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CREATION AND PROPAGATION OF LOGARITHMIC SINGULARITIES BY INTERACTION OF TWO PIECEWISE SMOOTH PROGRESSING WAVES

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ABSTRACT. Our aim is to understand the non-conservation of the piecewise smooth regularity by a semi-linear interaction of two transverse progressing waves. Indeed, we know that this phenomenon occurs when the number of characteristic hypersurfaces passing through the locus of interaction, that is, a two-codimensional variety, is strictly inferior to the size of the considered first order strictly hyperbolic system. Thanks to the study of a significant example, we explain the obstruction to the piecewise smooth propagation by a loss of transmission property for the symbols describing the conormal singularities, which originates logarithmic singularities.

1. INTRODUCTION

We are interested in the local phenomena which are the creation and the propagation of singularities by the interaction of two transversal conormal waves for a semi-linear partial differential system. More precisely, the purpose is to analyze a phenomenon of a loss of piecewise smooth regularity shown by G. Métivier and J. Rauch in [6] and [7].

We consider a first order semi-linear system

$$Lu = f(x, u). \tag{1.1}$$

It is supposed strictly hyperbolic with respect to a timelike function t. The coefficients of L are $M \times M$ smooth real matrix. The real function f is smooth. The variable x describes an open neighborhood Ω of 0 into $\mathbb{R}^n$, for which the past is assumed to be a domain of determinacy.

Let Σ_1, Σ_1 be two characteristic hypersurfaces intersecting transversely along a 2 codimension manifold Γ.

If Σ is a smooth hypersurface, we write pC^∞_Σ as the space of the piecewise smooth functions with respect to Σ. Locally, their restriction to each open half-space defined by Σ extends in a C^∞ function defined on the whole space.

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We work under the following hypotheses:

1. The rays of the bicharacteristics issued from characteristic conormal vectors to Γ are transverse to Γ.
2. The $\mathbb{R}^M$ valued function u is a solution of (1.1) in L^∞_{loc}.
3. $u_{|t<0} \in pC^\infty_{\Sigma_1} + pC^\infty_{\Sigma_2}$.

The hypothesis 1. implies there is a finite number of characteristic hypersurfaces $\Sigma_1, ..., \Sigma_m$ $(2 \leq m \leq M)$ passing through Γ. They are two-by-two transverse.

Thanks to J.M. Bony and G. Métivier's works (see [1] (smooth case) and [5] (discontinuous case)), we know that the singularities of u are conormal with respect to Σ_i for $i = 1, ..., m$ and Γ, that is, $WF(u) \subset \bigcup_{i=1,...,m} N(\Sigma_i) \cup N(\Gamma)$ if we write $WF(u)$ as the wave front of u and $N(\Sigma_i)$ (respectively $N(\Gamma)$) as the conormal space of Σ_i (respectively Γ) deprived its vanished section.

More precisely G. Métivier and J. Rauch [7] demonstrated notably that if Γ is included in a spacelike hypersurface, then u is piecewise smooth with respect to $\bigcup_{i=1,...,m} \Sigma_i$. Note that $m = M$ in this case. Moreover they built an example showing that the hypothesis $m = M$ is essential (see [6]).

In this work we propose a symbolic study of the singularities showing the microlocal mechanism of the creation and the propagation of non-piecewise smooth type singularities. We assume that the number m of characteristic hypersurfaces passing through the locus of interaction is strictly inferior to the size M of the system (1.1). We will see, by a significant example, that the elliptic part of the system originates the apparition of a non-piecewise smooth type singularity on the edge Γ which can propagate along the characteristic hypersurfaces $\Sigma_1, ..., \Sigma_m$ if the semi-linearity permits it.

The example we will present is inspired by [6]. Nevertheless our approach is microlocal: we explain the loss of piecewise smooth regularity by a loss of transmission property in the complete symbol of the solution. Note the symbolic forms which appear have generalizations (see [4]) which should allow us to treat more general cases.

We consider the following 5×5 first order system on $\mathbb{R}^3$:

$$\left.\begin{aligned} \sqrt{2}(\partial_t + \partial_{x_1})u_1 + (\partial_t + \partial_{x_1} + \partial_{x_2})u_2 = 0 \\ (\partial_t + \partial_{x_1} + \partial_{x_2})u_1 + \sqrt{2}(\partial_t + \partial_{x_2})u_2 = 0 \end{aligned}\right\} \tag{1.2}$$

$$\partial_t v = u_1 w_1 \tag{1.3}$$

$$\left.\begin{aligned} (\partial_t + 2\partial_{x_1})w_1 + 2\partial_{x_2}w_2 = 0 \\ 2\partial_{x_2}w_1 + (\partial_t - 2\partial_{x_1})w_2 = \psi(t)u_1u_2 \end{aligned}\right\} \tag{1.4}$$

with $\psi(t) = \left| \begin{array}{l} 0 \text{ if } t < -\varepsilon \\ 1 \text{ if } t > -\frac{\varepsilon}{2} \end{array} \right.$ for a $\varepsilon > 0$.

