A COMPACT SET WITH NONCOMPACT DISC-HULL

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Abstract. The disc-hull of a set is the union of the set and all $H^1$ discs whose boundaries lie in the set. We give an example of a compact set in $\mathbb{C}^2$ whose disc-hull is not compact, answering a question posed by P. Ahern and W. Rudin.

The polynomial hull of a compact set $X \subset \mathbb{C}^n$ is the set $b_X$ of all points $x \in \mathbb{C}^n$ at which the inequality $|P(x)| \leq \max\{|P(z)| : z \in X\}$ holds for every polynomial $P$. Let $U$ denote the unit disc in $\mathbb{C}$. In [1] P. Ahern and W. Rudin introduced the following definition.

“If $\Phi : U \to \mathbb{C}^n$ is a non-constant map whose components are in $H^\infty(U)$, its range $\Phi(U)$ is called an $H^\infty$-disc, parametrized by $\Phi$. If $\lim_{r \to 1}(\Phi(re^{i\theta})) \in X$ for almost all $e^{i\theta}$ on the unit circle $T$, then $\Phi(U)$ is an $H^\infty$-disc whose boundary lies in $X$.”

They further define the disc-hull $D(X)$ to be the union of $X$ and all $H^\infty$-discs whose boundaries lie in $X$. One of the questions posed in [1] (see p. 25) is whether the disc-hull $D(X)$ is always compact for a compact set $X \subset \mathbb{C}^n$.

Below we answer this question negatively by constructing a counter-example in $\mathbb{C}^2$.

1. Define $\omega = \{z \in U : \text{Re } z > \frac{1}{2}\}$. Let $\varphi : \overline{U} \to \mathbb{C}$ be the Riemann map satisfying $\varphi(3i) = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ and $\varphi(1) = 1$. Therefore $\text{Re } \varphi(e^{i\theta}) = \frac{1}{2}$ for $e^{i\theta} \leq 0$. Also, $0 \notin \varphi(U)$, $|\varphi(0)| < 1$, and hence $\lim_{n \to \infty} \varphi^n(0) = 0$.

2. Let $X = \{(\zeta, \eta) \in \mathbb{C}^2 : \zeta \in T, \eta \in \Gamma_\zeta\}$, where the fiber $\Gamma_\zeta$ is defined as follows. $\Gamma_\zeta = T$ for $\text{Re } \zeta > 0$, $\Gamma_\zeta = \overline{U}$ for $\zeta = \pm i$, and $\Gamma_\zeta = \{\varphi^n(\zeta) : n \in \mathbb{N}\} \cup \{0\}$ for $\text{Re } \zeta < 0$.

One can check that the complement $\mathbb{C}^2 \setminus X$ of $X$ is an open set and, since $X$ is also bounded, it is compact. One can also notice that $X$ is connected.

3. For each $n$ consider

$$\Phi_n(z) = (z, \varphi^n(z)) : U \to \mathbb{C}^2$$

By construction, $\Phi_n(T) \subset X$, so, $\Phi_n(U) \subset D(X)$. One can see that $\lim_{n \to \infty} \Phi_n(0) = (0, 0)$; therefore $(0, 0) \in D(X)$. 
4. To complete the example we now need to show that \((0,0) \notin D(X)\). If not, then there is an \(H^\infty\)-disc \(\Phi(U)\), \(\Phi(z) = (\alpha(z), \beta(z)) : U \to \mathbb{C}^2\), such that \(\lim_r \Phi(rt) \in X\) for almost all \(t \in T\) and \((0,0) \in \Phi(U)\). Without loss of generality we may assume that \(\Phi(0) = (\alpha(0), \beta(0)) = (0,0)\) (one can consider \(\Phi \circ \psi\) in place of \(\Phi\) for a suitable Möbius transformation \(\psi\)). Since by construction \(\lim_r \Phi(re^{i\theta}) \in T\) for almost all \(e^{i\theta} \in T\). Let \(u(t)\) denote the corresponding limit. The following property of an \(H^\infty\) function \(u(z)\) (see [2], p. 339) will be used:

\[
\text{If } u(t) = 0 \text{ for } t \in T' \subset T, \text{ and } T' \text{ has positive measure,}
\]
\[
\text{then } u(z) = 0 \text{ for all } z \in U.
\]
(1)

For the function \(\alpha(z)\) we introduce the set

\[T_0 = \{t \in T : \alpha(t) \text{ exists and } \alpha(t) \in T\},\]

so \(T_0\) is almost all of \(T\). Consider the following sets:

\[S^- = \alpha^{-1}\{e^{i\theta} : \text{Re} e^{i\theta} < 0\} \cap T_0,\]
\[S^+ = \alpha^{-1}\{e^{i\theta} : \text{Re} e^{i\theta} > 0\} \cap T_0,\]
\[S^0 = \alpha^{-1}\{e^{i\theta} : \text{Re} e^{i\theta} = 0\} \cap T_0.\]

Our main goal now is to prove that \(S^-\) has positive measure. Notice that \(S^- \cup S^+ \cup S^0 = T_0\) which has full circle measure.

The set \(\alpha(S^0)\) consists of two points and if \(S^0\) had positive measure, then according to (1), \(\alpha(z)\) would be constant. Therefore, \(S^0\) has measure 0. If \(S^+\) had the full measure, then \(\text{Re } \alpha(0)\) would be positive since it is the average of its values on \(T\), but \(\alpha(0) = 0\). Therefore, the measure of \(S^-\) is positive.

Introduce now the following functions: \(u_p(z) = \beta(z) - \varphi^p(\alpha(z))\) for \(p = 1, 2, \ldots\); \(u_0(z) = \beta(z)\). All of them are in \(H^\infty(U)\). Define \(S_p = \{t \in T : u_p(t) = 0\}\). One can see that by construction almost all points of \(S^-\) lie in \(\bigcup S_p\). Therefore there exists a \(q\) such that \(S_q\) has positive measure.

If \(q = 0\), then by (1), \(\beta(z) = u_0(z) = 0\) on \(U\). This implies that \(X\) (containing almost all of \(\Phi(T)\)) contains almost all points \((e^{i\theta}, 0)\). This is impossible since for all \(\text{Re } e^{i\theta} > 0\), the point \((e^{i\theta}, 0) \notin X\).

Therefore \(q > 0\). So, by (1), \(u_q(z) = \beta(z) - \varphi^q(\alpha(z)) = 0\) for all of \(U\). Now \(0 = \beta(0) = \varphi^q(\alpha(0))\), and \(\varphi(0) = 0\), contradicting \(0 \notin \varphi(U)\).

Remark. One can see that the entire disc \((U, 0) \subset \tilde{X}\) and all the points of this disc belong to \(\tilde{D}(\tilde{X})\) but not to \(D(X)\).

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\text{Added after posting}
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After the galley proofs were returned, the authors were informed by J. Globevnik that a different counterexample was published by Herb Alexander in “A disc-hull in \(\mathbb{C}^{2n}\), Proc. Amer. Math. Soc. 120 (1994), 1207–1209.
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