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THE EXPECTED L_p NORM OF RANDOM POLYNOMIALS

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ABSTRACT. The results of this paper concern the expected L_p norm of random polynomials on the boundary of the unit disc (equivalently of random trigonometric polynomials on the interval $[0, 2\pi]$). Specifically, for a random polynomial

$$q_n(\theta) = \sum_0^{n-1} X_k e^{ik\theta}$$

let

$$||q_n||_p^p = \int_0^{2\pi} |q_n(\theta)|^p \, d\theta/(2\pi).$$

Assume the random variables $X_k, k \geq 0$, are independent and identically distributed, have mean 0, variance equal to 1 and, if $p > 2$, a finite p^{th} moment $\mathrm{E}(|X_k|^p)$. Then

$$\frac{\mathrm{E}(||q_n||_p^p)}{n^{p/2}} \to \Gamma(1 + p/2)$$

and

$$\frac{\mathrm{E}(||q_n^{(r)}||_p^p)}{n^{(2r+1)p/2}} \to (2r+1)^{-p/2}\Gamma(1 + p/2)$$

as $n \to \infty$.

In particular if the polynomials in question have coefficients in the set $\{+1, -1\}$ (a much studied class of polynomials), then we can compute the expected L_p norms of the polynomials and their derivatives

$$\frac{\mathrm{E}(||q_n||_p)}{n^{1/2}} \to (\Gamma(1 + p/2))^{1/p}$$

and

$$\frac{\mathrm{E}(||q_n^{(r)}||_p)}{n^{(2r+1)/2}} \to (2r+1)^{-1/2}(\Gamma(1 + p/2))^{1/p}.$$

This complements results of Fielding in the $p := 0$ case, Newman and Byrnes in the $p := 4$ case, and Littlewood et al. in the $p = \infty$ case.

1. INTRODUCTION

There are a number of difficult old conjectures that concern the possible rates of growth of polynomials with all coefficients in the set $\{+1, -1\}$. Since many of these were raised by Littlewood we denote the set of such polynomials by $\mathcal{L}_n$ and

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refer to them as Littlewood polynomials. Specifically

$$\mathcal{L}_n := \left\{ p : p(x) = \sum_{j=0}^{n} a_j x^j \ , \ \ a_j \in \{-1, 1\} \right\}.$$

In [Li-66] Littlewood conjectures that it is possible to find $p_n \in \mathcal{L}_n$ so that

$$C_1 \sqrt{n+1} \le |p_n(z)| \le C_2 \sqrt{n+1}$$

for all complex z of modulus 1. Such polynomials are often called "flat". Because the L_2 norm of a polynomial from $\mathcal{L}_n$ is exactly $\sqrt{n+1}$ the constants must satisfy $C_1 \le 1$ and $C_2 \ge 1$. This is discussed in some detail in problem 19 of Littlewood's delightful monograph [Li-68]. A sequence of polynomials that satisfies just the upper bound is given by the Rudin-Shapiro polynomials. No sequence is known that satisfies the lower bound.

This conjecture is complemented by a conjecture of Erdős [Er-62] that the constant C_2 is bounded away from 1 (independently of n). This is also still open. Although a remarkable result of Kahane [Kah-85, Kah-80] shows that if the polynomials are allowed to have complex coefficients of modulus 1, then "flat" polynomials exist and indeed that it is possible to make C_1 and C_2 asymptotically arbitrarily close to 1. Equally remarkable is a result of Beck [Bec-91] who proves that "flat" polynomials exist from the class of polynomials of degree n whose coefficients are 400th roots of unity.

The relationship between these problems and Barker polynomials is discussed in [Sa-90]. The most famous problem concerning polynomials with coefficients in the set $\{0, -1, +1\}$ is the celebrated, now resolved, Littlewood conjecture [Ko-80]. It asserts that the L_1 norm of a polynomial $\sum_{j=0}^{n} \pm x^{k_j}$ must grow at least like $\log(n)$. Here and in what follows the L_p norms are on the boundary of the unit disk in the complex plane.