This system is strictly hyperbolic with respect to the time variable t. Writing $\mathbf{H}$ the Heaviside function, we impose the following conditions in the past $\{t < -\varepsilon\}$:

$$\text{for } t < -\varepsilon, \quad \begin{aligned} &u_1 = (t - x_1)^{k_1 - 1}\mathbf{H}(t - x_1) \text{ with } k_1 \geq 1, \\ &u_2 = (t - x_2)^{k_2 - 1}\mathbf{H}(t - x_2) \text{ with } k_2 \geq 1, \\ &v = w_1 = w_2 = 0. \end{aligned} \tag{1.5}$$

In fact, the system (1.2) yields explicitly $u_j = (t - x_j)^{k_j - 1}\mathbf{H}(t - x_j)$ for $j = 1, 2$. If we consider successively (1.4) as a linear system with respect to w_1, w_2, and (1.3) as a linear equation with respect to v, we obtain a global solution $u = (u_1, u_2, v, w_1, w_2)$ which is in $pC^\infty_{\Sigma_1} + pC^\infty_{\Sigma_2}$ in the past with $\Sigma_j = \{t - x_j\}$ for $j = 1, 2$. Both

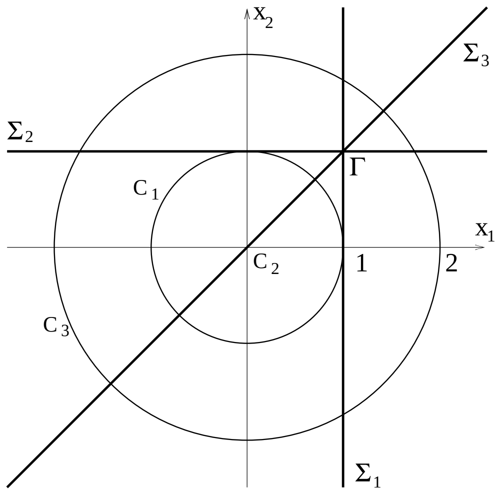

Figure 1.

characteristic hypersurfaces Σ_1 and Σ_2 intersect transversely along the edge $\Gamma = \{t = x_1 = x_2\}$.

Figure 1 shows us that only three characteristic hypersurfaces contain $\Gamma : \Sigma_1$, Σ_2 and $\Sigma_3 = \{x_1 = x_2\}$. We drew on the plan $\{t = 1\}$ the traces of the light cones C_1, C_2, C_3 issued from 0 respectively associated with (1.2), (1.3), (1.4).

There is not any characteristic hypersurface for (1.4) passing through the edge Γ. In other words the system (1.4) is microlocaly elliptic on a neighborhood of $N(\Gamma)$. This partial ellipticity of the system (1.2), (1.3), (1.4) and the "sufficient nonlinearity" of the source term will create non-piecewise smooth type singularities after interaction.

We will prove

Theorem 1.1. *On a neighborhood of* 0*, near* $\Sigma_3^+ = \{x_1 = x_2\} \cap \{t > x_1\}$*, we have* $v(x) \equiv c(x_1 - x_2)^{2k_1+k_2-1} ln|x_2 - x_1|$ *modulo* $pC^\infty_{\Sigma_3} \cap C^{k_1+k_2-1}$ *with* $c \neq 0$.

2. Symbols

By the change of variables $X_1 = t - x_1$, $X_2 = t - x_2$, $X_3 = x_1 + x_2$, and writing again x_1, x_2, x_3 the new variables, we obtain

$$\left.\begin{array}{r}\sqrt{2}(\partial_{x_2} + \partial_{x_3})u_1 + 2\partial_{x_3}u_2 = 0\\ 2\partial_{x_3}u_1 + \sqrt{2}(\partial_{x_1} + \partial_{x_3})u_2 = 0\end{array}\right\} \tag{2.1}$$

$$(\partial_{x_1} + \partial_{x_2})v = u_1 w_1 \tag{2.2}$$

$$\left.\begin{array}{r}(-\partial_{x_1} + \partial_{x_2} + 2\partial_{x_3})w_1 + 2(-\partial_{x_2} + \partial_{x_3})w_2 = 0\\ 2(-\partial_{x_2} + \partial_{x_3})w_1 + (3\partial_{x_1} + \partial_{x_2} - 2\partial_{x_3})w_2 = \Psi(\frac{1}{2}[x_1 + x_2 + x_3])u_1 u_2\end{array}\right\} \tag{2.3}$$

with the new conditions in the past:

$$\text{for } x_1 + x_2 + x_3 < -2\varepsilon, \quad \begin{array}{l} u_1 = x_1^{k_1-1}\mathbf{H}(x_1) \text{ with } k_1 \geq 2,\\ u_2 = x_2^{k_2-1}\mathbf{H}(x_2) \text{ with } k_2 \geq 2,\\ v = w_1 = w_2 = 0.\end{array} \tag{2.4}$$

