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AN EXAMPLE IN THE THEORY OF AC-OPERATORS

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ABSTRACT. AC-operators are a generalization in the context of well-boundedness of normal operators on Hilbert space. It was shown by Doust and Walden that compact AC-operators have a representation as a conditionally convergent sum reminiscent of the spectral representations for compact normal operators. In this representation, the eigenvalues must be taken in a particular order to ensure convergence of the sum. Here we show that one cannot replace the ordering given by Doust and Walden by the more natural one suggested in their paper.

1. INTRODUCTION

There is a long history of results showing that a compact Banach space operator T with a suitable functional calculus has a representation of the form

$$T = \sum_{j=1}^{\infty} \lambda_j P_j \tag{$*$}$$

where $\{\lambda_j\}$ is the set of non-zero eigenvalues of T, and P_j is the Riesz projection onto the eigenspace corresponding to λ_j. (Of course, T may have only finitely many non-zero eigenvalues.)

For self-adjoint or normal compact operators the sum $(*)$ converges unconditionally. To provide a theory which covered operators whose spectral expansions may only converge conditionally, Smart [Sm] introduced the concept of a well-bounded operator. Let X denote a complex Banach space. An operator $T \in B(X)$ is said to be *well-bounded* if it admits an absolutely continuous functional calculus, i.e. if there exist $K > 0$ and a compact interval $[a,b] \subset \mathbb{R}$ such that

$$\|g(T)\| \le K \left\{ |g(a)| + \int_a^b |g'(t)|\, dt \right\} \equiv K \|g\|_{AC[a,b]}$$

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for all polynomials g. One restriction of well-bounded operators is that they necessarily have real spectrum. The analogue of a normal operator in this context is known as an AC-operator. An operator $T \in B(X)$ is an *AC-operator* if $T = A+iB$ for some pair $\{A, B\}$ of commuting well-bounded operators, or equivalently, if T has a functional calculus for the absolutely continuous functions on some rectangle $[a,b] \times [c,d] \subset \mathbb{R}^2$. We refer the reader to [Dow] for background on well-bounded operators and to [BG] for background on AC-operators.

It was shown by Cheng and Doust [CD1] that a compact well-bounded operator has a representation in the form $(*)$ where the sum is ordered so that

$$|\lambda_1| \geq |\lambda_2| \geq \cdots . \tag{O-1}$$

Examples show that we may indeed get only conditional convergence for the sum. Doust and Walden [DW] extended this result to cover compact AC-operators. In this case they could only prove convergence under a more complicated ordering of the eigenvalues.

For a complex number $\lambda = x+iy$ with $x, y \in \mathbb{R}$, let $|\lambda|_\infty = \max\{|x|, |y|\}$. Define the order $\prec$ on $\mathbb{C}$ by setting $\lambda \prec \mu$ if

(i) $|\lambda|_\infty < |\mu|_\infty$, or,
(ii) if $|\lambda|_\infty = |\mu|_\infty = \alpha$ and μ lies on that part of the square $|z|_\infty = \alpha$ between $-\alpha + i\alpha$ and λ going from $-\alpha + i\alpha$ in a clockwise direction.

In [DW] it was shown that if T is a compact AC-operator, then T has a representation in the form $(*)$ where the eigenvalues are ordered by

$$\lambda_1 \succ \lambda_2 \succ \ldots . \tag{O-2}$$

They also showed that under certain circumstances the sum also converges to T under the order (O-1), but they left open the question as to whether this is always the case.

In this note we construct an example which shows that in general one cannot use the order (O-1) for compact AC-operators.

2. The example

The following general result will be used in the construction. We leave the proof to the reader.

Lemma. *Suppose that $\{z_j\}_{j=1}^\infty$ is a sequence in a Banach space Z and that $z = \sum_{j=1}^\infty z_j$ converges. Given a sequence $0 = m_0 < m_1 < m_2 < \ldots$ of integers, define the sequence $\{w_j\}_{j=1}^\infty$ in Z by setting $w_{m_{k-1}+\ell} = z_{m_k+1-\ell}$ for $k = 1, 2, \ldots$ and $\ell = 1, \ldots, m_k - m_{k-1}$. Then $\sum_{j=1}^\infty w_j$ converges and equals z.*

Theorem. *Suppose that X is an infinite dimensional Banach space with a basis. Then there exists a compact AC-operator $T \in B(X)$ for which the sum $(*)$ above does not converge under the order* (O-1).

Proof. As X has a basis, it also has a conditional basis $\{x_j\}$ [PS]. Denote the corresponding coordinate projections by $\{P_j\}$ and for $n \geq 1$ set $Q_n = \sum_{j=1}^n P_j$. We shall apply the convention that $P_0 = Q_0 = 0$. We note that the sets $\{P_j\}$ and $\{Q_j\}$ are uniformly bounded by say K.

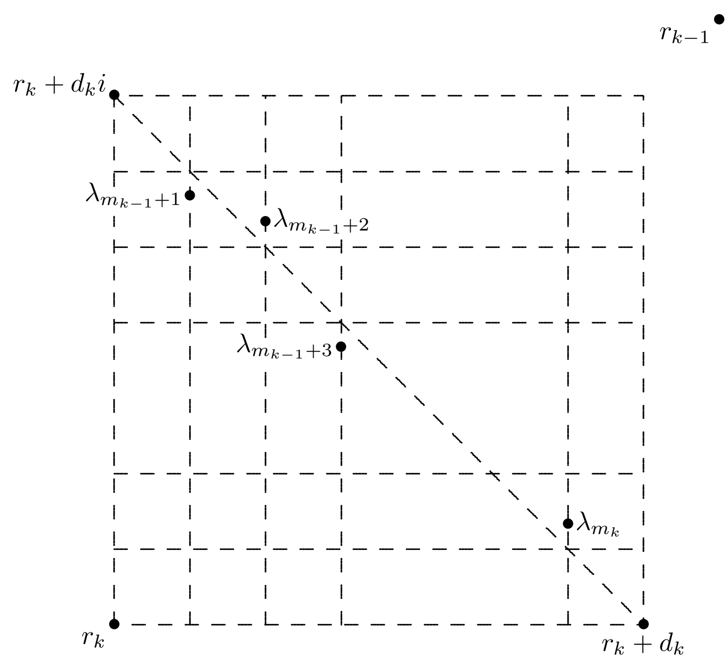

FIGURE 1.