In [Sa-54] it is shown that for all but $o(2^n)$ Littlewood polynomials the supremum on the unit disc lies between $c_1\sqrt{n \log n}$ and $c_2\sqrt{n \log n}$. In fact, Halász [Ha-73] shows that $\lim ||q_n/\sqrt{n \log n}||_\infty = 1$ almost surely. See also [An-83].

The expected L_4 to the fourth power of a Littlewood polynomial of degree n is computed by Newman and Byrnes [Ne-90]. They show that

$$\mathrm{E}(||p||_4^4) = 2(n+1)^2 - (n+1)$$

where p is a random element of $\mathcal{L}_n$.

In the L_0 case Fielding [Fi-70] computes the expected norm (which in this case is the Mahler Measure) over the polynomials with *complex* coefficients of modulus 1. He proves that

$$\mathrm{E}(||p||_0) \ge \exp(-\gamma/2)\sqrt{n}(1 + O(n^{-1/2+\delta}))$$

where γ is Euler's constant. See also [Ul-88].

Our principal aim in this paper is to to compute the expected L_p norms of Random Littlewood polynomials. The complete results are stated in the next sections. For random Littlewood polynomials, $q_n \in \mathcal{L}_n$, we have

$$\frac{\mathrm{E}(||q_n||_p)}{n^{1/2}} \to (\Gamma(1 + p/2))^{1/p}$$

and for their derivatives

$$\frac{\mathrm{E}(||q_n^{(r)}||_p)}{n^{(2r+1)/2}} \to (2r+1)^{-1/2}(\Gamma(1+p/2))^{1/p}.$$

From this and the inequality [Bor-95, p. 406]

$$\frac{||q_n'||_p}{n||q_n||_p} \leq 1$$

we can deduce an expected Bernstein Inequality for Littlewood polynomials, namely

$$\mathrm{E}\left(\frac{||q_n'||_p}{n||q_n||_p}\right) \to \frac{1}{\sqrt{3}}\,.$$

This should be compared to interesting results of Nazarov and Queffélec and Saffari [Qe-96] which say that

$$\max_{q_n \in \mathcal{L}_n} \frac{||q_n'||_p}{n||q_n||_p} \to 1$$

for all $p > 1$ except $p = 2$ where the lim sup is $1/\sqrt{3}$.

Of course, because of the monotonicity of the L_p norms it is relevant to rephrase Littlewood's conjecture in other norms. It has been conjectured that

$$||p||_4^4 \geq (7-\delta)n^2/6$$

for $p \in \mathcal{L}_n$ and n sufficiently large. This would be best possible and would imply Erdős' conjecture above. See [Bor-98] for a discussion of this.

Random polynomials have been much looked at, particularly the location of their roots. See for example [Bh-86], [Bri-75] and [Kac-48].

2. Results

Consider a random polynomial

$$q_n(\theta) = \sum_0^{n-1} X_k e^{ik\theta}$$

for $0 \leq \theta \leq 2\pi$. We study the p^{th} power of the L_p norm of q_n; that is,

$$||q_n||_p^p = \int_0^{2\pi} |q_n(\theta)|^p \, d\theta/2\pi.$$

Theorem 1. *Fix $0 < p < \infty$. Assume that the random variables $X_k, k \geq 0$, are independent and identically distributed, have mean 0, variance equal to 1 and, if $p > 2$, a finite p^{th} moment $\mathrm{E}(|X_k|^p)$. Then*

$$\frac{\mathrm{E}(||q_n||_p^p)}{n^{p/2}} \to \Gamma(1+p/2)$$

as $n \to \infty$. If, in addition, $\mathrm{E}(|X_k|^{2p}) < \infty$, then

$$\frac{||q_n||_p}{n^{1/2}} \to \Gamma(1+p/2)^{1/p}$$

in probability and

$$\frac{\mathrm{E}(||q_n||_p)}{n^{1/2}} \to (\Gamma(1+p/2))^{1/p}\,.$$

For ± 1 coefficients randomly chosen the moment conditions are trivially satisfied for all finite p. Numerical confirmation of the principle results based on computations up to degree 24 is presented in Figures 1 and 2. The calculations are courtesy of Lesley Robinson [Ro-97]. Figure 1 shows the average L_3 and L_4 norms (normalized by division by $\sqrt{n+1}$) for the Littlewood polynomials up to degree 24. Figure 2 is a similar graph for L_3^3 and L_4^4 (normalized by division by $(n+1)^{3/2}$ and $(n+1)^2$ respectively).