Now we have $\Gamma = \{x_1 = x_2 = 0\}$, $\Sigma_1 = \{x_1 = 0\}$, $\Sigma_2 = \{x_2 = 0\}$, $\Sigma_3 = \{x_1 = x_2\}$, and for $j = 1, 2$,

$$u_j = x_j^{k_j - 1}\mathbf{H}(x_j) = U_j + g_j \tag{2.5}$$

with g_j C^∞, $U_j = \int e^{ix_j\xi_j} c_j \frac{\chi(\xi_j)}{\xi_j^{k_j}} d\xi_j$, and $c_j = \frac{1}{2\pi}(-i)^{k_j}(k_j - 1)!$.

Using a microlocal parametrix of (2.3) on a neighborhood of $N(\Gamma)$, and working near 0 so that $\Psi(\frac{1}{2}[x_1 + x_2 + x_3]) = 1$, we obtain

$$w_1 \equiv P(u_1 u_2) \text{ modulo } C^\infty \tag{2.6}$$

where P is a pseudo-differential operator of order -1. Near $N(\Gamma)$ the symbol of P is equal to $\frac{2(\xi_2-\xi_3)}{i[-(\xi_1+\xi_2)^2+4(\xi_1-\xi_3)^2+4(\xi_2-\xi_3)^2]}$.

The microlocal study of the conormal singularities which appear after interaction necessitates the description of diverse class of symbols.

Symbols with one variable of frequency. One recalls that $S^\mu(\mathbb{R}^n \times \mathbb{R})$ is the set of the symbols of order μ that is the C^∞ function $a(x, \xi)$ with $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}$, such that for $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}$, and $\mathbf{K}$ a compact of $\mathbb{R}^n$,

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \le C(1 + |\xi|)^{\mu - \beta} \text{ for all } (x, \xi) \in \mathbf{K} \times \mathbb{R}.$$

For a hypersurface Σ and $\mu \in \mathbb{R}$, we write $I^\mu(\Sigma)$ the space of the distributions f smooth out of Σ and which can be written

$$f(x) = \int e^{ix_1\xi_1} a(x, \xi_1) d\xi_1, \quad a \in S^\mu(\mathbb{R}^n \times \mathbb{R}) \tag{2.7}$$

near each point of Σ in coordinates so that $\Sigma = \{x_1 = 0\}$. Note the wave front of such distributions is included in $N(\Sigma)$.

With this notion of symbols we can characterize the piecewise smooth regularity by a transmission property of symbols. Indeed we have the well-known following result:

Lemma 2.1. *Denoting $x = (x_1, ..., x_n) = (x_1, x')$ as the variable in $\mathbb{R}^n$ and $\Sigma = \{x_1 = 0\}$, pC_Σ^∞ is the set of the distributions f of $I^{-1}(\Sigma)$ satisfying (2.7) with*

$$a(x, \xi) \sim \sum_{j \ge 1} \frac{a_j(x')}{\xi_1^j} \text{ for } |\xi_1| > 1. \tag{2.8}$$

In this case $a_j(x') = \frac{1}{2\pi i^j}[\partial_{x_1}^j f]_\Sigma$ where $[\,]_\Sigma$ designates the jump through Σ.

We write $pC_k^\infty(\Sigma) = pC_\Sigma^\infty \cap I^{-k}(\Sigma)$ for $k \in \mathbb{N}^*$.

One recalls (see [3]) that the asymptotic expansion (2.8) means that for all $N \in \mathbb{N}^*$,

$$a(x, \xi) - \sum_{j=1}^{N-1} a_j(x') \frac{\chi(\xi_1)}{\xi_1^j} \in I^{-N}(\Sigma)$$

where χ is a C^∞ function vanishing near 0 so that $\chi(\xi_1) = 1$ if ξ_1 is tall enough.

Symbols with two variables of frequency. One calls $\tilde{S}^{\mu_1,\mu_2}$ the set of the symbols with two variables of frequency defined as the C^∞ functions $b(x,\xi_1,\xi_2)$ with $x \in \mathbb{R}^n$ and $\xi_1,\xi_2 \in \mathbb{R}$, so that for $\alpha \in \mathbb{N}^n$, $\beta_1,\beta_2 \in \mathbb{N}$, and $\mathbf{K}$ a compact of $\mathbb{R}^n$,

$$|\partial_x^\alpha \partial_{\xi_1}^{\beta_1} \partial_{\xi_2}^{\beta_2} b(x,\xi_1,\xi_2)| \leq C(1+|\xi_1|)^{\mu_1-\beta_1}(1+|\xi_2|)^{\mu_2-\beta_2}$$

for all $(x,\xi_1,\xi_2) \in \mathbf{K} \times \mathbb{R}^2$.

