}(I_2 A).\end{aligned}$$

Transform this expression such that it only contains (iterated) integrals over *holomorphic* forms. Observe $\operatorname{diag}(ZA\bar{Z}) = \operatorname{diag}(Z(\bar{Z} - Z)^{-1}(\bar{Z} - Z)) + \operatorname{diag}(ZAZ) = \operatorname{diag}(Z) + \operatorname{diag}(ZAZ)$ and similarly $2Z\operatorname{diag}(A\bar{Z}) \equiv 2Z\operatorname{diag}(AZ) \mod (I, Z)\mathbb{Z}^{2g}$ and $2\operatorname{tr}(I_1 A\bar{Z}) = 2\operatorname{tr}(I_1) + 2\operatorname{tr}(I_1 AZ)$. Using these identities we continue

$$\begin{aligned}u \circ \Psi(k_{pp}) &\equiv \operatorname{diag}(Z) + \operatorname{diag}(ZAZ) - Z\operatorname{diag}(A) \\ &+ 2\operatorname{diag}(ZA) - 2Z\operatorname{diag}(AZ) \\ &- 2\operatorname{tr}(I_1) - 2\operatorname{tr}(I_1 AZ) + 2\operatorname{tr}(I_2 A) \mod (I, Z)\mathbb{Z}^{2g}.\end{aligned}$$

Notice: $\operatorname{diag}(ZA) - \operatorname{diag}(AZ) = 0$. When we apply Theorem 1.6, we finally get

$$u \circ \Psi(k_{pp}) \equiv \operatorname{diag} Z - 2\operatorname{tr}(I_1) \mod (I, Z)\mathbb{Z}^{2g}.$$

Writing this out, we find $u \circ \Psi(k_{pp}) \equiv 2\kappa_p \mod (I, Z)\mathbb{Z}^{2g}$ by virtue of formula (1.5). This is the proof of Theorem 1.3.

2. A pointed Torelli theorem for punctured Riemann surfaces

Here we want to show that the extension w_{pq} or respectively m_{pq} determines p and q. Finally, we will combine this with results of Hain and Pulte [Hai87b], [Pul88], which we briefly sketch first.

2.1. **The pointed Torelli theorem.** The pointed Torelli theorem of Hain and Pulte is based on the following.

Theorem 2.1 (Hain, Pulte)**.** *The map from* $\operatorname{Pic}^0 \bar{X}$ *to* $\operatorname{Ext}_{\text{MHS}}(K; H^1)$ *which maps* $(p - p')$ *to* $m_p - m_{p'}$ *is well-defined and injective.*

We write $(\bar{X}, p) \cong (\bar{X}, p')$ if there is an automorphism $\phi : \bar{X} \to \bar{X}$ that maps p to p'. For a point p on $\bar{X}$ we define *the set of alternatives for* p as

$$a_{\bar{X}}(p) := \{p\} \cup \left\{p' \in \bar{X} \mid m_{p'} = -m_p \text{ in } \operatorname{Ext}_{\text{MHS}}(K; H^1) \text{ and } (\bar{X}, p) \ncong (\bar{X}, p')\right\}.$$

The following is a consequence of Theorem 2.1. Let us give a short proof of it.

Corollary 2.2. *For a pointed compact Riemann surface* $(\bar{X}, p)$*, the set* $a_{\bar{X}}(p)$ *consists of at most two points. Up to automorphism of* $\bar{X}$*, there cannot be more than one pair of different points* $\{p, p'\}$ *on* $\bar{X}$ *such that* $a_{\bar{X}}(p) = \{p, p'\} = a_{\bar{X}}(p')$.

