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A NOTE ON p-ADIC NEVANLINNA THEORY

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Abstract. In this paper, we show that the First Main Theorem in p-adic Nevanlinna theory implies the Second Main Theorem without the ramification term.

1. Introduction

Nevanlinna theory is a far-reaching generalization of Picard's theorem. The theory, roughly speaking, is based on two main theorems. The First Main Theorem is just the reformulation of the Poisson-Jensen Formula for meromorphic functions. The Second Main Theorem, however, is an elegant, deep theorem which is the central heart of Nevanlinna theory. Because of this theorem, Nevanlinna theory becomes a rich, non-trivial theory. Recently, Nevanlinna theory has been extended (see [Bo], [Ch1], [Ch2], [CY], [Co1], [Kh], [KQ], and [KT]) to the p-adic meromorphic functions on $\mathbb{C}_p$, the p-adic completion of the algebraic closure of the field of p-adic numbers $\mathbb{Q}_p$. These generalizations have often used a parallel approach to Nevanlinna theory due to H. Cartan. In particular, their methods follow Cartan's original method by using the logarithmic derivative lemma as did Nevanlinna himself. The main purpose of this note is to point out that, unlike the case of standard Nevanlinna theory, the Second Main Theorem in p-adic Nevanlinna theory follows directly from the First Main Theorem. Secondly, we want to indicate that, since we don't require the logarithmic derivative lemma in the proof, the Second Main Theorem with Moving Targets actually shares the same proof with the constant case. This is again different from the standard Nevanlinna theory case. Therefore, from this point of view, results in p-adic Nevanlinna theory are less interesting unless the ramification term is involved. This note is based on the following simple observation: for any radius $r \geq 1$, there is only one i, among $1, 2, \ldots, q$, such that $|f - a_i|_r$ is small (cf. the proof of Theorem 2.1); this is completely alien to the standard Nevanlinna theory situation. Once this observation is made, everything else is transparent. This method also yields the result on p-adic holomorphic curves intersecting hypersurfaces in $\mathbb{P}^n(\mathbb{C}_p)$. See section 3 for details.

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2. Nevanlinna theory for p-adic meromorphic functions

Let p be a prime number, let $\mathbb{Q}_p$ be the field of p-adic numbers, and let $\mathbb{C}_p$ be the p-adic completion of the algebraic closure of $\mathbb{Q}_p$. Let f be a meromorphic function on $\mathbb{C}_p$. The absolute value $|\ |_p$ is normalized so that $|p|_p = p^{-1}$. Let $\mathbf{B}(r)$ be the open disc, which is defined by $\mathbf{B}(r) = \{z \ ||z|_p < r\}$. We use $\mathbf{B}[r]$ to denote the closed disc. We recall the following definitions and results. See [CY] for reference.

p-adic analytic functions. An infinite sum converges in a non-archimedean norm if and only if its general term approaches zero. So an expression of the form

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, \ a_n \in \mathbb{C}_p,$$

is well defined whenever

$$|a_n z^n|_p \to 0, \text{ as } n \to \infty.$$

Such functions are called *p-adic analytic functions.* If h is analytic on $\mathbb{C}_p$, then h is called a p-adic entire function.

Let $h = \sum_{n=0}^{\infty} a_n z^n$ be a non-constant p-adic analytic function on $\mathbf{B}(\rho)$. For $0 < r < \rho$, define

$$|h|_r = \max_{n\geq 0} |a_n|_p r^n.$$

Note that $|\ |_r$ has the following properties:

Proposition 2.1. *Let h_1, h_2 be analytic functions. Then*
(1) $|h_1 + h_2|_r \leq |h_1|_r + |h_2|_r$.
(2) $|h_1 h_2|_r = |h_1|_r |h_2|_r$.

We also define

$$\nu(r, h) = \max_{n\geq 0}\{n \mid |a_n|_p r^n = |h|_r\},$$

$\nu(r, h)$ is called the *central index*.

We have the following Poisson-Jensen Formula (Theorem 3.1 in [CY]).

Poisson-Jensen Formula. *The central index $\nu(r, h)$ increases as $r \to \rho$, and satisfies the formula*

$$\log |h|_r = \log |a_{\nu(0,h)}|_p + \int_0^r \frac{\nu(t, h) - \nu(0, h)}{t} dt + \nu(0, h) \log r,$$

where $\nu(0, h) = \lim_{r\to 0^+} \nu(r, h)$.

We also have the following theorem (Theorem 2.2 in [CY]).