For $\mu_1,\mu_2 \in \mathbb{R}$, let $\tilde{I}^{\mu_1,\mu_2}$ be the space of the distributions f on $\mathbb{R}^n$ smooth out of $\Sigma_1 = \{x_1 = 0\}$ and $\Sigma_2 = \{x_2 = 0\}$ such that $f \in I^{\mu_i}(\Sigma_i)$ near each point of $\Sigma_i \backslash \Gamma$ for $i = 1,2$ where $\Gamma = \Sigma_1 \cap \Sigma_2$, and

$$f(x) = \int e^{i(x_1\xi_1+x_2\xi_2)} b(x,\xi_1,\xi_2) d\xi_1 d\xi_2, \quad b \in \tilde{S}^{\mu_1,\mu_2} \tag{2.9}$$

near each point of the edge Γ. The wave front of such distributions is contained by $N(\Sigma_1) \cup N(\Sigma_2) \cup N(\Gamma)$.

We write $pC^\infty_{\Sigma_1\cup\Sigma_1}$ as the space of the piecewise smooth functions with respect to $\Sigma_1 \cup \Sigma_2$. Locally, their restriction to each quarter space defined by $\Sigma_1 \cup \Sigma_2$ extends in a C^∞ function defined on the whole space.

In this way we have a result analogous to Proposition 2.1:

Lemma 2.2. *We denote $x = (x_1,...,x_n) = (x_1,x_2,x'')$ as the variable in $\mathbb{R}^n$, $\Sigma_i = \{x_i = 0\}$ for $i = 1,2$.*

Near each point of the edge $\Gamma = \{x_1 = x_2 = 0\}$, the functions of $pC^\infty_{\Sigma_1\cup\Sigma_1}$ are the distributions $f \in \tilde{I}^{-1,-1}$ satisfying (2.9) *with*

$$b(x,\xi_1,\xi_2) \sim \sum_{j_1,j_2\geq 1} \frac{b_{j_1,j_2}(x'')}{\xi_1^{j_1}\xi_2^{j_2}} \text{ for } |\xi_1|,|\xi_2| > 1. \tag{2.10}$$

In this case $b_{j_1,j_2}(x'') = \frac{1}{4\pi^2 i^{j_1+j_2}}[\partial_{x_1}^{j_1}\partial_{x_2}^{j_2} f]_\Gamma$ where $[\ ]_\Gamma$ designates the jump through Γ, that is,

$$[f]_\Gamma = f(0^+,0^+,x'') - f(0^+,0^-,x'') + f(0^-,0^+,x'') - f(0^-,0^-,x'').$$

Let's specify the meaning of the asymptotic expansion (2.10) : for all $P,N \in \mathbb{N}$,

$$b(x,\xi_1,\xi_2) - \sum_{\substack{1\leq j_1\leq P-1\\ 1\leq j_2\leq N-1}} b_{j_1,j_2}(x'')\frac{\chi(\xi_1)\chi(\xi_2)}{\xi_1^{j_1}\xi_2^{j_2}}$$
$$\equiv \sum_{1\leq j_1\leq P-1} b^N_{j_1}(x'',\xi_2)\frac{\chi(\xi_1)}{\xi_1^{j_1}} + \sum_{1\leq j_2\leq N-1} {}^P b_{j_2}(x'',\xi_1)\frac{\chi(\xi_2)}{\xi_1^{j_2}}$$

modulo $\tilde{S}^{-P,-N}$ with $b^N_{j_1} \in S^{-N}(\mathbb{R}^{n-2}\times\mathbb{R})$ and ${}^P b_{j_2} \in S^{-P}(\mathbb{R}^{n-2}\times\mathbb{R})$.

For $k,k' \geq 1$ we say that the symbol b verifying (2.10) is k,k'-classical if $b_{j_1,j_2} = 0$ for $j_1 \leq k$ or $j_2 \leq k'$, that is $b \in \tilde{S}^{-k,-k'}$.

Let's remark that the asymptotic expansion does not depend on x_1 in (2.8) and x_1,x_2 in (2.10). In fact we know (see [3]) that every distribution of $I^\mu(\Sigma)$ can be written under the form (2.7) with a symbol a not depending on x_1. We have an analogous result of reduction of amplitude for the elements of $\tilde{I}^{\mu_1,\mu_2}$:

Lemma 2.3. *Working near a point of the edge Γ, a distribution $f \in \tilde{S}^{\mu_1,\mu_2}$, $\mu_1,\mu_2 \in \mathbb{R}$, verifying* (2.10) *can be written in the form*

$$f(x) = \int e^{i(x_1\xi_1+x_2\xi_2)} c(x'',\xi_1,\xi_2) d\xi_1 d\xi_2, \quad c \in \tilde{S}^{\mu_1,\mu_2},$$

with

$$c \sim \sum_{\alpha_1,\alpha_2\in\mathbb{N}} \frac{1}{\alpha_1!\alpha_1!} D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} b_{|x_1=x_2=0}. \tag{2.11}$$