Proof. The first assertion is an obvious consequence of 2.1. To prove the second assertion, assume that $\tilde{p}$ and $\tilde{p}'$ is another such pair with $a_{\bar{X}}(\tilde{p}) = \{\tilde{p}, \tilde{p}'\} = a_{\bar{X}}(\tilde{p})$. Then by 2.1, the divisors $p + p' = \tilde{p} + \tilde{p}'$ are linearly equivalent. It follows that either $\{p, p'\} = \{\tilde{p}, \tilde{p}'\}$ or $\bar{X}$ is hyperelliptic and the hyperelliptic involution maps p to p' and q to q', which contradicts the assumptions on p, p' and q, q'. □

Together with the classical Torelli theorem, Hain and Pulte used Theorem 2.1 to prove the following *pointed Torelli theorem*. For a pointed compact Riemann surface $(\bar{Z}, z_0)$ denote by $J_{z_0}(\bar{Z})$ the augmentation ideal in $\mathbb{Z}\pi_1(\bar{Z}, z_0)$.

Theorem 2.3 (Hain, Pulte)**.** *Suppose that* $(\bar{X}, p)$ *and* $(\bar{Y}, r)$ *are two pointed compact Riemann surfaces. If there is a ring homomorphism*

$$\mathbb{Z}\pi_1(\bar{X}, p)\Big/ J_p(\bar{X})^3 \xrightarrow{\cong} \mathbb{Z}\pi_1(\bar{Y}, r)\Big/ J_r(\bar{Y})^3$$

which induces an isomorphism of MHS's, then there is an isomorphism $f : \bar{X} \to \bar{Y}$ *with* $f(p) \in a_{\bar{Y}}(r)$.

Remark 2.4. As far as the author knows, still no example is known of a pointed compact Riemann surface $(\bar{X}, p)$ with $|a_{\bar{X}}(p)| = 2$. M. Pulte [Pul88] has shown that such an $(\bar{X}, p)$ with $|a_{\bar{X}}(p)| = 2$ must have zero harmonic volume. B. Harris [Har83] proved that a generic smooth projective complex curve has nonzero harmonic volume. Moreover, Pulte showed (loc. cit.) that, if there are two points p, p' with $a_{\bar{X}}(p) = \{p, p'\} = a_{\bar{X}}(p')$, then $(g - 1)(p + p') - K = 0 \in \operatorname{Pic}^0 \bar{X}$, where K is the canonical divisor. For pointed hyperelliptic curves $(\bar{X}, p)$ always holds: $a_{\bar{X}}(p) = \{p\}$, since here $m_p = -m_{p'}$ implies $(\bar{X}, p) \cong (\bar{X}, p')$ by the hyperelliptic involution.

2.2. **A punctured pointed Torelli theorem.** The following theorem will follow directly from Lemma 2.9, which we prove at the end of this section.

Theorem 2.5. *For all $p \in \bar{X}$, the map* $\operatorname{Pic}^0 \bar{X} \to \operatorname{Ext}_{\text{MHS}}(H^1 \otimes H^1; H^1)$ *which maps $(q - q')$ to $m_{pq} - m_{pq'}$ is well-defined and injective.*

Let Δ be the diagonal in $\bar{X} \times \bar{X}$. Combining Theorem 2.5 with the results of Hain and Pulte we find

Proposition 2.6. *The map from $(\bar{X} \times \bar{X}) \setminus \Delta$ to* $\operatorname{Ext}_{\text{MHS}}(H^1 \otimes H^1; H^1)$ *given by $(p, q) \mapsto m_{pq}$ is well-defined, extends to the diagonal Δ and is injective.*

Proof of 2.6. Note that the map of complex tori

$$\operatorname{Ext}_{\text{MHS}}(H^1 \otimes H^1; H^1) \to \operatorname{Ext}_{\text{MHS}}(K \oplus Q; H^1)$$

is a covering map, since $\operatorname{Hom}(H^1 \otimes H^1; H^1)_{\mathbb{Z}} \hookrightarrow \operatorname{Hom}(K \oplus Q; H^1)_{\mathbb{Z}}$. Moreover, we have the commutative diagram:

$$\begin{array}{ccc}
(\bar{X} \times \bar{X}) \setminus \Delta & \xrightarrow{\tilde{\varphi}} & \operatorname{Ext}_{\text{MHS}}(H^1 \otimes H^1; H^1) \\
\downarrow & & \downarrow{\scriptstyle \text{covering map}} \\
\bar{X} \times \bar{X} & \xrightarrow{\varphi} & \operatorname{Ext}_{\text{MHS}}(K; H^1) \oplus \operatorname{Pic}^0 \bar{X} \\
(p, q) & \longmapsto & \big(m_p,\, (2g\, q - 2p - K)\big).
\end{array}$$

