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INVARIANT PROJECTIONS AND CONVOLUTION OPERATORS

JACQUES DELAPORTE AND ANTOINE DERIGHETTI

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ABSTRACT. We prove the existence of invariant projections $\mathcal{P}$ from the Banach space $PM_p(G)$ of p-pseudomeasures onto $PM_p(H)$ with $\operatorname{supp}\mathcal{P}(T) \subset \operatorname{supp} T$ for H closed neutral subgroup of a locally compact group G. As a main application we obtain that every closed neutral subgroup is a set of p-synthesis in G and in fact locally p-Ditkin in G. We also obtain an extension theorem concerning the Fourier algebra.

1. INTRODUCTION

For a locally compact group G, let $CV_p(G)$ be the Banach algebra of all convolution operators of $L^p(G)$ where $1 < p < \infty$. In 1974 Lohoué [10] proved, for an amenable closed subgroup H of G (and G σ-compact), the existence of a projection of $CV_p(G)$ onto $CV_p(H)$. We obtain the existence of a projection $\mathcal{P}$ for the class of closed neutral subgroups. This class includes the following situations: (i) the normalizer of H in G is open in G; (ii) $G \in [SIN]_H$. We write $G \in [SIN]_H$ if there is a fundamental system of neighborhoods U of e in G such that $hUh^{-1} = U$ for every $h \in H$. The class $[SIN]_H$ has been thoroughly investigated by Henrichs [5].

The existence of such a $\mathcal{P}$ for H normal in G is already in [1]. In fact we are now able to show that $\operatorname{supp}\mathcal{P}(T) \subset \operatorname{supp} T$ and $\mathcal{P}(uT) = (\operatorname{Res}_H u)\mathcal{P}(T)$ for $u \in A_p(G)$ and $T \in CV_p(G)$. The existence of a $\mathcal{P}$ with these properties is new even for G abelian and $p \neq 2$; if $p = 2$ and G is abelian the result is due to C. Herz [6]. As a main application we prove that every closed neutral subgroup is a set of p-synthesis in G and locally p-Ditkin in G. A closed subset F of G is locally p-Ditkin in G (see [3], p. 102) if for every $\varepsilon > 0$ and every $u \in A_p(G) \cap C_{00}(G)$ with $\operatorname{Res}_F u = 0$ there is $v \in A_p(G) \cap C_{00}(G)$ with $\operatorname{supp} v \cap F = \emptyset$ and $||u - uv||_{A_p(G)} < \varepsilon$. F is p-Ditkin in G if for every $\varepsilon > 0$ and every $u \in A_p(G)$ with $\operatorname{Res}_F u = 0$ there is $v \in A_p(G) \cap C_{00}(G)$ with $\operatorname{supp} v \cap F = \emptyset$ and $||u - uv||_{A_p(G)} < \varepsilon$. The method used in the construction of $\mathcal{P}$ gives the following extension theorem concerning the Figà-Talamanca Herz algebra of G: given $u \in A_p(H) \cap C_{00}(H)$, $\varepsilon > 0$, and an open subset Ω of G with $\operatorname{supp} u \subset \Omega$, there exists $v \in A_p(G) \cap C_{00}(G)$ with $\operatorname{Res}_H v = u$, $||v||_{A_p(G)} \leq ||u||_{A_p(H)} + \varepsilon$ and $\operatorname{supp} v \subset \Omega$.

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2. Definitions and preliminaries

For the precise definitions of $CV_p(G)$, $PM_p(G)$, $A_p(G)$ and all the unexplained notations we refer to [1], p. 38. Let H be an arbitrary closed subgroup of G: for $T \in CV_p(H)$, $i(T)$ denotes the image of T under the inclusion i of H in G, as defined in [2], p. 76. We assume that G/H admits a G-invariant measure. For $k, \ell \in C_{00}(G)$ the relation

$$\langle \Lambda_{k,\ell}(T)\varphi, \psi\rangle_{L^p(H),L^{p'}(H)} = \langle T\tau_p(k *_H \tau_p\varphi), \tau_{p'}(\ell *_H \tau_{p'}\psi)\rangle_{L^p(G),L^{p'}(G)}$$

defines a linear continuous map $\Lambda_{k,\ell}$ from $CV_p(G)$ into $CV_p(H)$. We have $||\Lambda_{k,\ell}|| \leq ||T_H|k|\,||_p\, ||T_H|\ell|\,||_{p'}$ and $\Lambda_{k,\ell}(PM_p(G)) \subset PM_p(H)$ ([1], pp. 38–39).