Here, the asymptotic expansion means that, for all $N \in \mathbb{N}^*$,

$$c - \sum_{|(\alpha_1,\alpha_2)|\leq N-1} \frac{1}{\alpha_1!\alpha_1!} D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} b_{|x_1=x_2=0} \in \sum_{k+k'=N} \tilde{S}^{\mu_1-k,\mu_2-k'}.$$

Action of the pseudo-differential operators. We write $\xi = (\xi_1, ..., \xi_n)$ as the cotangent variables of $\mathbb{R}^n$. We need a new space of symbols : $\bar{S}^{\mu_1,\mu_2,\nu} = \tilde{S}^{\mu_1+\nu,\mu_2} \cap \tilde{S}^{\mu_1,\mu_2+\nu}$ for $\mu_1, \mu_2, \nu \in \mathbb{R}$, corresponding to a new space of conormal distributions: $\bar{I}^{\mu_1,\mu_2,\nu} = \tilde{I}^{\mu_1+\nu,\mu_2} \cap \tilde{I}^{\mu_1,\mu_2+\nu}$.

Let's describe the action of a pseudo-differential operator on the conormal distributions (see [3] for an idea of the proof).

Lemma 2.4. *Let P be a proper pseudo-differential operator of degree $d \leq 0$. Let's write $p(x,\xi)$ as its symbol.*

1) *If $f \in I^{\mu}(\Sigma)$, $\mu \in \mathbb{R}$, is locally given by (2.7), then $Pf \in I^{\mu+d}(\Sigma)$ can be locally written as*

$$Pf = \int e^{ix_1\xi_1} c(x,\xi_1) d\xi_1$$

with

$$c(x,\xi_1) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} p(x,\xi_1,0,...,0) D_x^{\alpha} a(x,\xi_1). \tag{2.12}$$

2) *If $f \in \tilde{I}^{\mu_1,\mu_2}$, $\mu_1, \mu_2 \in \mathbb{R}$, is locally given by (2.9), then $Pf \in \bar{I}^{\mu_1,\mu_2,d}$ can be locally written as*

$$Pf = \int e^{i(x_1\xi_1+x_2\xi_2)} c(x,\xi_1,\xi_2) d\xi_1 d\xi_2$$

with

$$\text{with} \quad c(x,\xi_1,\xi_2) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} p(x,\xi_1,\xi_2,0,...,0) D_x^{\alpha} b(x,\xi_1,\xi_2). \tag{2.13}$$

As defined in [3], the asymptotic expansion (2.12) means that for all $N \in \mathbb{N}^*$,

$$c(x,\xi_1) - \sum_{|\alpha|\leq N-1} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} p(x,\xi_1,0,...,0) D_x^{\alpha} a(x,\xi_1) \in I^{\mu+d-N}(\Sigma).$$

The meaning of the asymptotic expansion (2.13) is the following : for all $N \in \mathbb{N}^*$,

$$c(x,\xi_1,\xi_2) - \sum_{|\alpha|\leq N-1} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} p(x,\xi_1,\xi_2,0,...,0) D_x^{\alpha} b(x,\xi_1,\xi_2) \in \bar{I}^{\mu_1,\mu_2,d-N}.$$

In that way, if P is a parametrix of a partial differential system elliptic on a neighborhood of $N(\Gamma)$, then its action on the functions of $pC^{\infty}_{\Sigma_1\cup\Sigma_1}$ yields the apparition of a new type of symbols. These symbols are an asymptotic sum of terms $\sigma_{j_1,j_2,j} = b_{j_1,j_2,j}(x)\frac{\rho_j(x,\xi_1,\xi_2)}{\xi_1^{j_1}\xi_2^{j_2}}$ where ρ_j is a rational function for ξ_1, ξ_2 of degree $-j \leq 0$, without any real pole, and smooth with respect to x, and $b_{j_1,j_2,j}$ is a smooth function.