The map φ is continuous (m_p is — in a coordinate system — an expression of iterated integrals over paths with basepoint p). As the map $\tilde{\varphi}(p, q) = m_{pq}$ is a lifting of φ, we see that $\tilde{\varphi}$ is continuous too. The fact that the map $m_{pq} \mapsto (m_p,\, k_{pq})$ is a covering map tells us moreover that we may extend $\tilde{\varphi}$ to the diagonal Δ. Since the extension m_{pq} determines m_p it determines by Theorem 2.1 of Hain and Pulte also p. By virtue of Theorem 2.5 it determines q. □

Pulling back the intersection form $H_1(\bar{X}, \mathbb{Z}) \otimes H_1(\bar{X}, \mathbb{Z}) \to \mathbb{Z}$ along the natural isomorphism $J\big/J^2 \xrightarrow{\cong} \bar{J}\big/\bar{J}^2$ induces a polarization on $\operatorname{Gr}^W_{-1}(J\big/J^3) = H^1(X)$. We can also put a polarization on $\operatorname{Gr}^W_{-2}(J\big/J^3) = J^2\big/J^3 \cong J\big/J^2 \otimes J\big/J^2 = \operatorname{Gr}^W_{-1}(J\big/J^3) \otimes \operatorname{Gr}^W_{-1}(J\big/J^3)$, by taking the tensor product of the polarized Hodge structure $H^1(X)$ in the category of polarized Hodge structures. In that sense, $J\big/J^3$ becomes a *graded polarized MHS*, i.e. each Gr^W_l is a polarized Hodge structure.

For points p and q on $\bar{X}$ we define

$$A_{\bar{X}}(p, q) := \{(p, q)\} \cup \left\{ (p', q') \in \bar{X} \times \bar{X} \;\middle|\; \begin{array}{l} m_{p'q'} = -m_{pq} \text{ in } \operatorname{Ext}_{\text{MHS}}((H^1)^{\otimes 2}; H^1) \\ \text{and } \big(\bar{X} \setminus \{q\}, p\big) \not\cong \big(\bar{X} \setminus \{q'\}, p'\big) \end{array} \right\}.$$

The following is then a consequence of Proposition 2.6.

Corollary 2.7. *$A_{\bar{X}}(p, q)$ consists of at most two elements.* □

Our results lead to the following *punctured pointed Torelli theorem.*

Theorem 2.8. *Suppose that $(\bar{X} \setminus \{q\}, p)$ and $(\bar{Y} \setminus \{s\}, r)$ are two punctured compact Riemann surfaces with basepoint. If there is a ring isomorphism*

$$\mathbb{Z}\pi_1(\bar{X} \setminus \{q\}, p)\Big/J_p(\bar{X} \setminus \{q\})^3 \xrightarrow{\cong} \mathbb{Z}\pi_1(\bar{Y} \setminus \{s\}, r)\Big/J_r(\bar{Y} \setminus \{s\})^3,$$

which induces an isomorphism of graded polarized MHS's, then there is a biholomorphism $f : \bar{X} \to \bar{Y}$ with $(f(p), f(q)) \in A_{\bar{Y}}(r, s)$.