Weierstrass Preparation Theorem. *There exist a unique monic polynomial P of degree $\nu(r, h)$ and a p-adic analytic function g on $\mathbf{B}[r]$ such that $h = gP$, where g does not have any zero inside $\mathbf{B}[r]$, and P has exactly $\nu(r, h)$ zeros, counting multiplicity.*

Let $n_h(r,0)$ denote the number of zeros of h in $\mathbf{B}[r]$, counting multiplicity. Define the *valence function* of h by

$$N_h(r,0) = \int_0^r \frac{n_h(t,0) - n_h(0,0)}{t} dt + n_h(0,0) \log r.$$

The Weierstrass Preparation Theorem shows that

$$n_h(r,0) = \nu(r,h),$$

and the Poisson-Jensen Formula implies that

$$N_h(r,0) = \log |h|_r - \log |a_{n_h(0,0)}|_p.$$

p-adic meromorphic functions. A *p-adic meromorphic function* f on $\mathbf{B}[\rho]$ is the quotient of two p-adic analytic functions h/g such that h, g do not have common zeros on $\mathbf{B}[\rho]$. Therefore, we can uniquely extend the above notions to f, that is,

$$|f|_r = \frac{|h|_r}{|g|_r};$$

$$n_f(r,0) = n_h(r,0),\ n_f(r,\infty) = n_g(r,0);$$

and

$$N_f(r,0) = N_h(r,0),\ N_f(r,\infty) = N_g(r,0).$$

We also define $m_f(r,\infty) = \log^+ |f|_r$, $m_f(r,a) = \log^+(1/|f-a|_r)$, and the *characteristic function*

$$T_f(r) = m_f(r,\infty) + N_f(r,\infty).$$

As in the classical case, the Poisson-Jensen Formula implies the following:

First Main Theorem. *Let f be a non-constant p-adic meromorphic function in $\mathbf{B}[\rho]$. Then for every $a \in \mathbb{C}_p$, and every real number r with $0 < r < \rho$,*

$$m_f(r,a) + N_f(r,a) = T_f(r) + O(1),$$

where $O(1)$ is a constant independent of r.

Our main purpose is to show that the First Main Theorem implies the following Second Main Theorem.

Theorem 2.1 (Second Main Theorem)**.** *Let f be a non-constant p-adic meromorphic function on $\mathbf{B}(\rho)$ with $\rho \geq 1$ and let $a_1, \ldots, a_q$ be distinct numbers in $\mathbb{C}_p$. Then, for any real number $1 \leq r < \rho$, we have*

$$\sum_{j=1}^q m_f(r,a_j) + m_f(r,\infty) \leq T_f(r) + O(1). \tag{2.1}$$

Proof. Let $\delta = \min_{i \neq j} |a_i - a_j|_p$. By definition,

$$\sum_{j=1}^q m_f(r,a_j) + m_f(r,\infty) = \sum_{j=1}^q \log^+ \frac{1}{|f-a_j|_r} + \log^+ |f|_r.$$

Given a real number $0 < r < \rho$, we first consider the case that

$$|f - a_j|_r > \frac{1}{2}\delta,$$

for $1 \le j \le q$. In this case,

$$\sum_{j=1}^{q} m_f(r, a_j) + m_f(r, \infty) \le q \log^+ 2/\delta + m_f(r, \infty) \le T_f(r) + O(1).$$

So the theorem holds. Now let $i, 1 \le i \le q$, be the index among $\{1, 2, \ldots, q\}$ such that

$$|f - a_i|_r \le \frac{1}{2}\delta.$$

Note that here i depends on r. For any $j \ne i$, $1 \le j \le q$, we have

$$\delta \le |a_i - a_j|_p \le |f - a_i|_r + |f - a_j|_r \le \frac{1}{2}\delta + |f - a_j|_r.$$

So

$$|f - a_j|_r \ge \frac{1}{2}\delta.$$

Therefore, for $j \ne i$,

$$m_f(r, a_j) = \log^+ \frac{1}{|f - a_j|_r} \le \log^+ \frac{2}{\delta}.$$

On the other hand, since $|f - a_i|_r \le \frac{1}{2}\delta$,

$$|f|_r \le |f - a_i|_r + |a_i|_p \le \frac{1}{2}\delta + |a_i|_p.$$

So

$$m_f(r, \infty) \le \log^+(\frac{1}{2}\delta + |a_i|_p).$$

Thus

$$\sum_{j=1}^{q} m_f(r, a_j) + m_f(r, \infty) \le (q-1)\log^+ \frac{2}{\delta} + m_f(r, a_i) + \log^+(\frac{1}{2}\delta + |a_i|_p).$$

By the First Main Theorem, $m_f(r, a_j) \le T_f(r, f) + O(1)$. So the theorem is proved.