The following unpublished result is due to Roelcke. Let H be a closed neutral subgroup of G ([12]). Then e admits a fundamental system of neighborhoods V with $HV = VH$. Indeed let U be an open neighborhood of e in G. There is an open W with $e \in W$, $W = W^{-1}$ and $HWH \subset UH$. Then for $V = (HWH) \cap U$ one has precisely $HV = VH$.

A one-sided version of the following lemma is already in [2], p. 71.

Lemma 1. *Let G be a locally compact group, H an arbitrary closed subgroup, U a neighborhood of e in G, and W a neighborhood of e in H. Then there exists $k \in C_{00}^+(G)$ with $\operatorname{supp} k \subset U$, $(\operatorname{supp} k) \cap H \subset W$ and*

$$\int_H k(h)dh = 1,\ \int_H k(xh)dh \leq 1,\ \int_H \Delta_H(h^{-1})k(hx)dh \leq 1$$

for every $x \in G$.

Proof. There is an open neighborhood U_1 of e in G with $U_1 \cap H \subset W$ and an open neighborhood U_2 of e in G with $U_2 = U_2^{-1}$ and $U_2 \subset U \cap U_1$. Let K be a compact neighborhood of e in G with $K = K^{-1}$ and $K \subset U_2$. We choose $\varphi' \in C_{00}^+(G)$ with $\varphi'(K) = \{1\}$ and $\operatorname{supp}\varphi \subset U_2$. Consider also $\psi' \in C_{00}^+(G/H)$ with $\psi' \leq 1_{G/H}$, $\psi'(H) = 1$ and $\operatorname{supp}\psi' \subset \omega(K)$. ($\omega$ is the canonical map from G to G/H.) Let

$$\varphi = \frac{\varphi' + \varphi'^{\vee}}{2} \quad (\varphi'^{\vee}(x) = \varphi'(x^{-1})) \text{ and } k'(x) = \frac{\varphi(x)\psi'(\omega(x))}{\int_H \varphi(xh)dh} \text{ for } x \in AH,$$

where $A = \{y \in G \mid \varphi(y) > 0\}$ and $k'(x) = 0$ if $x \in G\backslash AH$. Taking into account that $G = AH \cup (G\backslash HK)$, we obtain that $k' \in C_{00}^+(G)$, $\operatorname{supp} k' \subset \operatorname{supp}\varphi$ and $\int_H k'(xh)dh \leq 1$ for every $x \in G$.

Now let ω' be the canonical map from G onto $H\backslash G$. We similarly choose $\psi'' \in C_{00}^+(H\backslash G)$ with $\psi'' \leq 1_{H\backslash G}$, $\psi''(H) = 1$, $\operatorname{supp}\psi'' \subset \omega'(K)$ and

$$k''(x) = \frac{\varphi(x)\psi''(\omega'(x))}{\int_H \Delta_H(h^{-1})\varphi(hx)dh} \quad \text{for } x \in HA,$$

$k''(x) = 0$ for $x \in G \setminus HA$. We obtain that $k = \min\{k', k''\}$ satisfies all the required properties.

3. Projections of $CV_p(G)$ onto $CV_p(H)$

According to Leischner and Roelcke [9] H is said to be locally neutral in G if there is a compact neighborhood U_0 of e such that for every neighborhood U of e there is a neighborhood V of e with $(HVH) \cap U_0 \subset UH$.

Proposition 2. *Let G be a locally compact group and H a closed subgroup locally neutral in G. We assume that G/H admits an invariant measure. Let $(r_n^{(j)})_{n=1}^{\infty}$ be m sequences of $L^p(H)$ and $(s_n^{(j)})_{n=1}^{\infty}$ be m sequences of $L^{p'}(H)$. We assume that for every $1 \leq j \leq m$, $\sum_{n=1}^{\infty} ||r_n^{(j)}||_p ||s_n^{(j)}||_{p'} < \infty$. Then, for every $\varepsilon > 0$ and every open neighborhood U, there is $k, \ell \in C_{00}^{+}(G)$ with $\operatorname{supp} k$, $\operatorname{supp} \ell \subset U$, $||\Lambda_{k,\ell}|| \leq 1$ and*

$$\sum_{n=1}^{\infty} \left| \langle \Lambda_{k,\ell}(i(S)) r_n^{(j)}, s_n^{(j)} \rangle_{L^p(H), L^{p'}(H)} - \langle S r_n^{(j)}, s_n^{(j)} \rangle_{L^p(H), L^{p'}(H)} \right| \leq \varepsilon |||S|||_p$$

for every $S \in CV_p(H)$ and every $1 \leq j \leq m$.