More precisely,

Definition 2.5. For $k, k', l \in \mathbb{N}^*$, we say that a symbol $b \in \bar{S}^{-k,-k',-l}$ is k, k', l-prelogarithmic if

$$b \sim \sum_{\substack{k \le j_1 \\ k' \le j_2 \\ l \le j}} b_{j_1,j_2,j}(x) \frac{\rho_j(x,\xi_1,\xi_2)}{\xi_1^{\,j_1} \xi_2^{\,j_2}} \text{ for } |\xi_1|, |\xi_2| > 1. \tag{2.14}$$

The asymptotic expansion (2.14) means : for all $P, N \in \mathbb{N}^*$,

$$\begin{aligned} b(x,\xi_1,\xi_2) - \sum_{\substack{k \le j_1 \le P-1 \\ k' \le j_2 \le N-1 \\ l \le j \le P+N-j_1-j_2-1}} & \rho_{j_1,j_2,j}(x,\xi_1,\xi_2) \frac{\chi(\xi_1)\chi(\xi_2)}{\xi_1^{\,j_1} \xi_2^{\,j_2}} \\ \equiv \sum_{\substack{k \le j_1 \le P-1 \\ l \le j \le P-j_1-1}} & b_{j_1}^N(x,\xi_2) \rho_j(x,\xi_1,\xi_2) \frac{\chi(\xi_1)}{\xi_1^{\,j_1}} \\ & + \sum_{\substack{k' \le j_2 \le N-1 \\ l \le j \le N-j_2-1}} {}^P b_{j_2}(x,\xi_1) \rho_j(x,\xi_1,\xi_2) \frac{\chi(\xi_2)}{\xi_1^{\,j_2}} \end{aligned}$$

modulo $\tilde{S}^{-P,-N}$ with $b_{j_1}^N \in S^{-N}(\mathbb{R}^n \times \mathbb{R})$ and ${}^P b_{j_2} \in S^{-P}(\mathbb{R}^n \times \mathbb{R})$.

If b is such a symbol, the distribution f given by (2.9) is $pC^\infty_{\Sigma_j}$ near each point of $\Sigma_j \backslash \Gamma$ for $j = 1, 2$. Indeed, near a point of $\Sigma_1 \backslash \Gamma$ for example, the variable x_2 does not vanish so we can use the Taylor expansion of b at 0 with respect to ξ_2 in order to make f under the form (2.7). This Taylor expansion yields a symbol like (2.8).

Nevertheless, f is not $pC^\infty_{\Sigma_1 \cup \Sigma_1}$ near the edge Γ in general. For example, if $b = \frac{\chi(\xi_1)\chi(\xi_2)}{\xi_1 \xi_2 (\xi_1^{\,2} + \xi_2^{\,2})}$, then we have $\Delta f \equiv 4\pi^2 \mathbf{H}(x_1)\mathbf{H}(x_2)$ modulo C^∞, so $\Delta \frac{\partial^2 f}{\partial x_1 \partial x_2} \equiv 4\pi^2 \delta$ and $\frac{\partial^2 f}{\partial x_1 \partial x_2} \equiv \pi \ln(x_1^{\,2} + x_2^{\,2})$ modulo C^∞.

Integration transverse to $\Sigma_1, \Sigma_2, \Gamma$. The following result originates from the properties of integration stated in [2]. Working on a neighborhood of a point on Γ, we have

Lemma 2.6. *Let f be under the form (2.9) with a symbol $b(x'', \xi_1, \xi_2) \in \tilde{S}^{\mu_1,\mu_2}$ and $\mu_1, \mu_2 < -1$. If g satisties*

$$(\partial_{x_1} + \partial_{x_2}) g = f, \tag{2.15}$$

and if g is C^∞ on a neighborhood of each point of $\Sigma_3^- = \{x_1 = x_2,\ x_1 < 0\}$, then

$$g(x) \equiv \int e^{i(x_1 - x_2)\eta} b(x'', \eta, -\eta) d\eta \text{ modulo } C^\infty \tag{2.16}$$

on a neighborhood of each point of $\Sigma_3^+ = \{x_1 = x_2,\ x_1 > 0\}$. We note that $b(x'', \eta, -\eta) \in S^{\mu_1+\mu_2}(\mathbb{R}^{n-2} \times \mathbb{R})$.

So if b is k, k'-classical or k, k', l-prelogarithmic, then g defined by (2.16) is respectively $pC^\infty_{k+k'}(\Sigma_3)$ or $pC^\infty_{k+k'+l}(\Sigma_3)$ near each point of Σ_3^+.

3. Application to the proof of Theorem 1.1

From the expression (2.5) of u_1 and u_2, we have $u_1u_2 = U + \tilde{u}_1 + \tilde{u}_2$ with $\tilde{u}_j \in pC^\infty_{k_j}(\Sigma_j)$ for $j = 1, 2$ and

$$U(x) = \int e^{i(x_1\xi_1 + x_2\xi_2)} c_1c_2 \frac{\chi(\xi_1)\chi(\xi_2)}{\xi_1^{k_1}\xi_2^{k_2}} d\xi_1 d\xi_2.$$

Using the expression (2.6) we obtain $w_1 = W + P\tilde{u}_1 + P\tilde{u}_2$, where $W = PU$ and $P\tilde{u}_j \in pC^\infty_{k_j+1}(\Sigma_j)$ for $j = 1, 2$ thanks to Proposition 2.4 i).