Proof. In what follows all unlabelled sums run from 0 through $n-1$. Define

$$\sigma_{n,c}^2(\theta) = \sum \cos^2(j\theta)$$

and

$$\sigma_{n,s}^2(\theta) = \sum \sin^2(j\theta)$$

and write

$$a_{k,n}(\theta) = \frac{\cos(k\theta)}{\sigma_{n,c}(\theta)}$$

and, for θ not a multiple of π,

$$b_{k,n}(\theta) = \frac{\sin(k\theta)}{\sigma_{n,s}(\theta)}\,.$$

Then

Lemma 2.1. *There is a constant M, free of n, k and θ such that*

$$|a_{k,n}(\theta)| + |b_{k,n}(\theta)| \le \frac{M}{\sqrt{n}}\,.$$

Postpone the proof of the lemma and consider an arbitrary sequence θ_n. The lemma permits application of the Lindeberg central limit theorem to show that for an arbitrary sequence θ_n

$$C_n(\theta_n) \equiv \frac{\sum_0^{n-1} X_k \cos(k\theta_n)}{\sigma_{n,c}(\theta_n)} = \sum_0^{n-1} a_{k,n}(\theta_n) X_k$$

converges in distribution to standard normal and, provided no θ_n is an integer multiple of π,

$$S_n(\theta_n) \equiv \frac{\sum_0^{n-1} X_k \sin(k\theta_n)}{\sigma_{n,s}(\theta_n)}$$

converges in distribution to standard normal. Moreover, for any fixed θ not an integer multiple of π, the elementary convergences $\sigma_{n,c}^2/n \to 1/2$, $\sigma_{n,s}^2/n \to 1/2$ and $\sum \cos(k\theta)\sin(k\theta)/n \to 0$ show that the covariance matrix of $(C_n(\theta), S_n(\theta))$ converges to the 2-by-2 identity so that $(C_n(\theta), S_n(\theta))$ converges in distribution to (Z_1, Z_2) where the Z_i are independent standard normals. It follows for such θ that $|q_n(\theta)|^2/n$ converges in distribution to $(Z_1^2 + Z_2^2)/2$ which has a standard exponential distribution.

