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INDECOMPOSABILITY OF CERTAIN LEFSCHETZ FIBRATIONS

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Abstract. We prove that Lefschetz fibrations admitting a section of square -1 cannot be decomposed as fiber sums. In particular, Lefschetz fibrations on symplectic 4-manifolds found by Donaldson are indecomposable. This observation also shows that symplectic Lefschetz fibrations are not necessarily fiber sums of holomorphic ones.

1. Introduction

According to recent results, the study of Lefschetz fibrations is of central importance in the research of topological properties of symplectic 4-manifolds (cf. Theorem 1.6). Since the topology of Kähler surfaces is fairly well-understood, in investigating symplectic manifolds one would like to extend techniques of Kähler geometry to the largest extent. The first examples of Lefschetz fibrations without supporting complex structures were constructed by fiber summing holomorphic Lefschetz fibrations [FS], [OS]. This fact naturally raised the question:

Question 1.1. Is every Lefschetz fibration the fiber sum of holomorphic Lefschetz fibrations?

Examples found by I. Smith (genus-3 fibrations with noncomplex total spaces that do not decompose as fiber sums, and fibrations on blow-ups of noncomplex homotopy $K3$-surfaces) already answered this question in the negative [Sm1], [Sm2]. In the following we will prove an observation leading us to realize that many symplectic Lefschetz fibrations (at least the ones originated from Theorem 1.6) do not decompose as fiber sums. Since there are many of them not supporting any complex structure, we get

Corollary 1.2. *For any $g \in \mathbf{N}$ there are infinitely many Lefschetz fibrations of genus $\geq g$ that are indecomposable (into fiber sum) and do not admit Kähler structure.*

The main result in proving Corollary 1.2 is the following theorem.

Theorem 1.3. *If a Lefschetz fibration $f\colon X \to S^2$ admits a section with self-intersection -1, then X cannot be decomposed as $X = X_1 \#_f X_2$ unless X_1 or X_2 is the trivial fibration $\Sigma_g \times S^2 \to S^2$.*

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The proof of Theorem 1.3 rests on the following theorem, which follows from Taubes' work on Seiberg-Witten invariants of symplectic 4-manifolds together with a vanishing result for those invariants (see, e.g., [Sa]).

Theorem 1.4. *If X is a symplectic 4-manifold with $b_2^+(X) > 1$, then X does not contain any smoothly embedded sphere $\Sigma \subset X$ with $0 \neq [\Sigma] \in H_2(X; \mathbf{Z})$ such that $[\Sigma]^2 \geq 0$.* □

We will also make use of the following corollary of the existence of an irreducible singular fiber in a nontrivial Lefschetz fibration.

Proposition 1.5 ([S2])**.** *For a relatively minimal, nontrivial genus-g Lefschetz fibration $f\colon X \to S^2$ we have $b_1(X) < 2g$, consequently $b_2^+(X \#_f X) = 2b_2^+(X) - 1 + (2g - b_1(X)) > 1$.* □

(In this last proposition we consider the fiber sum $X\#_f X$ using the identification of the boundaries of the two copies of $X - \nu$(regular fiber) given by {complex conjugation} $\times$ $\mathrm{id}_{\text{fiber}}$, i.e., the one corresponding to the double branched cover along a pair of regular fibers [GS].)

Finally, we recall a recent result of Donaldson and Gompf, establishing a relationship between Lefschetz fibrations and symplectic 4-manifolds.

Theorem 1.6 (Donaldson [D], and Gompf [GS])**.** *If (X, ω) is a symplectic 4-manifold, then there exists a positive integer n such that the n-fold blow-up $X \# n\overline{\mathbf{CP}^2}$ admits a Lefschetz fibration $X\#n\overline{\mathbf{CP}^2} \to S^2$. Conversely, a genus-g Lefschetz fibration $f\colon X \to \Sigma$ with fiber-genus $g \geq 2$ over the Riemann surface Σ admits a symplectic structure.* □

2. Fiber sum decompositions

We will make use of the following lemma in our subsequent discussions.

Lemma 2.1. *If the relatively minimal genus-g Lefschetz fibration $f\colon X \to S^2$ (with $g \geq 2$) admits a section $\sigma\colon S^2 \to X$ with $[\sigma(S^2)]^2 \geq 0$, then X is trivial.*

Proof. Suppose that $X \to S^2$ is a nontrivial Lefschetz fibration with section $\sigma = \sigma(S^2)$ of square $\sigma^2 = a \geq 0$. Then $X\#_f X \to S^2$ has a section $\sigma\#\sigma$ of square $2a$; hence $X\#_f X$ contains a sphere of nonnegative square. Notice, however, that (by Theorem 1.6) the manifold $X\#_f X$ is a symplectic 4-manifold, and since X is nontrivial (according to Proposition 1.5), we have $b_2^+(X\#_f X) > 1$. Now Theorem 1.4 completes the proof. □

Now we are ready to prove our main technical result.