So we can write $u_1w_1 = u_1W + r_1 + r_2$ with $r_1 = u_1P\tilde{u}_1 \in pC^\infty_{k_1}(\Sigma_1)$ and $r_2 = u_1P\tilde{u}_2 \in pC^\infty_{\Sigma_1\cup\Sigma_1} \cap \tilde{I}^{-k_1,-k_2-1}$.

From equation (2.2) and the condition $v = 0$ in the past, we obtain that v is smooth near Σ_3^-. So we can study the singularities of v along Σ_3^+ using Proposition 2.6. The functions r_1, r_2 originate $pC^\infty_{k_1+k_2+1}(\Sigma_3)$ singularities for v.

So it remains to study the contribution of the product u_1W. Proposition 2.4 shows that $W = W_0 + r$ where r is C^∞ and

$$W_0(x) = \int e^{i(x_1\xi_1+x_2\xi_2)} c_3 \frac{\chi(\xi_1)\chi(\xi_2)}{\xi_1^{k_1}\xi_2^{k_2-1}(\xi_1 - \alpha\xi_2)(\xi_1 - \bar{\alpha}\xi_2)} d\xi_1 d\xi_2,$$

with $\alpha = \frac{1}{3} + i\frac{2\sqrt{2}}{2}$ and $c_3 = \frac{2c_1c_2}{3i}$.

Hence from (2.5) we have $u_1W = (U_1+g_1)(W_0+r)$. Because u_1r is just piecewise smooth with respect to Σ_1, it does not contribute to any singularity of v along Σ_3^+. Let's treat the term g_1W_0. The symbol describing its singularities is $k_1, k_2 - 1, 2$-prelogarithmic. Proposition 2.3 allows us to assume that this prelogarithmic symbol does not depend on the variable x_3. So we can apply Proposition 2.6 to conclude that the contribution of g_1W_0 to calculate v just yields $pC^\infty_{k_1+k_2+1}(\Sigma_3)$ singularities near Σ_3^+.

Finally we just need to study the contribution of the term U_1W_0. By convolution in the ξ_1 variable, the symbol of U_1W_0 is:

$$\tilde{b}(\xi_1, \xi_2) = c_1c_3 \frac{\chi(\xi_1)}{\xi_1^{k_1}} *_{\xi_1} \frac{\chi(\xi_1)\chi(\xi_2)}{\xi_1^{k_1}\xi_2^{k_2-1}(\xi_1 - \alpha\xi_2)(\xi_1 - \bar{\alpha}\xi_2)}. \tag{3.1}$$

We have

Lemma 3.1. *Let's write* $g(z) = \left[(\xi_1 - z)^{k_1} z^{k_1} \xi_2^{k_2-1}(z - \alpha\xi_2)(z - \bar{\alpha}\xi_2)\right]^{-1}$.

Then the convolution defined by (3.1) *is equivalent to*

$$\sigma(\xi_1, \xi_2) = 2i\pi c_1c_3 \left| \begin{array}{l} Res(g, \alpha\xi_2) \ \text{if } \xi_2 > 0 \\ Res(g, \bar{\alpha}\xi_2) \ \text{if } \xi_2 < 0 \end{array} \right.$$

modulo a $k_1, k_2 - 1, 2$*-prelogarithmic symbol plus a* $k_1, k_2 + 1$*-classical symbol.*

The notation $Res(g, z_0)$ *designates the residue of* g *at the pole* z_0.

Let's assume this result provisionally. We obtain:

$$\sigma(\xi_1, \xi_2) = 2i\pi c_1c_3 \left| \begin{array}{l} \dfrac{\chi(\xi_2)}{(\xi_1 - \alpha\xi_2)^{k_1} \alpha^{k_1}(\alpha - \bar{\alpha})\xi_2^{k_1+k_2}} \ \text{if } \xi_2 > 0 \\ \dfrac{\chi(\xi_2)}{(\xi_1 - \bar{\alpha}\xi_2)^{k_1} \bar{\alpha}^{k_1}(\bar{\alpha} - \alpha)\xi_2^{k_1+k_2}} \ \text{if } \xi_2 < 0, \end{array} \right.$$

and

$$\sigma(-\eta,\eta) = 2i\pi c_1 c_3 \left| \begin{array}{ll} \dfrac{\chi(\eta)}{2i[(-1-\alpha)\alpha]^{k_1} Im(\alpha)\eta^{\,2k_1+k_2}} & \text{if } \eta > 0 \\ \dfrac{\chi(\eta)}{-2i[(-1-\bar{\alpha})\bar{\alpha}]^{k_1} Im(\alpha)\eta^{\,2k_1+k_2}} & \text{if } \eta < 0 \end{array} \right.$$

is a symbol in $S^{-2k_1-k_2}(\mathbb{R})$ which does not verify the transmission property characterizing the symbols of the piecewise smooth functions.