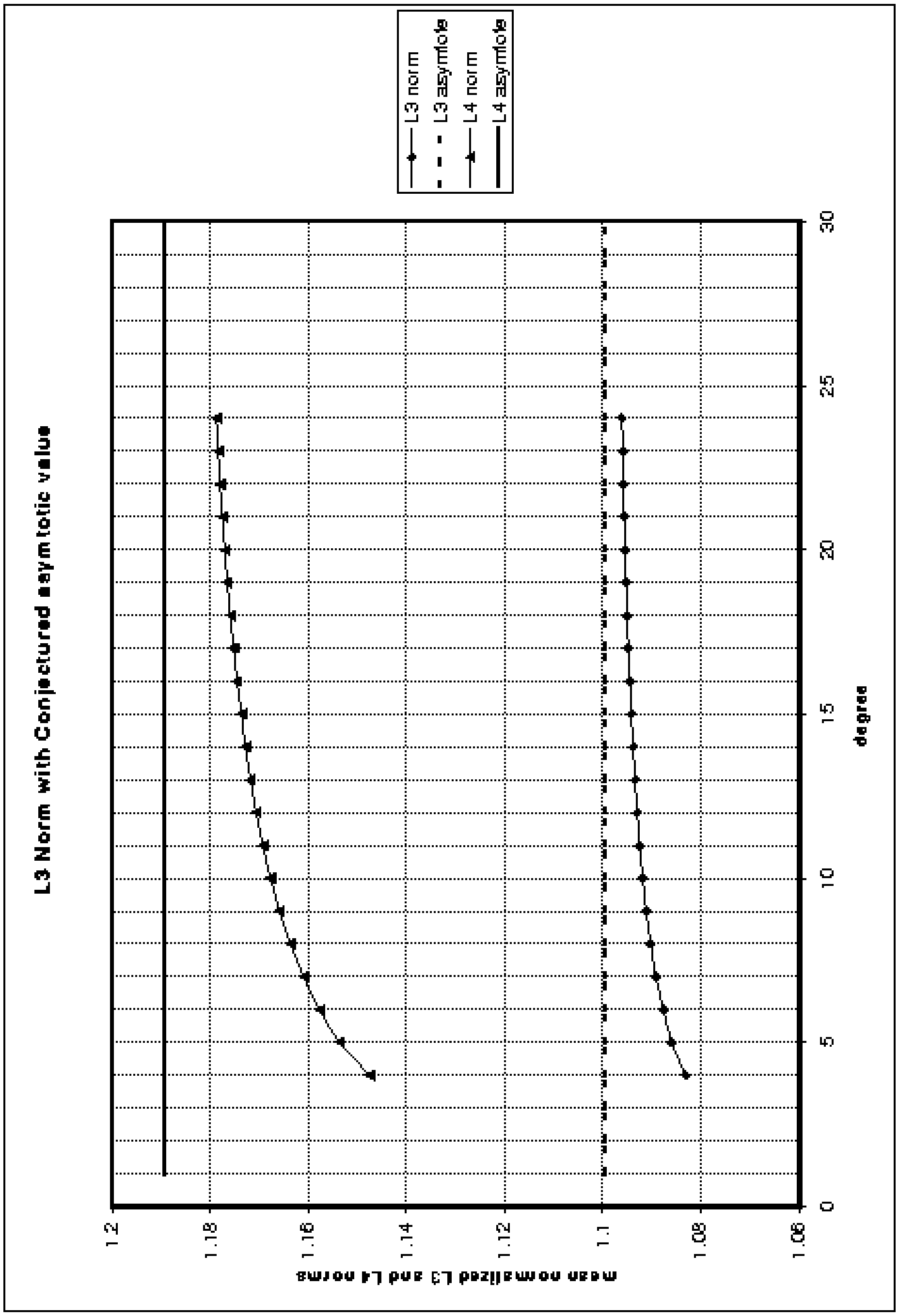
L3 Norm with Conjectured asymtotic value
L3 norm
L3 asymtote
L4 norm
L4 asymtote
degree
mean normalized L3 and L4 norms
0
5
10
15
20
25
30
1.2
1.18
1.16
1.14
1.12
1.1
1.08
1.06

FIGURE 1.