Proof. To avoid unessential technical difficulties, we suppose $m = 1$ and $r_n^{(1)}, s_n^{(1)} \in C_{00}(H)$ for every $n \in \mathbb{N}$. We put $r_n = r_n^{(1)}$ and $s_n = s_n^{(1)}$. There is $N \in \mathbb{N}$ such that $\sum_{n=1+N}^{\infty} ||r_n||_p ||s_n||_{p'} < \frac{\varepsilon_1}{8}$ where $0 < \varepsilon_1 < \min\{1, \varepsilon\}$. We can find a relatively compact neighborhood V of e in H with $||r_n - (r_n)_{h^{-1}} \Delta_H(h^{-1})||_p$, $||s_n - (s_n)_{h^{-1}} \Delta_H(h^{-1})||_{p'} < \frac{\varepsilon_2}{2}$ for every $h \in V$ and every $1 \leq n \leq N$. We have chosen ε_2 such that

$$0 < \varepsilon_2 < \min \left\{ \frac{\varepsilon_1}{2^{j+2}(1 + ||r_j||_p + ||s_j||_{p'})} \,\middle|\, 1 \leq j \leq N \right\}.$$

Let U_0 be a compact neighborhood of e in G with $U_0 = U_0^{-1}$ which guarantees the local neutrality of H in G: for every neighborhood U' of e in G, there is a neighborhood V' of e in G with $U_0 \cap (HV'H) \subset U'H$.

Let

$$0 < \varepsilon_3 < \frac{\varepsilon_1}{40(1 + \sum_{n=1}^{\infty} ||r_n||_p \, ||s_n||_{p'})}.$$

There is a compact neighborhood U_1 of e in G with $\Delta_G(x) < 1 + \varepsilon_3$ for every $x \in U_1$. We choose an open neighborhood U_2 of e in G with $U_2^2 \subset U_1$. According to Lemma 1, there is $k' \in C_{00}^{+}(G)$ with $\int_H k'(h) dh = 1$, $\int_H k'(hx) \Delta_H(h^{-1}) dh \leq 1$ for every $x \in G$, $\operatorname{supp} k' \subset U_0 \cap U_2 \cap U^{-1}$ and $\operatorname{supp} k' \cap H \subset V$. This implies, for every $1 \leq n \leq N$, $||r_n - \operatorname{Res}_H(r_n *_H k')||_p \leq \frac{\varepsilon_2}{2}$ and $||s_n - \operatorname{Res}_H(s_n *_H k')||_{p'} \leq \frac{\varepsilon_2}{2}$. There is a relatively compact open neighborhood U_3 of e in G such that, for every $x \in U_3$ and every $1 \leq n \leq N$, $||(r_n *_H k')_{x,H} - \operatorname{Res}_H(r_n *_H k')||_p < \frac{\varepsilon_2}{2}$ and $||(s_n *_H k')_{x,H} - \operatorname{Res}_H(s_n *_H k')||_{p'} < \frac{\varepsilon_2}{2}$. (We recall that, for a function f on G and for $x \in G$, $f_{x,H}$ denotes the function defined on H by $f_{x,H}(h) = f(xh)$.)

Now let A be an arbitrary open neighborhood of e in G with $A \subset U_3$. Then, for every $S \in CV_p(H)$ and every $1 \leq n \leq N$, we have

$$\begin{aligned} &\frac{\langle i(S)1_{AH}(r_n *_H k'), 1_{AH}(s_n *_H k')\rangle_{L^p(G),L^{p'}(G)}}{\dot{m}(\omega(A))} - \langle Sr_n, s_n\rangle_{L^p(H),L^{p'}(H)} \\ &\quad = \frac{1}{\dot{m}(\omega(A))}\int_G 1_{AH}(x)\beta(x)\Big(\langle S(r_n *_H k')_{x,H}, (s_n *_H k')_{x,H}\rangle_{L^p(H),L^{p'}(H)} \\ &\qquad\qquad - \langle Sr_n, s_n\rangle_{L^p(H),L^{p'}(H)}\Big)\,dx, \end{aligned}$$

where $\dot{m}(\omega(A)) = \int_{\omega(A)} d\dot{x}$ and β is a Bruhat function for H, G.

Taking into account that we have, for $x = uh$ with $u \in A$, $h \in H$,

$$\begin{aligned} &\langle S(r_n *_H k')_{x,H}, (s_n *_H k')_{x,H}\rangle_{L^p(H),L^{p'}(H)} \\ &\quad = \langle S(r_n *_H k')_{u,H}, (s_n *_H k')_{u,H}\rangle_{L^p(H),L^{p'}(H)}, \end{aligned}$$

we obtain

$$\begin{aligned} &\left|\frac{\langle i(S)1_{AH}(r_n *_H k'), 1_{AH}(s_n *_H k')\rangle_{L^p(G),L^{p'}(G)}}{\dot{m}(\omega(A))} - \langle Sr_n, s_n\rangle_{L^p(H),L^{p'}(H)}\right| \\ &\quad \leq |||S|||_p\,\varepsilon_2(1 + ||r_n||_p + ||s_n||_{p'}) \end{aligned}$$

for every $1 \leq n \leq N$.