Finally Proposition 2.6 implies that near any point of Σ_3^+, we can write

$$v(x) \equiv \int e^{i(x_2-x_1)\eta}\sigma(-\eta,\eta)d\eta$$

modulo a function in $pC^\infty_{k_1+k_2+1}(\Sigma_3)$.

The calculus of Fourier's transformation completes the proof of Theorem 1.1.

Remark 3.2. We need the non-linearity to be sufficient if we want the loss of transmission property to occur. For example, if we replace (1.3) by $\partial_t v = w_1$ the solution is piecewise smooth along Σ_3^+ and other types of singularities just appear on the edge Γ.

Remark 3.3. Proposition 3.1 creates a new type of symbol, called "logarithmic". A systematic study of the classical, prelogarithmic and logarithmic symbols ough to allow us to generalize our approach.

Proof of Lemma 3.1. Our aim is to calculate

$$I_{\mathbb{R}} = \int_{\mathbb{R}} \chi(\xi_1 - z)\chi(z)g(z)dz.$$

Let $\mathbf{K}$ be a compact neighborhood of 0 in $\mathbb{R}$ so that $\chi = 1$. At first we study $I_{\mathbf{K}} = \int_{\mathbf{K}} \chi(\xi_1 - z)\chi(z)g(z)dz$. The compactness of the domain of integration allows us to expand in power series the functions $z \mapsto [(z-\alpha\xi_2)(z-\bar{\alpha}\xi_2)]^{-1}$ and $z \mapsto \frac{\chi(\xi_1-z)}{(\xi_1-z)^{k_1}}$. We obtain that $I_{\mathbf{K}}$ is a k_1, k_2+1-classical symbol.

Let's treat $I_{\mathbf{K}(\xi_1)} = \int_{\xi_1 - z\,\in\mathbf{K}} \chi(\xi_1-z)\chi(z)g(z)dz = \int_{\mathbf{K}} \chi(z)\chi(\xi_1-z)g(\xi_1-z)dz$. We carry out the study of $I_{\mathbf{K}}$ by replacing the first expansion by the expansion of $z \mapsto [(\xi_1 - z - \alpha\xi_2)(\xi_1 - z - \bar{\alpha}\xi_2)]^{-1}$. So $I_{\mathbf{K}(\xi_1)}$ is a $k_1, k_2-1, 2$-prelogarithmic symbol.

Finally we prove that $I_{\mathbb{R}} \equiv \int_{\mathbb{R}\setminus(\mathbf{K}\cup\mathbf{K}(\xi_1))} g(z)dz$ modulo the classical and prelogarithmic symbols. The function g is analytic on $\mathbb{C}\setminus\{0, \xi_1, \alpha\xi_2, \bar{\alpha}\xi_2\}$ and can be integrated in a complex path.

As for $I_{\mathbf{K}}$ and $I_{\mathbf{K}(\xi_1)}$, $I_\gamma = \int_\gamma g(z)dz$ is still a k_1, k_2+1-classical symbol and $I_{\gamma(\xi_1)} = \int_{\xi_1 - z\,\in\gamma} g(z)dz$ a $k_1, k_2-1, 2$-prelogarithmic one, when γ is the half circle defined by Figure 2.

Accordingly, for a tall enough $R > 0$, $I_{\mathbb{R}}$ is equivalent to $I_{\gamma_R} = \int_{\gamma_R} g(z)dz$ modulo classical and prelogarithmic symbols, where γ_R is the closed path defined on the figure.

The Residue Theorem completes the proof. □

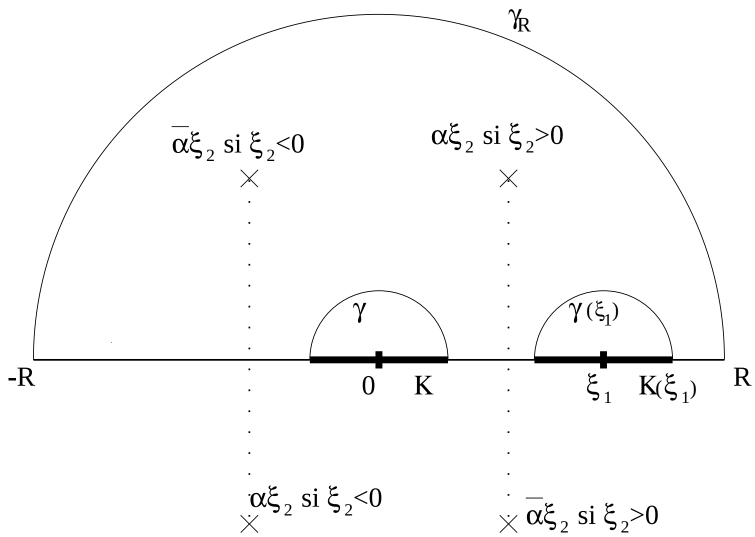

FIGURE 2.

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