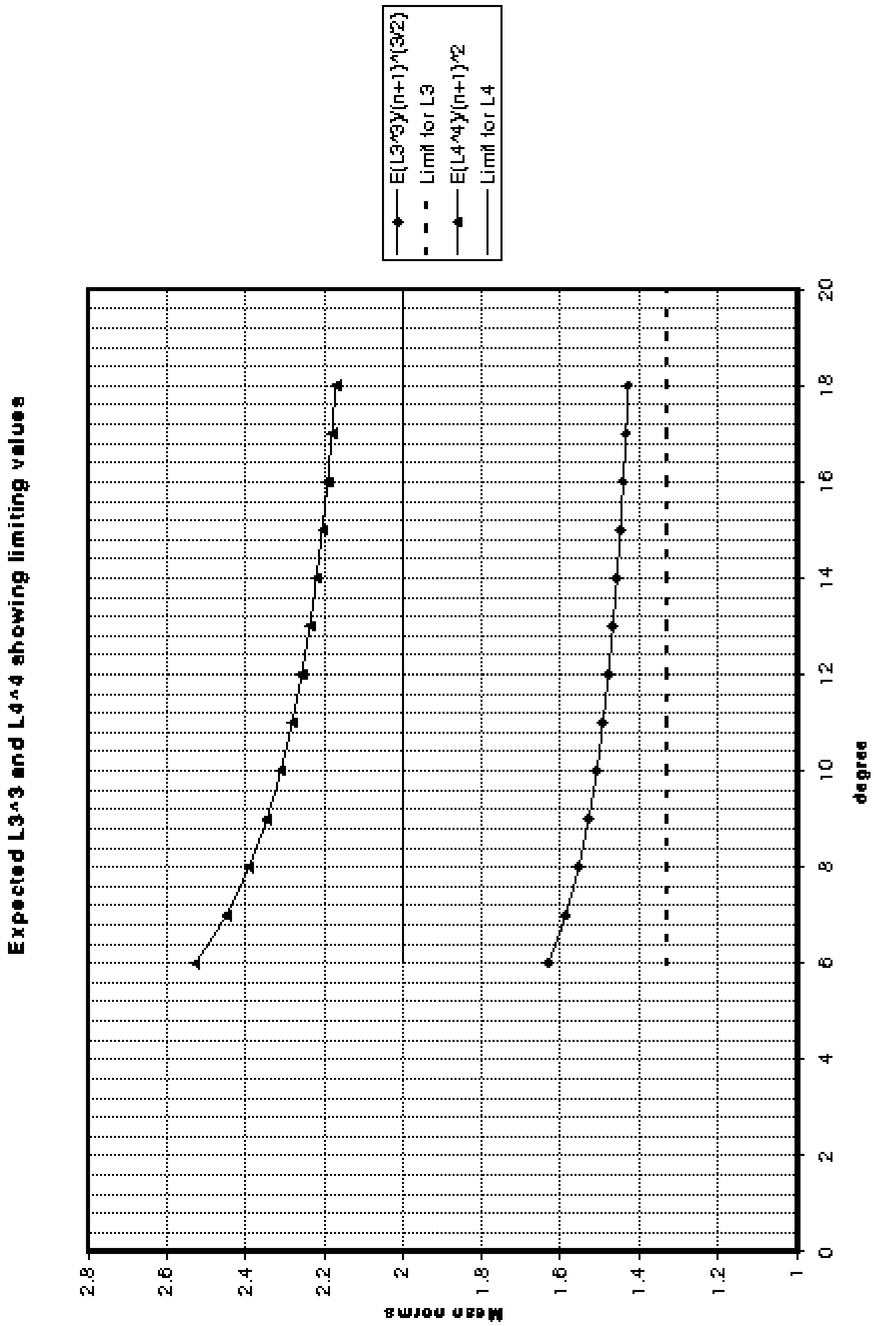
Expected L3^3 and L4^4 showing limiting values
E(L3^3)/(n+1)^(3/2)
Limit for L3
E(L4^4)/(n+1)^2
Limit for L4
Mean norms
degree
2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1
0
2
4
6
8
10
12
14
16
18
20

FIGURE 2.

For $p > 2$, a theorem of Bernstein ([Be-39], [Br-70]) asserts that, in the central limit theorem, p^{th} moments converge to the p^{th} moment of the normal distribution provided an analogue of the Lindeberg condition holds. In particular,

$$\mathrm{E}(|C_n(\theta_n)|^p) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |u|^p \exp(-u^2/2)\, du$$

provided

$$\sum \mathrm{E}(|a_{k,n}|^p |X_k|^p 1(a_{k,n}^2 X_k^2 > \eta)) \to 0$$

for each positive η (and analogously for S_n replacing $a_{k,n}$ by $b_{k,n}$). (For $p \le 2$ we could proceed straight from the central limit theorem.) The quantity in question is bounded by

$$\sum |a_{k,n}|^p \mathrm{E}(|X_0|^p 1(X_0^2 > n\eta/M^2)) \le \sum |a_{k,n}|^p \mathrm{E}(|X_0|^p 1(X_0^2 > n\eta/M^2))\,.$$

But $\sum |a_{k,n}|^p$ is, for $p \ge 2$, bounded by $\sum a_{k,n}^2 = 1$ and X_0 has a finite p^{th} moment so the bound converges to 0.