Let K be a compact subset of H with $e \in K$ and $\operatorname{supp} r_n \cup \operatorname{supp} s_n \subset K$ for every $1 \leq n \leq N$. There is an open neighborhood U_4 of e in G such that $U_4^{-1} = U_4$ and $U_4 \subset U_2 \cap U_3$. There is an open neighborhood U_5 of e in G such that $kU_5k^{-1} \subset U_4$ for every $k \in K$. By assumption we can find an open neighborhood U_6 of e in G such that $(HU_6H) \cap U_0 \subset U_5H$. This implies $(HU_6H) \cap KU_0H \subset U_4H$. Consider $U_8 = (HU_7H) \cap U_4$ where U_7 is an open neighborhood of e in G with $U_7^{-1} = U_7$ and $U_7 \subset U_6$. Roelcke's argument gives $(KU_0K^{-1}) \cap U_8H = (KU_0K^{-1}) \cap HU_8$. If we take into account that $\operatorname{supp}(r_n *_H k')$, $\operatorname{supp}(s_n *_H k') \subset KU_0K^{-1}$ $(1 \leq n \leq N)$, we obtain, for $1 \leq n \leq N$,

$$1_{U_8H}(r_n *_H k') = 1_{HU_8}(r_n *_H k') \quad\text{and}\quad 1_{U_8H}(s_n *_H k') = 1_{HU_8}(s_n *_H k').$$

Let

$$k'' = \frac{\tau_p(1_{HU_8}k')}{\dot{m}(\omega(U_8))^{1/p}} \quad\text{and}\quad \ell'' = \frac{\tau_{p'}(1_{HU_8}k')}{\dot{m}(\omega(U_8))^{1/p'}}.$$

Then, for $1 \leq n \leq N$,

$$\begin{aligned} &\langle \Lambda_{k'',\ell''}(i(S))r_n, s_n\rangle_{L^p(H),L^{p'}(H)} \\ &\quad = \frac{\langle i(S)1_{U_8H}(r_n *_H k'), 1_{U_8H}(s_n *_H k')\rangle_{L^p(G),L^{p'}(G)}}{\dot{m}(\omega(U_8))}. \end{aligned}$$

From the inequality

$$\sum_{n=1}^{\infty} \left|\langle Sr_n, s_n\rangle_{L^p(H),L^{p'}(H)} - \langle \Lambda_{k'',\ell''}(i(S))r_n, s_n\rangle_{L^p(H),L^{p'}(H)}\right|$$
$$\leq \sum_{n=1}^{N} |||S|||_p\, \varepsilon_2(1+||r_n||_p+||s_n||_{p'}) + \sum_{n=N+1}^{\infty} \left|\langle Sr_n, s_n\rangle_{L^p(H),L^{p'}(H)}\right|$$
$$+ \sum_{n=N+1}^{\infty} \left|\langle \Lambda_{k'',\ell''}(i(S))r_n, s_n\rangle_{L^p(H),L^{p'}(H)}\right|,$$

we get

$$\sum_{n=1}^{\infty} \left|\langle Sr_n, s_n\rangle_{L^p(H),L^{p'}(H)} - \langle \Lambda_{k'',\ell''}(i(S))r_n, s_n\rangle_{L^p(H),L^{p'}(H)}\right|$$
$$\leq \frac{3\varepsilon_1|||S|||_p}{8} + \frac{\varepsilon_1|||S|||_p||T_H k''||_p||T_H\ell''||_{p'}}{8}.$$

To estimate $||T_H k''||_p||T_H\ell''||_{p'}$, observe that, for $x = uh'$ with $u \in U_8$, $h' \in H$, $\int_H (\tau_p k')(xh)dh = \int_H k'(hu^{-1})\Delta_G(hu^{-1})^{1/p}\Delta_H(h^{-1})dh$. But $u^{-1} \in U_1$ implies

$$\int_H (\tau_p k')(xh)dh \leq (1+\varepsilon_3)^{1/p} \int_H k'(hu^{-1})\Delta_G(h)^{1/p}\Delta_H(h^{-1})dh.$$

