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NON-HYPERBOLIC COMPLEX SPACE WITH A HYPERBOLIC NORMALIZATION

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Abstract. We construct an example of a non-hyperbolic singular projective surface X whose normalization V is the square of a genus 3 curve C and hence, hyperbolic.

Let C be a smooth irreducible projective curve of genus $g \geq 2$. Then the smooth projective surface $V = C \times C$ is Kobayashi hyperbolic, that is, the Kobayashi pseudodistance on it is a distance [2]. Let $V \hookrightarrow \mathbb{P}^N$ be a projective embedding. Consider a generic projection $\pi : V \to \mathbb{P}^3$. By Bertini's theorem, the singular locus S (i.e. the closure of the set of double points) of the image surface $X = \pi(V) \subset \mathbb{P}^3$ is an irreducible curve, and $\pi : V \to X$ is a normalization map (see [4]). The question arises whether the surface X is also hyperbolic. The answer is positive [5], and hence by the stability of hyperbolicity theorem [6], any (smooth) surface X' in $\mathbb{P}^3$ close enough to X is hyperbolic, as well. In that way examples of degree 32 smooth hyperbolic surfaces in $\mathbb{P}^3$ were produced [5].

By Proposition 1.1 in [5], hyperbolicity of a (singular) surface X as above is equivalent to hyperbolicity of its double curve S. Actually, in [5] it is shown that the geometric genus of the curve S is ≥ 225, which provides that X is hyperbolic.

On the other hand, by the Kobayashi-Kwack theorem [2, 3], a normalization of a hyperbolic complex space is also hyperbolic. In this note we give an example which shows that in general, the converse is not true. To describe this example, denote by C the Fermat quartic $x^4 + y^4 + z^4 = 0$ in $\mathbb{P}^2$. Then the Cartesian square $V = C \times C \subset \mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$ (the Segre embedding) is a smooth surface of degree 32 in $\mathbb{P}^8$. We construct a singular projective surface X whose normalization is V, which has a fibration $X \to C$ over C with general fibre isomorphic to C and with four degenerate fibres $C_i,\ i = 1, \ldots, 4$, isomorphic to $\mathbb{P}^1$. The "double curve" $S = C_1 \cup \cdots \cup C_4 \subset X$ of X is neither irreducible nor hyperbolic, in contrast with the situation studied in [5]. Thus, the assumption in [5] that the projection π is generic, is likely to be essential to provide hyperbolicity of the image surface $X = \pi(V)$.

Actually, in our example the surface X does not appear as a projection of $V = C \times C$; but it has a natural embedding into a four-dimensional Brauer-Severi variety Y (see [1]) which is a smooth projective fiber bundle over $\mathbb{P}^2$ with general fibre $\mathbb{P}^2$.

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More precisely, let the surface $V = C \times C \subset \mathbb{P}^2 \times \mathbb{P}^2$ be given as

$$V = \begin{cases} y^4 + x^4 + z^4 &= 0, \\ u^4 + v^4 + w^4 &= 0. \end{cases}$$

Evidently, it is hyperbolic; indeed, C is a smooth genus 3 curve.

Set $C_1 = C \setminus \{z = 0\}$ and $C_2 = C \setminus \{x = 0\}$. Let $q : E \to C$ be an algebraic fiber bundle over C with the fiber $\mathbf{P}^2$ such that $q^{-1}(C_k) \cong C_k \times F_k$ where $F_k \cong \mathbf{P}^2$ with a homogeneous coordinate system $(u_k : v_k : w_k),\ k = 1, 2$, and the transition mapping is given in $q^{-1}(C_1 \cap C_2)$ as follows: $(u_2 : v_2 : w_2) = (zu_1 : xv_1 : xw_1)$. Let $X \subset E$ be the surface defined by the equations

$$X_1 = X \cap q^{-1}(C_1) = \begin{cases} y^4 + x^4 + z^4 = 0, \\ z^4 u_1^4 + y^4(v_1^4 + w_1^4) = 0 \end{cases}$$

and

$$X_2 = X \cap q^{-1}(C_2) = \begin{cases} y^4 + x^4 + z^4 = 0, \\ x^4 u_2^4 + y^4(v_2^4 + w_2^4) = 0. \end{cases}$$

Then the intersection $S := X \cap \{y = 0\}$ consists of four disjoint smooth rational curves (whereas any other fibre of the natural projection $X \to C$, that is, the restriction to X of the projection of the Cartesian square $\mathbb{P}^2 \times \mathbb{P}^2$ to the first factor, is isomorphic to the curve C). Thus, the surface X is not hyperbolic (cf. Remark 1 below).

Put $V_1 = V \setminus \{z = 0\},\ V_2 = V \setminus \{x = 0\}$, and consider further the morphisms $\nu_i : V_i \to X_i,\ i = 1, 2$, given as

$$(u_1 : v_1 : w_1) = (yu : zv : zw) \qquad \text{resp., } (u_2 : v_2 : w_2) = (yu : xv : xw)\,.$$

It is easily seen that these formulas define a birational morphism $\nu : V \to X$ which makes V a normalization of X. Indeed, since V is a smooth surface, ν can be factorized through the normalization $\nu' : V' \to X$ of X, that is, $\nu = \mu \circ \nu'$ where $\mu : V \to V'$ is a birational morphism. It is easily seen that μ is a bijection, and then by Zariski's Main Theorem, it is an isomorphism. Thus, $\nu : V \to X$ is a normalization of X. This gives a desired example.

Remarks. **1**. In fact, the surface X is hyperbolic modulo the "double curve" S. This follows from the fact that any holomorphic disc $f : \Delta \to X$ (where Δ denotes the unit disc) whose image is not contained in S can be lifted to the normalization, that is, there exists a holomorphic disc $\widehat{f} : \Delta \to V$ such that $f = \nu \circ \widehat{f}$. Hence, since V is hyperbolic, the Kobayashi-Royden pseudometric on X can be estimated from below outside of the directions tangent to S.

2. We can easily get a similar example of a non-hyperbolic affine algebraic surface which has a smooth hyperbolic affine normalization. Indeed, let $V_0 \subset \mathbb{C}^4$ resp., $X_0 \subset \mathbb{C}^4$ be the surface given by the equations

$$\begin{cases} y^4 + x^4 + 1 = 0, \\ u^4 + v^4 + 1 = 0, \end{cases}$$

resp.,

$$\begin{cases} y^4 + x^4 + 1 = 0, \\ u^4 + y^4(v^4 + 1) = 0. \end{cases}$$

Then, as above, the restriction to V_0 of the birational morphism

$$\sigma : \mathbb{C}^4 \to \mathbb{C}^4, \quad \sigma : (x, y, u, v) \longmapsto (x, y, yu, v)$$

(which consists of blowing up with center at the plane $y = u = 0$ and then deleting the proper transform of the divisor $y = 0$) makes V_0 a normalization of X_0. The intersection $S_0 := X_0 \cap \{y = 0\}$ consists of four complex affine lines $\simeq \mathbb{C}$ and hence, the surface X_0 is not hyperbolic, whereas its normalization V_0 is hyperbolic being the Cartesian square of a hyperbolic affine curve $C_0 = \{x^4 + y^4 + 1 = 0\}$.

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