Remarks. (1) Because we did not use the derivative of f in the proof, Theorem 2.1 holds, by the same proof, if a_j is replaced by slowly growing meromorphic functions. This is different from the case of standard Nevanlinna theory.

(2) In the right side of the inequality (2.1), the coefficient before $T_f(r)$ is 1, comparing to $2 + \epsilon$ in standard Nevanlinna theory. This is sharp from looking at the function $f(z) = 1/z$ or $f(z) = z$. However, if we include the ramification term, then the coefficient before $T_f(r)$ in Theorem 2.1 would be "2", rather than "1". This is clear from looking at polynomials.

Theorem 2.1 implies the following:

Theorem 2.2. *Let f be a p-adic meromorphic function on $\mathbb{C}_p$. If $f(\mathbb{C}_p)$ omits two points in $\mathbb{C}_p \cup \{\infty\}$, then f is constant.*

Remark. Theorem 2.2, as well as the Nevanlinna defect relation, was already contained in Cherry-Ye's paper [CY]. They also provide, in [CY], an answer to the p-adic Nevanlinna inverse problem.

3. Nevanlinna theory for p-adic holomorphic curves

Let $f : \mathbb{C}_p \to \mathbb{P}^n(\mathbb{C}_p)$ be a p-adic holomorphic curve. Let $\tilde{f} = (f_0, \dots, f_n)$ be a reduced representative of f, where $f_0, \dots, f_n$ are p-adic entire functions on $\mathbb{C}_p$ and have no common zeros. The Nevanlinna-Cartan characteristic function $T_f(r)$ is defined by

$$T_f(r) = \log ||f||_r$$

where

$$\|f\|_r = \max\{|f_0|_r, \dots, |f_n|_r\}.$$

Let Q be a homogeneous polynomial (form) in $n+1$ variables with coefficients in $\mathbb{C}_p$. We consider the p-adic entire function $Q \circ f = Q(f_0, \dots, f_n)$ on $\mathbb{C}_p$. Let $n_f(r, Q)$ be the number of zeros of $Q \circ f$ in the disk $\mathbf{B}[r]$, counting multiplicity. Set

$$N_f(r, Q) = \int_0^r \frac{n_f(t, Q) - n_f(0, Q)}{t} dt + n_f(0, Q) \log r;$$

$$m_f(r, Q) = \log \frac{\|f\|_r^d}{|Q \circ f|_r};$$

if $Q \circ f \not\equiv 0$. The Poission-Jensen formula implies:

First Main Theorem. *Let $f : \mathbb{C}_p \to \mathbb{P}^n(\mathbb{C}_p)$ be a p-adic holomorphic curve, and let Q be homogeneous forms of degree d. If $Q(f) \not\equiv 0$, then for every real number r with $0 < r < \infty$*

$$m_f(r, Q) + N_f(r, Q) = dT_f(r) + O(1),$$

where $O(1)$ is a constant independent of r.

Forms $Q_1, \dots, Q_q$, $q > n$, are said to be admissible if no set of $n+1$ forms in this system has common zeros in $\mathbb{C}^{n+1} - \{0\}$.

Theorem 3.1. *Let $f : \mathbb{C}_p \to \mathbb{P}^n(\mathbb{C}_p)$ be a non-constant p-adic holomorphic curve, and let $Q_1, \dots, Q_q$ be an admissible system of homogeneous forms with coefficients in $\mathbb{C}_p$ of degree $d \geq 1$. If $Q_j(f)$ are not identically 0 for $1 \leq j \leq q$, then, for any real number $r \geq 1$,*

$$\sum_{j=1}^{q} m_f(r, Q_j) \leq ndT_f(r) + O(1). \tag{3.1}$$

The proof of Theorem 3.1 is similar to Theorem 2.1. For completeness, we include the proof below.