From $\int_H k'(hu^{-1})\Delta_G(h)^{1/p}\Delta_H(h^{-1})dh = \int_{U_1\cap H} k'(hu^{-1})\Delta_G(h)^{1/p}\Delta_H(h^{-1})dh$, we therefore get $||T_H k''||_p \leq (1+\varepsilon_3)^{2/p}$ and similarly $||T_H \ell''||_{p'} \leq (1+\varepsilon_3)^{2/p'}$. This gives, for every $S \in CV_p(H)$,

$$\sum_{n=1}^{\infty} \left|\langle Sr_n, s_n\rangle_{L^p(H),L^{p'}(H)} - \langle \Lambda_{k'',\ell''}(i(S))r_n, s_n\rangle_{L^p(H),L^{p'}(H)}\right| \leq \frac{7\varepsilon_1}{8}|||S|||_p.$$

Consider now $f, g \in C_{00}^+(G/H)$ with $\left|\left|f - \frac{1_{\omega(U_8)}}{\dot m(\omega(U_8))^{1/p}}\right|\right|_p$, $\left|\left|g - \frac{1_{\omega(U_8)}}{\dot m(\omega(U_8))^{1/p'}}\right|\right|_{p'}$ both smaller than $\dfrac{\varepsilon_4}{(1+||T_H\tau_p k'||_\infty)(1+||T_H\tau_{p'}k'||_\infty)}$, where

$$0 < \varepsilon_4 < \frac{\varepsilon_3}{(1+2^{2/p}+2^{2/p'})\left(1+\sum_{n=1}^{\infty}||r_n||_p||s_n||_{p'}\right)}.$$

Putting $k''' = (f\circ\omega)\,\tau_p k'$ and $\ell''' = (g\circ\omega)\,\tau_{p'}k'$, we have successively

$$||T_H(|k'''-k''|)\,||_p, ||T_H(|\ell'''-\ell''|)\,||_{p'} < \varepsilon_4$$

$||T_H k'''\,||_p < \varepsilon_4 + (1+\varepsilon_3)^{2/p}$ and $||T_H \ell'''\,||_{p'} < \varepsilon_4 + (1+\varepsilon_3)^{2/p'}$. Finally it suffices to choose $k = \dfrac{k'''}{\varepsilon_4 + (1+\varepsilon_3)^{2/p}}$ and $\ell = \dfrac{\ell'''}{\varepsilon_4+(1+\varepsilon_3)^{2/p'}}$ to obtain

$$\sum_{n=1}^{\infty} \left|\langle Sr_n, s_n\rangle_{L^p(H),L^{p'}(H)} - \langle \Lambda_{k,\ell}(i(S))r_n, s_n\rangle_{L^p(H),L^{p'}(H)}\right| \leq \varepsilon|||S|||_p$$

with $||\Lambda_{k,\ell}|| \leq 1$ and $\operatorname{supp} k,\ \operatorname{supp}\ell \subset U$.

We are now ready to prove our first result concerning projections of $CV_p(G)$ onto $CV_p(H)$.

Theorem 3. *Let G be a locally compact group and H a closed subgroup, locally neutral in G. We assume that G/H admits an invariant measure. Then there is a linear contraction Q from $\mathcal{L}(L^p(G))$ (the Banach space of all bounded operators of $L^p(G)$) into $\mathcal{L}(L^p(H))$ such that:*

(1) *$Q(T) \in CV_p(H)$ for every $T \in CV_p(G)$,*
(2) *$\operatorname{supp} Q(T) \subset \operatorname{supp} T$ for every $T \in CV_p(G)$,*
(3) *$Q(i(S)) = S$ for every $S \in CV_p(H)$,*
(4) *$Q(T) \in PM_p(H)$ for every $T \in PM_p(G)$.*

Proof. Let $\mathcal{A}$ be the set of all pairs $\big((r_n)_{n=1}^\infty, (s_n)_{n=1}^\infty\big)$ where $(r_n)_{n=1}^\infty$ is a sequence of $L^p(H)$ and $(s_n)_{n=1}^\infty$ is a sequence of $L^{p'}(H)$ with $\sum_{n=1}^\infty ||r_n||_p ||s_n||_{p'} < \infty$. We denote by $\mathcal{E}$ the set of all maps F from $\mathcal{L}(L^p(G)) \times L^p(H) \times L^{p'}(H)$ to $\mathbb{C}$, linear in the first two variables, conjugate linear in the third one, and for which there is a positive real number C with $|F(T, \varphi, \psi)| \le C |||T|||_p ||\varphi||_p ||\psi||_{p'}$. For $F \in \mathcal{E}$ we put $||F|| = \sup\{|F(T,\varphi,\psi)| \mid |||T|||_p \le 1, ||\varphi||_p \le 1, ||\psi||_{p'} \le 1\}$. For $k, \ell \in C_{00}(G)$, $F_{k,\ell}(T,\varphi,\psi) = \langle \Lambda_{k,\ell}(T)\varphi, \psi\rangle_{L^p(H), L^{p'}(H)}$ is an element of $\mathcal{E}$ with $||F_{k,\ell}|| \le ||T_H|k|\,||_p \,||T_H|\ell|\,||_{p'}$.