Proof of Theorem 1.3. Suppose that X admits a section σ with $\sigma^2 = -1$ and consider a fiber sum decomposition $X = X_1\#_f X_2$ of X. After possibly isotoping the section σ, we find sections σ_1 and σ_2 of $X_1 - \nu$(regular fiber) and $X_2 - \nu$(regular fiber) resp., such that $\sigma_1\#\sigma_1$ and $\sigma_2\#\sigma_2$ give rise to sections of $X_1\#_f X_1$ and $X_2\#_f X_2$ respectively. Since $(\sigma_1\#\sigma_1)^2 + (\sigma_2\#\sigma_2)^2 = 2\sigma^2 = -2$ and both $(\sigma_1\#\sigma_1)^2$ and $(\sigma_2\#\sigma_2)^2$ are even numbers, we conclude that one of them, say $(\sigma_1\#\sigma_1)^2$, is nonnegative. In light of Lemma 2.1, however, this implies that $X_1\#_f X_1 \to S^2$, and so $X_1 \to S^2$ is the trivial fibration $\Sigma_g \times S^2 \to S^2$; hence X does not admit a nontrivial fiber sum decomposition. □

Remarks 2.2. **(a)** In fact, the evenness of the numbers $(\sigma_1\#\sigma_1)^2$ and $(\sigma_2\#\sigma_2)^2$ is not needed in the above argument: if $(\sigma_1\#\sigma_1)^2 + (\sigma_2\#\sigma_2)^2 = -2$, then either one of them is nonnegative (leading to the triviality of X_1 or X_2 as above) or both are equal to -1. It was shown in [S2], however, that the fiber sum $X_i\#_f X_i$ is minimal for any Lefschetz fibration $X_i \to S^2$, consequently cannot contain a smoothly embedded -1-sphere.

(b) The above-mentioned result of [S2] (showing that $X\#_f X$ is always minimal) together with the result of Theorem 1.3 (proving that a fiber sum cannot contain a -1-section) naturally led us to the more general conjecture:

Conjecture 2.3. *For nontrivial Lefschetz fibrations $X_1, X_2 \to S^2$ the fiber sum $X_1\#_f X_2$ is a minimal symplectic 4-manifold; i.e., it does not contain any smoothly embedded sphere of square -1.*

Now we discuss the proof of Corollary 1.2.

Proof of Corollary 1.2. Consider a symplectic 4-manifold X which admits no complex structure. (The existence of such manifolds is established in [G], [GS], [S1], for example.) Now the construction of Donaldson quoted in Theorem 1.6 provides a Lefschetz fibration on $X\#n\overline{\mathbf{CP}^2}$ with at least n (> 0) distinct -1-sections. If $X\#n\overline{\mathbf{CP}^2}$ carries a complex structure, then so does X, in contradiction to our choice. Now Theorem 1.3 shows that the resulting nonholomorphic fibration cannot be decomposed as a fiber sum; hence the proof is complete. By further blow-ups of the same X we get infinitely many examples with the same property. $\square$

A genus-g Lefschetz fibration $X \to S^2$ determines a word $w = D(a_1)\dots D(a_l)$ in the mapping class group $\mathcal{M}_g$ representing $1 \in \mathcal{M}_g$. (Here $D(a_i)$ denotes the right-handed Dehn twist along the simple closed curve a_i — the vanishing cycle of the corresponding singular fiber — in the typical fiber Σ_g of the fibration.) Our previous result implies that if $X \to S^2$ admits a section with square -1, then w cannot be written as a product $w_1 w_2$ where w_i have the same properties as w (i.e., represent $1 \in \mathcal{M}_g$ and are products of right-handed Dehn twists along simple closed curves). We conjecture that, in fact, the converse of the above statement is also true:

Conjecture 2.4. *If the Lefschetz fibration $X \to S^2$ cannot be decomposed as a fiber sum (i.e., the corresponding word w in the mapping class group cannot be written as a nontrivial product $w_1 w_2$ with words representing $1 \in \mathcal{M}_g$), then X admits a section of square -1.*

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