Now note

$$\begin{aligned}
|q_n(\theta)|^p/n^{p/2} &= \left(\frac{\sigma_{n,c}^2(\theta) C_n^2(\theta) + \sigma_{n,s}^2(\theta) S_n^2(\theta)}{n}\right)^{p/2} \\
&\le 2^{p/2-1}\left[\left(\frac{\sigma_{n,c}^2(\theta)}{n}\right)^{p/2} |C_n(\theta)|^p + \left(\frac{\sigma_{n,s}^2(\theta)}{n}\right)^{p/2} |S_n(\theta)|^p\right] \\
&\le 2^{p/2-1}\left[|C_n(\theta)|^p + |S_n(\theta)|^p\right].
\end{aligned}$$

Putting $g_n(\theta) = \mathrm{E}(|q_n(\theta)|^p/n^{p/2})$ and $h_n(\theta) = 2^{p/2-1}\mathrm{E}(|C_n(\theta)|^p + |S_n(\theta)|^p)$ we see that $0 \le g_n \le h_n$ almost everywhere and that h_n converges uniformly on $(0,\pi) \cup (\pi, 2\pi)$ to $2^{p/2}\mathrm{E}(|Z|^p)$ where Z is standard normal. If we establish that g_n converges almost everywhere to $\Gamma(1+p/2)$, then the theorem will follow by dominated convergence.

To do so we can apply the following result. Suppose U_n and V_n are random variables with U_n converging in distribution to some U, V_n converging in distribution to some V and $0 \le U_n \le V_n$. If $\mathrm{E}(V_n) \to \mathrm{E}(V) < \infty$, then $\mathrm{E}(U_n) \to \mathrm{E}(U)$. (This is the Dominated Convergence Theorem for convergence in distribution.)

Use the result with $U_n = |q_n(\theta)|^p/n^{p/2}$, which converges in distribution, for θ not an integer multiple of π, to $U = ((Z_1^2 + Z_2^2)/2)^{p/2}$, and

$$V_n = 2^{p/2-1}\left[|C_n(\theta_n)|^p + |S_n(\theta_n)|^p\right]$$

which converges in distribution to $V = 2^{p/2-1}(|Z_1|^p + |Z_2|^p)$. Since $\mathrm{E}(U) = \Gamma(1+p/2)$ we are done.

To establish the convergence in probability we compute the variance of $||q_n||_p^p/n^{p/2}$ and show that this converges to zero. It suffices to show that

$$\frac{\mathrm{E}(||q_n||_p^{2p})}{n^p} \to \Gamma^2(1+p/2)\,.$$

However,

$$\frac{\mathrm{E}(||q_n||_p^{2p}))}{n^p} = \int_0^{2\pi}\int_0^{2\pi} g_n^{(*)}(\theta_1,\theta_2)\, d\theta_1\, d\theta_2,$$

where $g_n^{(*)}(\theta_1,\theta_2) = \mathrm{E}(|q_n(\theta_1)q_n(\theta_2)|^p)/n^p$. For fixed θ_1 and θ_2, neither an integer multiple of π, it is easily checked that the variance covariance matrix of $(C_n(\theta_1), S_n(\theta_1), C_n(\theta_2), S_n(\theta_2))$ converges to the 4-by-4 identity matrix. In view of the Lindeberg condition already checked the random vector converges in distribution to the standard normal and $|q_n(\theta_1)q_n(\theta_2)|^p)/n^p$ converges in distribution to $(Z_1^2+Z_2^2)^p(Z_3^2+Z_4^2)^p/2^{2p}$ where $Z_1,\ldots,Z_4$ are independent standard normal. Since

$$g_n^{(*)}(\theta_1,\theta_2) \le h_n^{(*)}(\theta_1,\theta_2) \equiv \left(\frac{\mathrm{E}(|q_n(\theta_1)|^{2p})}{n^p} + \frac{\mathrm{E}(|q_n(\theta_2)|^{2p})}{n^p}\right)/2$$

we may apply the first part of the theorem and the dominated convergence theorem to conclude that $g_n^{(*)}$ converges almost everywhere to $\Gamma^2(1+p/2)$. Moreover we have already checked that $h_n^{(2)}$ converges almost everywhere to $\Gamma(1+p)$ and that

$$\int_0^{2\pi}\int_0^{2\pi} h_n^{(*)}(\theta_1,\theta_2)\,d\theta_1\,d\theta_2/(4\pi^2) \to \Gamma(1+p)\,.$$