Let A be a finite subset of $\mathcal{A}$, B a finite subset of $CV_p(H)$, U an open neighborhood of e in G and $\varepsilon > 0$. Proposition 2 implies precisely that the set

$$K_{A,B,U,\varepsilon} = \Bigg\{ F_{k,\ell} \;\Bigg|\; k, \ell \in C_{00}^+(G) \;\; ||F_{k,\ell}|| \le 1,\ \operatorname{supp} k,\ \operatorname{supp} \ell \subset U,$$
$$\sum_{n=1}^\infty \big| F_{k,\ell}(i(S), r_n, s_n) - \langle S r_n, s_n\rangle_{L^p(H), L^{p'}(H)} \big| < \varepsilon$$
$$\text{for every } \big((r_n)_{n=1}^\infty, (s_n)_{n=1}^\infty\big) \in A \text{ and every } S \in B \Bigg\}$$

is nonempty. Let $\overline{K}_{A,B,U,\varepsilon}$ be the closure of $K_{A,B,U,\varepsilon}$ with respect to the topology $\sigma(\mathcal{E}, \mathcal{L}(L^p(G)) \times L^p(H) \times L^{p'}(H))$. The set $\cap\{\overline{K}_{A,B,U,\varepsilon} \mid A$ finite subset of $\mathcal{A}$, B finite subset of $CV_p(H)$, $0 < \varepsilon < 1$, U open neighborhood of e in $G\}$ is not empty. Choose J in this set. There is a linear map Q from $\mathcal{L}(L^p(G))$ to $\mathcal{L}(L^p(H))$ with $J(T,\varphi,\psi) = \langle Q(T)\varphi, \psi\rangle_{L^p(H), L^{p'}(H)}$ for $T \in \mathcal{L}(L^p(G))$, $\varphi \in L^p(H)$, $\psi \in L^{p'}(H)$. Clearly Q satisfies conditions (1) to (4).

Let H be a closed subgroup of G for which there is a linear map Q from $CV_p(G)$ onto $CV_p(H)$ satisfying conditions (3) and (4) of Theorem 3. Then H is a set of p-synthesis in G. Indeed let $T \in PM_p(G)$ with $\operatorname{supp} T \subset H$ and $u \in A_p(G)$ with $\operatorname{Res}_H u = 0$. According to Lohoué ([11], Théorème 5, p. 190), there is an $S \in CV_p(H)$ with $i(S) = T$. From $Q(i(S)) \in PM_p(H)$ we deduce that $S \in PM_p(H)$ and therefore $\langle u, T\rangle_{A_p(G), PM_p(G)} = \langle \operatorname{Res}_H u, S\rangle_{A_p(H), PM_p(H)} = 0$.

Corollary 4. *Let G be a locally compact group and H a closed subgroup, locally neutral in G, for which G/H admits an invariant measure. Then H is a set of p-synthesis of G.*

The following extension theorem was proved by C. Herz for G second countable and H normal in G ([7], p. 115).

Corollary 5. *Let G be a locally compact group and H a closed subgroup as in Theorem 3. Given $u \in A_p(H) \cap C_{00}(H)$, $\varepsilon > 0$ and an open subset Ω of G with $\operatorname{supp} u \subset \Omega$, there exists $v \in A_p(G) \cap C_{00}(G)$ with $\operatorname{Res}_H v = u$, $||v||_{A_p(G)} \leq ||u||_{A_p(H)} + \varepsilon$ and $\operatorname{supp} v \subset \Omega$.*

Proof. According to [7], p. 115, it suffices to find $v \in A_p(G) \cap C_{00}(G)$ with $\operatorname{supp} v \subset \Omega$, $||v||_{A_p(G)} \leq ||u||_{A_p(H)}$ and $||u - \operatorname{Res}_H v||_{A_p(H)} < \varepsilon$.