Proof of Theorem 3.1. Given a real number $0 < r < \infty$. By rearranging indices if necessary, we may assume that

$$|Q_1 \circ f|_r \leq |Q_2 \circ f|_r \leq \cdots \leq |Q_q \circ f|_r. \tag{3.2}$$

Since $Q_1, \dots, Q_q$ are admissible, by Hilbert's Nullstellensatz (cf. [W]) that for any integer $k, 0 \leq k \leq n$, there is an integer $m_k \geq d$ such that

$$x_k^{m_k} = \sum_{i=1}^{n+1} b_{ik}(x_0, \dots, x_n) Q_i(x_0, \dots, x_n),$$

where $b_{ik}, 1 \leq i \leq n+1, 0 \leq k \leq n$, are the homogeneous forms with coefficients in $\mathbb{C}_p$ of degree $m_k - d$. So

$$|f_k|_r^{m_k} \leq C\|f\|_r^{m_k - d} \max\{|Q_1(f)|_r, \dots, |Q_{n+1}(f)|_r\},$$

where C is a positive constant depends only on the coefficients of $b_{ik}, 1 \leq i \leq n+1, 0 \leq k \leq n$, thus depends only on the coefficients of $Q_i, 1 \leq i \leq n+1$. Therefore,

$$\|f\|_r^d \leq C \max\{|Q_1(f)|_r, \dots, |Q_{n+1}(f)|_r\}. \tag{3.3}$$

By (3.2) and (3.3),

$$\prod_{j=1}^{q} \frac{\|f\|_r^d}{|Q_j(f)|_r} \leq \frac{1}{C^{q-n}} \prod_{k=1}^{n} \frac{\|f\|_r^d}{|Q_k(f)|_r}.$$

Therefore

$$\sum_{j=1}^{q} m_f(r, Q_j) \leq \sum_{k=1}^{n} m_f(r, Q_k) + O(1). \tag{3.4}$$

(3.1) then follows from (3.4) and the First Main Theorem. This completes the proof of Theorem 3.1.

Remark. Here again, since we did not use derivatives, the moving target version follows trivially. We note that Hu-Yang [HY2] obtained the moving target version of Theorem 3.1 for $d = 1$ with the coefficient "$2n$" before $T_f(r)$, rather than "n". Moreover, their method, which is parallel to Ru-Stoll's approach in Standard Nevanlinna theory, seems too complicated. For $d > 1$, Hu-Yang, under the assumption that f is algebraically non-degenerate of degree d, obtained (3.1) with coefficient "$(n+1)d$", rather than "dn", before $T_f(r)$ in (3.1).

Theorem 3.1 implies the following theorem:

Theorem 3.2. *Let $D_1, \dots, D_q$ be a collection of admissible hypersurfaces of degree $d \geq 1$ in $\mathbb{P}^n(\mathbb{C}_p)$. Let $f : \mathbb{C}_p \to \mathbb{P}^n(\mathbb{C}_p)$ be a p-adic holomorphic curve. If $f(\mathbb{C}_p)$ omits D_j for $1 \leq j \leq q$ and $q \geq n+1$, then f must be constant.*

Remark. Theorem 3.1 and Theorem 3.2 are sharp by the following example: Let $D_1, \dots, D_{n+1}$ be the coordinate hyperplane and let $f = [1 : 1 : \cdots : 1 : z] : \mathbb{C}_p \to \mathbb{P}^n(\mathbb{C}_p)$. Then $f(\mathbb{C}_p)$ omits the first n coordinate hyperplanes, but f is non-constant.

To compare Theorem 3.1 with results in the Nevanlinna-Cartan theory for holomorphic curves, we recall the following results:

Cartan's Theorem (Compare to Theorem 3.1 with $d = 1$)**.** *Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a non-constant holomorphic curve, and let $H_1, \dots, H_q$ be the hyperplanes in $\mathbb{P}^n(\mathbb{C})$, located in general position. If $f(\mathbb{C})$ is not entirely contained in $H_j, 1 \leq j \leq q$, then, for any $\epsilon > 0$, we have*

$$\sum_{j=1}^{q} m_f(r, H_j). \leq .(2n + \epsilon) T_f(r),$$

where $. \leq .$ means the inequality holds for all $r \notin E$, where E is a set of finite Lebesgue measure.

Theorem (Eremenko-Sodin; compare to Theorem 3.1 with $d > 1$)**.** *Let* $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ *be a non-constant holomorphic curve, and let* $Q_1, \ldots, Q_q$ *be an admissible system of homogeneous forms of degree* d*. If* $Q_j(f)$ *are not identically* 0 *for* $1 \leq j \leq q$*, then, for any* $\epsilon > 0$*, we have*

$$\sum_{j=1}^{q} m_f(r, Q_j) . \leq . (2n + \epsilon) d T_f(r),$$

where $. \leq .$ *means the inequality holds for all* $r \notin E$*, where* E *is a set of finite Lebesgue measure.*

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