Hence

$$\int_0^{2\pi}\int_0^{2\pi} g_n^{(*)}(\theta_1,\theta_2)\,d\theta_1\,d\theta_2/(4\pi^2) \to \Gamma^2(1+p/2)$$

by dominated convergence. This establishes the convergence in probability.

The final statement of the theorem is simply dominated convergence applied via the elementary inequality $||q_n/n^{1/2}||_p \le 1 + ||q_n/n^{1/2}||_p^p$. □

Proof of Lemma. For $\theta \le \pi/(2n)$ and $1 \le j \le n-1$ we have $2j\theta/\pi \le \sin(j\theta) \le j\theta$ whence

$$|b_{k,n}| \le \frac{\pi k\theta}{2\sqrt{\sum_{j=1}^{n-1} j^2\theta^2}} = \frac{\sqrt{6}\pi k}{2\sqrt{(n-1)n(2n-1)}} \le \frac{\sqrt{6}\pi/2}{\sqrt{n}}\,.$$

Moreover, on $[0,\pi/(2n)]$ the function $\cos(k\theta)$ is monotone decreasing for all $k \le n$. Hence

$$\begin{aligned}|a_{k,n}(\theta)| &\le \frac{1}{\sigma_{n,c}(\theta)}\\ &\le \frac{1}{\sigma_{n,c}(\pi/(2n))}\\ &= \sqrt{\frac{2}{n+1}}\\ &\le \frac{\sqrt{2}}{\sqrt{n}}\,.\end{aligned}$$

On the other hand if $\theta > \pi/(2n)$, then

$$\sum\{\cos^2(k\theta) - \sin^2(k\theta)\} = \sin^2(n\theta) + \sin(n\theta)\cos(\theta)\cos(n\theta)/\sin(\theta).$$

Since

$$\frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta) \ge \theta$$

we see that for $\pi/(2n) \le \theta \le \pi/2$ we have

$$\left|\sum\{\cos^2(k\theta) - \sin^2(k\theta)\}\right| \le 1 + 1/\tan(\theta) \le 1 + 2n/\pi \le \delta_0 n$$

for a $\delta_0 < 1$ and all $n \geq 3$. It follows that

$$\sum \cos^2(k\theta) \geq \frac{1-\delta_0}{2} n$$

and

$$\sum \sin^2(k\theta) \geq \frac{1-\delta_0}{2} n$$

from which the lemma follows for $0 \leq \theta \leq \pi/2$. Use easy symmetries of the weights $a_{k,n}$ and $b_{k,n}$ to get all values of θ. □

Similar techniques permit the extension of our main result to the derivative $q_n^{(r)}$ of order r.

Theorem 2. *Fix $0 < p < \infty$. Assume that the random variables $X_k, k \geq 0$, are independent and identically distributed, have mean 0, variance equal to 1 and, if $p > 2$, a finite p^{th} moment $\mathrm{E}(|X_k|^p)$. Then*

$$\frac{\mathrm{E}(||q_n^{(r)}||_p^p)}{n^{(2r+1)p/2}} \to (2r+1)^{-p/2}\Gamma(1+p/2)$$

as $n \to \infty$. If, in addition, $\mathrm{E}(|X_k|^{2p}) < \infty$, then

$$\frac{||q_n^{(r)}||_p}{n^{(2r+1)/2}} \to (2r+1)^{-1/2}(\Gamma(1+p/2))^{1/p}$$

in probability and

$$\frac{\mathrm{E}(||q_n^{(r)}||_p)}{n^{(2r+1)/2}} \to (2r+1)^{-1/2}(\Gamma(1+p/2))^{1/p}.$$

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