There is $(r_n)_{n=1}^\infty$ and $(s_n)_{n=1}^\infty$, two sequences of $C_{00}(H)$, such that $u = \sum_{n=1}^\infty \overline{r}_n * \check{s}_n$ and $\sum_{n=1}^\infty ||r_n||_{L^p(H)} ||s_n||_{L^{p'}(H)} < \infty$. There also exists an open neighborhood U of e in G such that $U \operatorname{supp} u\, U^{-1} \subset \Omega$. There is $k, \ell \in C_{00}^+(G)$ with $\operatorname{supp} k,\ \operatorname{supp} \ell \subset U$, $||\Lambda_{k,\ell}|| \leq 1$ and

$$\sum_{n=1}^\infty \left| \langle \Lambda_{k,\ell}(i(S)) \tau_p r_n, \tau_{p'} s_n \rangle_{L^p(G), L^{p'}(G)} - \langle S \tau_p r_n, \tau_{p'} s_n \rangle_{L^p(H), L^{p'}(H)} \right| \leq \varepsilon |||S|||_p$$

for every $S \in CV_p(H)$. There exists a unique $v \in A_p(G)$ such that

$$\langle v, i(S) \rangle_{A_p(G), PM_p(G)} = \langle u, \Lambda_{k,l}(S) \rangle_{A_p(H), PM_p(H)}$$

for every $S \in PM_p(H)$. From

$$\begin{aligned} &\left| \langle u, S \rangle_{A_p(H), PM_p(H)} - \langle \operatorname{Res}_H v, S \rangle_{A_p(H), PM_p(H)} \right| \\ \leq &\sum_{n=1}^\infty \left| \langle S \tau_p r_n, \tau_{p'} s_n \rangle_{L^p(H), L^{p'}(H)} - \langle \Lambda_{k,l}(i(S)) \tau_p r_n, \tau_{p'} s_n \rangle_{L^p(G), L^{p'}(G)} \right| \leq \varepsilon |||S|||_p \end{aligned}$$

we get $||u - \operatorname{Res}_H v||_{A_p(H)} \leq \varepsilon$ with $\operatorname{supp} v \subset \operatorname{supp} k\ \operatorname{supp} u\ (\operatorname{supp} \ell)^{-1}$.

Remark. Suppose G is abelian. According to J. Inoue [8] for every neighborhood U of e in G there is a linear isometric map Ω of $A_2(H)$ into $A_2(G)$ with $\operatorname{Res}_H \circ \Omega = \operatorname{id}_{A_2(H)}$ (such a map is called a linear lifting) and $\operatorname{supp} \Omega(u) \subset (\operatorname{supp} u) U$. By duality we easily derive the existence of a projection of $PM_2(G)$ onto $PM_2(H)$ as in Theorem 3. On the other hand B. Forrest [4] has shown that for G amenable and H closed abelian normal subgroup of G a linear lifting does not always exist. Consequently the map Q of Theorem 3 can be considered as a substitute, for G nonabelian, to the nonexistence of linear liftings of $A_2(H)$ into $A_2(G)$.

4. Invariant projections

In [1] we proved for H normal in G the existence of a projection $\mathcal{P}$ of $CV_p(G)$ onto $CV_p(H)$ satisfying the condition $\mathcal{P}(uT) = (\operatorname{Res}_H u)\mathcal{P}(T)$; however condition (2) of Theorem 3 was out of our reach. A projection $\mathcal{P}$ with $\mathcal{P}(uT) = (\operatorname{Res}_H u)\mathcal{P}(T)$ will be called an invariant projection.

Theorem 6. *Let G be a locally compact group and H a closed neutral subgroup of G. There is an invariant projection $\mathcal{P}$ of $CV_p(G)$ onto $CV_P(H)$ which also satisfies all conditions of Theorem 3.*

Proof. Let X be the set of all maps f from $\mathcal{L}(L^p(G)) \times L^p(G) \times L^{p'}(G)$ to $\mathbb{C}$ which are linear in the first two variables and conjugate linear in the third one and for which there is a positive real number C with $|\ f(T, \varphi, \psi)\ | \leq C |||T|||_p ||\varphi||_p ||\psi||_{p'}$.

For $a, b \in S(G/H)$ ($S(G/H)$ is the set of all bounded mesurable functions on G/H with compact support), $T \in \mathcal{L}(L^p(G))$, $\varphi, \psi \in C_{00}(G)$ we define

$$g_{a,b}(T,\varphi,\psi) = \int_{G/H} \langle TB(\varphi,a)(\dot{t}), B(\psi,b)(\dot{t})\rangle_{L^p(G),L^{p'}(G)} d\dot{t}$$