Proof of 2.8. The proof goes along the lines of the proof of the pointed Torelli theorem in [Pul88] and [Hai87b]. Let $J_{pq} = J_p(\bar{X} \setminus \{q\})$ and $J_{rs} = J_r(\bar{Y} \setminus \{s\})$. We have an isomorphism of MHS's, $\lambda : J_{pq}/J_{pq}^3 \xrightarrow{\cong} J_{rs}/J_{rs}^3$ and in particular, λ induces an isomorphism of polarized Hodge structures

$$\lambda^* : H^1(\bar{Y}) = W_1\left(J_{rs}/J_{rs}^3\right)^* \to W_1\left(J_{pq}/J_{pq}^3\right)^* = H^1(\bar{X}).$$

By the classical Torelli theorem (cf. for instance [Mar63]) we know that there is a biholomorphism $f : \bar{X} \to \bar{Y}$ such that $f^* : H^1(\bar{Y}) \to H^1(\bar{X})$ is $\pm\lambda^*$. Since λ respects the *ring* structure, the λ induced map $\left(J_{rs}^2/J_{rs}^3\right)^* \to \left(J_{pq}^2/J_{pq}^3\right)^*$ is determined by $\lambda^* : H^1(\bar{Y}) \to H^1(\bar{X})$ and hence,

$$f^* : \left(J_{rs}^2/J_{rs}^3\right)^* = H^1(\bar{Y}) \otimes H^1(\bar{Y}) \to H^1(\bar{X}) \otimes H^1(\bar{X}) = \left(J_{pq}^2/J_{pq}^3\right)^*$$

is equal to $\lambda^* \otimes \lambda^*$. Without loss of generality, we may therefore assume that $(\bar{Y} \setminus \{s\}, r) = (\bar{X} \setminus \{q'\}, p')$ for two points p' and q' in $\bar{X}$ and that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1 & \longrightarrow & \left(J_{pq}/J_{pq}^3\right)^* & \longrightarrow & H^1 \otimes H^1 & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \pm id} & & \Big\downarrow{\scriptstyle \lambda^*} & & \Big\downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & H^1 & \longrightarrow & \left(J_{p'q'}/J_{p'q'}^3\right)^* & \longrightarrow & H^1 \otimes H^1 & \longrightarrow & 0. \end{array}$$

It follows that $m_{pq} = \pm m_{p'q'}$. This means that there either is an automorphism $\phi : (\bar{X} \setminus \{q\}, p) \to (\bar{X} \setminus \{q'\}, p')$ or $A_{\bar{X}}(p,q) = \{(p,q);\ (p',q')\} = A_{\bar{X}}(p',q')$. In both cases, the identity map is the map with the desired properties. □

2.3. **A technical lemma.** Theorem 2.5 is a consequence of the following:

Lemma 2.9. *For each element* $\sum_i (q_i - q_i') \in \operatorname{Pic}^0 \bar{X}$*, we have*

$$\sum\nolimits_i (q_i - q_i') = 0 \in \operatorname{Pic}^0 \bar{X} \;\Leftrightarrow\; \sum\nolimits_i (m_{pq_i} - m_{pq_i'}) = 0 \in \operatorname{Ext}_{\text{MHS}}\left((H^1)^{\otimes 2}; H^1\right).$$

Proof. Consider the isomorphism (cf. [Car80]) from $\operatorname{Ext}_{\text{MHS}}\left(H^1 \otimes H^1; H^1\right)$ to

$$\operatorname{Hom}(H^1 \otimes H^1; H^1)_{\mathbb{C}} \Big/ (F^0 \operatorname{Hom}(H^1 \otimes H^1; H^1)_{\mathbb{C}} + \operatorname{Hom}(H^1 \otimes H^1; H^1)_{\mathbb{Z}}).$$

