NON-HYPERBOLIC COMPLEX SPACE WITH A HYPERBOLIC NORMALIZATION

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Abstract. We construct an example of a non-hyperbolic singular projective surface $X$ whose normalization $V$ is the square of a genus 3 curve $C$ and hence, hyperbolic.

Let $C$ be a smooth irreducible projective curve of genus $g \geq 2$. Then the smooth projective surface $V = C \times C$ is Kobayashi hyperbolic, that is, the Kobayashi pseudodistance on it is a distance $\mathbb{D}$ [2]. Let $V \hookrightarrow \mathbb{P}^N$ be a projective embedding. Consider a generic projection $\pi : V \twoheadrightarrow \mathbb{P}^3$. By Bertini’s theorem, the singular locus $S$ (i.e. the closure of the set of double points) of the image surface $X = \pi(V) \subset \mathbb{P}^3$ is an irreducible curve, and $\pi : V \twoheadrightarrow X$ is a normalization map (see [4]). The question arises whether the surface $X$ is also hyperbolic. The answer is positive [5], and hence by the stability of hyperbolicity theorem [6], any (smooth) surface $X_0$ in $\mathbb{P}^3$ close enough to $X$ is hyperbolic, as well. In that way examples of degree 32 smooth hyperbolic surfaces in $\mathbb{P}^3$ were produced [5].

By Proposition 1.1 in [5], hyperbolicity of a (singular) surface $X$ as above is equivalent to hyperbolicity of its double curve $S$. Actually, in [5] it is shown that the geometric genus of the curve $S$ is $\geq 225$, which provides that $X$ is hyperbolic.

On the other hand, by the Kobayashi-Kwack theorem [2, 3], a normalization of a hyperbolic complex space is also hyperbolic. In this note we give an example which shows that in general, the converse is not true. To describe this example, denote by $C$ the Fermat quartic $x^4 + y^4 + z^4 = 0$ in $\mathbb{P}^2$. Then the Cartesian square $V = C \times C \subset \mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$ (the Segre embedding) is a smooth surface of degree 32 in $\mathbb{P}^8$. We construct a singular projective surface $X$ whose normalization is $V$, which has a fibration $X \rightarrow C$ over $C$ with general fibre isomorphic to $C$ and with four degenerate fibres $C_i, i = 1, \ldots, 4$, isomorphic to $\mathbb{P}^1$. The “double curve” $S = C_1 \cup \cdots \cup C_4 \subset X$ of $X$ is neither irreducible nor hyperbolic, in contrast with the situation studied in [4]. Thus, the assumption in [4] that the projection $\pi$ is generic, is likely to be essential to provide hyperbolicity of the image surface $X = \pi(V)$.

Actually, in our example the surface $X$ does not appear as a projection of $V = C \times C$; but it has a natural embedding into a four-dimensional Brauer-Severi variety $Y$ (see [4]) which is a smooth projective fiber bundle over $\mathbb{P}^2$ with general fibre $\mathbb{P}^2$. 

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More precisely, let the surface \( V = C \times C \subset \mathbb{P}^2 \times \mathbb{P}^2 \) be given as

\[
V = \left\{ \begin{array}{l}
y^4 + x^4 + z^4 = 0, \\
u^4 + v^4 + w^4 = 0.
\end{array} \right.
\]

Evidently, it is hyperbolic; indeed, \( C \) is a smooth genus 3 curve.

Set \( C_1 = C \setminus \{ z = 0 \} \) and \( C_2 = C \setminus \{ x = 0 \} \). Let \( q : E \to C \) be an algebraic fiber bundle over \( C \) with the fiber \( \mathbb{P}^2 \) such that \( q^{-1}(C_k) \cong C_k \times F_k \) where \( F_k \cong \mathbb{P}^2 \) with a homogeneous coordinate system \( (u_k : v_k : w_k) \), \( k = 1, 2 \), and the transition mapping is given in \( q^{-1}(C_1 \cap C_2) \) as follows: \((u_2 : v_2 : w_2) = (zu_1 : xv_1 : xw_1)\). Let \( X \subset E \) be the surface defined by the equations

\[
X_1 = X \cap q^{-1}(C_1) = \left\{ \begin{array}{l}
y^4 + x^4 + z^4 = 0, \\
z^4u_1^4 + y^4(v_1^4 + w_1^4) = 0
\end{array} \right.
\]

and

\[
X_2 = X \cap q^{-1}(C_2) = \left\{ \begin{array}{l}
y^4 + x^4 + z^4 = 0, \\
x^4u_2^4 + y^4(v_2^4 + w_2^4) = 0
\end{array} \right.
\]