with $B(\varphi,a)(\dot{t})(x) = \varphi(x)a(x^{-1}\dot{t})$ for $x, t \in G$. Let K be a compact subset of G with $K \cap H = \emptyset$ and $\varepsilon > 0$. At first we show that the set $D_{K,\varepsilon} = \Big\{ g_{a,b} \Big| a, b \in S(G/H),\ a, b \geq 0, ||a||_p ||b||_{p'} < 1+\varepsilon$, there is an open neighborhood U of e in G with $\mathrm{Res}_{HU}\, v = 1$, $\operatorname{supp} v \cap KU = \emptyset$ where $v(x) = \int_{G/H} a(x\dot{y})b(\dot{y})d\dot{y} \Big\}$ is nonempty. We indeed choose an open neighborhood U_1 of e in G with $U_1 = U_1^{-1}$ and $KU_1 \cap HU_1^2 = \emptyset$. There is an open neighborhood U_2 of e in G with $U_2^{-1} = U_2$, $U_2 \subset U_1$ and $U_2H = HU_2$. Let U_3 be an open neighborhood of e in G relatively compact with $\overline{U}_3 \subset U_2$ and $U_3H = HU_3$. This implies that $\overline{U}_3 H = H\overline{U}_3$. There is an open subset $\dot{U}_4$ of G/H with $\dot{U}_4 \supset \omega(\overline{U}_3)$ and $\dot{m}(\dot{U}_4 - \omega(\overline{U}_3)) < ((1+\varepsilon)^{p'} - 1)\dot{m}(\omega(\overline{U}_3))$. Let $U_5 = \omega^{-1}(\dot{U}_4) \cap U_2$. We consider an open neighborhood U_6 of e with $U_6^{-1} = U_6$ and $U_6\overline{U}_3 \subset U_5$. Let $v(x) = \int_{G/H} a(x\dot{y})b(\dot{y})d\dot{y}$ with $a = \dfrac{1_{\omega(\overline{U}_3)}}{\dot{m}(\omega(\overline{U}_3))}$ and $b = 1_{\omega(U_6\overline{U}_3)}$. We have

$$||a||_p ||b||_{p'} \leq \left(\frac{\dot{m}(\omega(U_5))}{\dot{m}(\omega(\overline{U}_3))} \right)^{\frac{1}{p'}} < 1 + \varepsilon.$$

Suppose $v(x) \neq 0$; there is $\dot{y} \in G/H$ with $x\dot{y} \in \omega(\overline{U}_3)$ and $\dot{y} \in \omega(U_6\overline{U}_3)$. This implies $x \in \overline{U}_3 H U_5^{-1}$ and consequently $x \in HU_1^2$. Let $x \in HU_6$ for every $\dot{y} \in \omega(\overline{U}_3)$ $x^{-1}\dot{y} \in \omega(U_6 H \overline{U}_3)$ but $\omega(U_6 H \overline{U}_3) = \omega(U_6\overline{U}_3)$; this implies precisely $v(x) = 1$. Let g be an element of $\cap\{\overline{D}_{K,\varepsilon} \mid K$ compact subset of G with $K \cap H = \emptyset,\ 0 < \varepsilon < 1\}$ where $\overline{D}_{K,\varepsilon}$ is the closure of $D_{K,\varepsilon}$ in X with respect to the topology $\sigma(X, \mathcal{L}(L^p(G)) \times L^p(G) \times L^{p'}(G))$. Let P be the corresponding map of $\mathcal{L}(L^p(G))$ to itself. It suffices to consider $Q \circ P$ where Q is the map of Theorem 3.

Corollary 7. *Let G be a locally compact group, H a closed neutral subgroup of G and F a closed subset of H. If F is locally p-Ditkin* ($1 < p < \infty$) *with respect to H, then F is locally p-Ditkin with respect to G.*

Proof. Let $u \in A_p(G)$, $T \in CV_p(G)$ with $\mathrm{Res}_F\, u = 0$ and $\operatorname{supp}(uT) \subset F$. It suffices to show that $uT = 0$. There is $S \in CV_p(H)$ such that $uT = i(S)$. Let $\mathcal{P}$ be the invariant projection of Theorem 6. From $\operatorname{supp} \mathcal{P}(uT) \subset \operatorname{supp} uT$ and $\mathcal{P}(uT) = (\mathrm{Res}_H\, u)\mathcal{P}(T) = S$ we deduce that $S = 0$ and therefore $uT = 0$.

Remark. For H normal in G, this corollary is already in [3], p. 103. Our proof there was completely different: it was based on the use of $A_p(G/H)$. The present approach is not only more conceptual but permits us to treat the case of certain interesting nonnormal subgroups: H compact or $G \in [SIN]_H$.

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Institut de Mathématiques, Faculté des Sciences, Université de Lausanne, CH-1015 Lausanne-Dorigny, Switzerland

E-mail address: jdelaporte@mail.vtx.ch

Institut de Mathématiques, Faculté des Sciences, Université de Lausanne, CH-1015 Lausanne-Dorigny, Switzerland

E-mail address: antoine.derighetti@ima.unil.ch