The image of an extension m_{pq} is $[\phi_{pq}]$ for a certain $\phi_{pq} \in \operatorname{Hom}(H^1 \otimes H^1; H^1)_{\mathbb{C}}$, which we now explain. On an element $[\varphi] \otimes [\psi] \in H^1 \otimes H^1$, the homomorphism ϕ_{pq} has the following property. There is an $\eta_q \in F^1E^1(X \log q)$ such that $\varphi \wedge \psi + d\eta_q = 0$ and $\phi_{pq}([\varphi] \otimes [\psi]) = \sum_{j=1}^{2g} (\int_{\gamma_j} \varphi\psi + \mu_q)[dx_j]$. If $[\varphi] \otimes [\psi] \in K$, then η_q can be chosen in $F^1E^1(X)$ and does not depend on q, which shows that $(\phi_{pq} - \phi_{pq'})$ is zero on K. Therefore it is determined by its value on one element of $\left(H^1 \otimes H^1\right) \setminus K$; for instance on $[dx_1] \otimes [dx_{g+1}]$.

Given a divisor $D = \sum_i (q_i - q_i')$ define the homomorphism $\Phi_D := \sum_i \left(\phi_{pq_i} - \phi_{pq_i'}\right)$: $H^1 \otimes H^1 \to H^1$. We will derive a series of equivalences. First, we have:

$$\begin{aligned} & \sum\nolimits_i \left(m_{pq_i} - m_{pq_i'}\right) = 0 \in \operatorname{Ext}_{\text{MHS}}\left(H^1 \otimes H^1; H^1\right) \\ \Leftrightarrow \quad & \Phi_D \in F^0 \operatorname{Hom}\left(H^1 \otimes H^1; H^1\right)_{\mathbb{C}} + \operatorname{Hom}\left(H^1 \otimes H^1; H^1\right)_{\mathbb{Z}}. \end{aligned}$$

Now let $\mathbf{w} \in H^{0,1} \otimes H^{0,1}$ be such that $[dx_1] \otimes [dx_{g+1}] - \mathbf{w} \in F^1(H^1 \otimes H^1) = H^{1,0} \otimes H^1 + H^1 \otimes H^{1,0}$. Note that $H^{0,1} \otimes H^{0,1} \subset K$ and hence $\Phi_D(\mathbf{w}) = 0$. Moreover,

$H^{1,0} \otimes H^{1,0} \subset K$ and $\Phi_D\left(H^{1,0} \otimes H^{1,0}\right) = 0$. Therefore, we may continue the series of equivalences by

$$\Leftrightarrow \Phi_D\left([dx_1] \otimes [dx_{g+1}] - \mathbf{w}\right) \in H^{1,0} + H^1_{\mathbb{Z}} \Leftrightarrow \Phi_D\left([dx_1] \otimes [dx_{g+1}]\right) \in H^{1,0} + H^1_{\mathbb{Z}}.$$

Let $\eta_{q_i} \in F^1E^1(X \log q_i)$ and $\eta_{q_i'} \in F^1E^1(X \log q_i')$ be such that $dx_1 \wedge dx_{g+1} + d\eta_{q_i} = 0$ and $dx_1 \wedge dx_{g+1} + d\eta_{q_i'} = 0$. Note that this implies $\operatorname{Res}_{q_i} \eta_{q_i} = \frac{1}{2\pi i} = \operatorname{Res}_{q_i'} \eta_{q_i'}$. Then a direct computation shows that we may go on:

$$\begin{aligned} &\Leftrightarrow \quad \sum\nolimits_{j=1}^{2g} \sum\nolimits_i \Big(\int_{\gamma_j} \mu_{q_i} - \mu_{q_i'}\Big)[dx_j] \in H^{1,0} + H^1_{\mathbb{Z}} \\ &\Leftrightarrow \quad \left(\Pi\left(\int dz_\nu;\ \int (\mu_{q_i} - \mu_{q_i'})\right)\right)_\nu \equiv 0 \mod \Omega\mathbb{Z}^{2g}. \end{aligned}$$

By the reciprocity law for differentials of the third kind (cf. [GH78]), we find as $(\mu_{q_i} - \mu_{q_i'})$ is meromorphic with simple poles $\Leftrightarrow \sum_i (q_i - q_i') = 0 \in \operatorname{Pic}^0 \bar{X}$. That proves the lemma. □

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