Then the intersection \( S := X \cap \{ y = 0 \} \) consists of four disjoint smooth rational curves (whereas any other fibre of the natural projection \( X \to C \), that is, the restriction to \( X \) of the projection of the Cartesian square \( \mathbb{P}^2 \times \mathbb{P}^2 \) to the first factor, is isomorphic to the curve \( C \)). Thus, the surface \( X \) is not hyperbolic (cf. Remark 1 below).

Put \( V_1 = V \setminus \{ z = 0 \} \), \( V_2 = V \setminus \{ x = 0 \} \), and consider further the morphisms

\[
\nu_i : V_i \to X_i, \ i = 1, 2,
\]

given as

\[
(u_1 : v_1 : w_1) = (yu : zw : zw) \quad \text{resp.,} \quad (u_2 : v_2 : w_2) = (yu : xv : xw).
\]

It is easily seen that these formulas define a birational morphism \( \nu : V \to X \) which makes \( V \) a normalization of \( X \). Indeed, since \( V \) is a smooth surface, \( \nu \) can be factorized through the normalization \( \nu' : V' \to X \) of \( X \), that is, \( \nu = \mu \circ \nu' \) where \( \mu : V \to V' \) is a birational morphism. It is easily seen that \( \mu \) is a bijection, and then by Zariski’s Main Theorem, it is an isomorphism. Thus, \( \nu : V \to X \) is a normalization of \( X \). This gives a desired example.

**Remarks.**

1. In fact, the surface \( X \) is hyperbolic modulo the “double curve” \( S \). This follows from the fact that any holomorphic disc \( f : \Delta \to X \) (where \( \Delta \) denotes the unit disc) whose image is not contained in \( S \) can be lifted to the normalization, that is, there exists a holomorphic disc \( \tilde{f} : \Delta \to V \) such that \( f = \nu \circ \tilde{f} \). Hence, since \( V \) is hyperbolic, the Kobayashi-Royden pseudometric on \( X \) can be estimated from below outside of the directions tangent to \( S \).

2. We can easily get a similar example of a non-hyperbolic affine algebraic surface which has a smooth hyperbolic affine normalization. Indeed, let \( V_0 \subset \mathbb{C}^4 \) resp., \( X_0 \subset \mathbb{C}^4 \) be the surface given by the equations

\[
\begin{cases}
y^4 + x^4 + 1 = 0, \\
u^4 + v^4 + 1 = 0,
\end{cases}
\]

resp.,

\[
\begin{cases}
y^4 + x^4 + 1 = 0, \\
u^4 + y^4(v^4 + 1) = 0.
\end{cases}
\]
Then, as above, the restriction to \( V_0 \) of the birational morphism
\[
\sigma : \mathbb{C}^4 \to \mathbb{C}^4, \quad \sigma : (x, y, u, v) \mapsto (x, y, yu, v)
\]
(which consists of blowing up with center at the plane \( y = u = 0 \) and then deleting the proper transform of the divisor \( y = 0 \)) makes \( V_0 \) a normalization of \( X_0 \). The intersection \( S_0 := X_0 \cap \{ y = 0 \} \) consists of four complex affine lines \( \simeq \mathbb{C} \) and hence, the surface \( X_0 \) is not hyperbolic, whereas its normalization \( V_0 \) is hyperbolic being the Cartesian square of a hyperbolic affine curve \( C_0 = \{ x^4 + y^4 + 1 = 0 \} \).

References


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[MR 39:1445]


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