As $\{x_j\}$ is conditional there exists $m_1 \in \mathbb{N}$ and a subset $N_1 \subset \{1, 2, \ldots, m_1\} = I_1$ such that $\left\|\sum_{j \in N_1} P_j\right\| \geq 2$. For $k \geq 2$ we can recursively find $m_k > m_{k-1}$ and $N_k \subset \{m_{k-1}+1, \ldots, m_k\} = I_k$ such that $\left\|\sum_{j \in N_k} P_j\right\| \geq k$. For convenience, set $m_0 = 0$.

For $k \geq 1$, let $r_k = e^{i\pi/4}/\sqrt{k}$ and suppose that d_k is a small positive number whose value will be fixed later in the proof. For the present it suffices to require that

(a) for $k \geq 2$, $k^{-1/2} + \sqrt{2} d_k < (k-1)^{-1/2}$,
(b) $d_k < (2\,|N_k|)^{-1}$.

Let

$$\delta_k = \frac{d_k}{m_k - m_{k-1} + 1}.$$

For $k \geq 1$ and $j = 1, 2, \ldots, m_k - m_{k-1}$, let

$$\lambda_{m_{k-1}+j} = r_k + d_k i + j\sqrt{2} e^{-i\pi/4} \delta_k - \frac{(-1)^{\kappa(k,j)}}{3} \delta_k i,$$

where $\kappa(k, j)$ is 1 if $m_{k-1} + j \in N_k$ and 0 otherwise. Figure 1 shows a typical arrangement of these values (with $\lambda_{m_{k-1}+2} \in N_k$ and $\lambda_{m_{k-1}+1}, \lambda_{m_{k-1}+3} \notin N_k$). Note that λ_ℓ will be above the diagonal drawn if and only if $\ell \in N_k$.

It is not too difficult to show that if we choose d_k small enough, then $|r_k + d_k i| < |\lambda_\ell|$ whenever $\ell \in N_k$ and hence

$$|\lambda_{\ell_1}| < |\lambda_{\ell_2}| \tag{1}$$

whenever $\ell_1 \in I_k \setminus N_k$ and $\ell_2 \in N_k$.

For $j = 1, 2, \ldots$, let $a_j = \mathrm{Re}(\lambda_j)$ and $b_j = \mathrm{Im}(\lambda_j)$. We wish to define $A = \sum_{j=1}^{\infty} a_j P_j$ and $B = \sum_{j=1}^{\infty} b_j P_j$. As $b_1 > b_2 > \ldots$, it follows from [CD2] that the second partial sum converges (in norm) and that B is well-bounded. The case of the first sum is just a little more delicate.

Let $\{\tilde{a}_j\}$ be the decreasing rearrangement of $\{a_j\}$ and let $\{\tilde{P}_j\}$ be the corresponding rearrangement of the coordinate projections. It is easy to check that the partial sum projections corresponding to this ordering are uniformly bounded (by $3K$). Thus $\tilde{A} = \sum_{j=1}^{\infty} \tilde{a}_j \tilde{P}_j$ converges to a well-bounded operator as well. It now follows from the lemma that $A = \sum_{j=1}^{\infty} a_j P_j$ converges and equals $\tilde{A}$, and so A is a well-bounded operator.

Clearly A and B commute and so $T = A + iB$ is a compact AC-operator with eigenvalues $\{\lambda_j\}$. It remains now to show that if we try to order these eigenvalues by (O-1), then the corresponding sum $(*)$ does not converge.

It follows from (1) that for each $k \geq 1$

$$S_k = \sum_{j \leq m_{k-1}} \lambda_j P_j + \sum_{j \in N_k} \lambda_j P_j$$

is a partial sum of $(*)$ under (O-1). Note that for each k, $\sum_{j \leq m_{k-1}} \lambda_j P_j$ is a partial sum of $(*)$ under (O-2) so these sums are uniformly bounded say by C. By property (b) above, $|\lambda_j - r_k| < 2d_k < 1/|N_k|$ for $j \in I_k$. Thus

$$\begin{aligned}
\|S_k\| &= \Big\| \sum_{j \leq m_{k-1}} \lambda_j P_j + r_k \sum_{j \in N_k} P_j + \sum_{j \in N_k} \lambda_j P_j - r_k \sum_{j \in N_k} P_j \Big\| \\
&\geq \Big\| r_k \sum_{j \in N_k} P_j \Big\| - \Big\| \sum_{j \leq m_{k-1}} \lambda_j P_j \Big\| - \Big\| \sum_{j \in N_k} (\lambda_j - r_k) P_j \Big\| \\
&\geq \frac{k}{\sqrt{k}} - C - \sum_{j \in N_k} |\lambda_j - r_k| \, K \\
&\geq \sqrt{k} - C - K.
\end{aligned}$$

As this sequence of partial sums is unbounded, the series cannot converge under (O-1). $\square$

The hypotheses on X could of course be relaxed somewhat. All that is required is that X admits a uniformly bounded increasing sequence of projections with bad unconditionality properties.

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