ON ABSORBING EXTENSIONS

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Abstract. Building on the work of Kasparov we show that there always exists a trivial absorbing extension of $A$ by $B \otimes K$, provided only that $A$ and $B$ are separable. If $A$ is unital there is a unital trivial extension which is unitally absorbing.

1. Introduction

Absorbing trivial extensions play an important role in the theory of extensions of $C^*$-algebras; cf. 15.12 in [B]. Recently the interest in such extensions has been renewed because of the way $KK$-theory comes into the classification program. In this connection, as well as in the proper theory of $C^*$-extensions, it is slightly disturbing that the existence of an absorbing trivial extension has only been established in the case where at least one of the $C^*$-algebras involved is nuclear; cf. Theorem 5 of [K]. The purpose of the present note is to show that such extensions always exist when both $C^*$-algebras are separable. The argument for this is a modification of Kasparov’s approach from [K]. The absorbing trivial extensions were constructed, in [K] as well as before Kasparov’s work, by taking the infinite direct sum of the same copy of a faithful unital representation of the separable $C^*$-algebra $A$ (for the moment assumed to be unital) which plays the role of the quotient in the extensions. The resulting representation $\pi : A \to \mathcal{B}(\mathcal{H})$ was then composed with the natural imbedding $\mathcal{B}(\mathcal{H}) \subseteq \mathcal{M}(B \otimes K)$, where $B \otimes K$ is the stable $C^*$-algebra which features as the ideal in the extensions. So in practice this means that the absorbing extension was constructed by taking a weak$^*$ dense sequence of states of $A$, repeating all states in the sequence infinitely often, and then adding the corresponding GNS-representations. This procedure has nothing to do with the $C^*$-algebra $B$, and it is a highly non-trivial task to show that it often results in an absorbing extension when prolonged to a map $A \to \mathcal{M}(B \otimes K)$; cf. [K]. The observation we offer here is that if one instead takes a sequence $s_n : A \to B \otimes K$ of completely positive contractions which is dense for the topology of pointwise norm-convergence among all completely positive contractions (such a sequence exists when both $A$ and $B$ are separable), repeats each $s_n$ infinitely often and add up the unital representations

$$\pi_n : A \to \mathcal{M}(B \otimes K) , \ n \in \mathbb{N} ,$$

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coming from the Kasparov-Stinespring decompositions
\[ s_n(\cdot) = W_n^* \pi_n(\cdot) W_n , \]
the resulting representation \( A \to \mathcal{M}(B \otimes K) \) will be a unitally absorbing trivial extension. The general trivial absorbing extensions can then be obtained (for a not necessarily unital \( C^* \)-algebra \( A \)) by taking a unitally absorbing representation \( \pi : A^+ \to \mathcal{M}(B \otimes K) \) and restricting it to \( A \).

In order to illustrate how the absorbing \(*\)-homomorphisms constructed here can be used to extend known results we prove a general version of the Paschke-Valette-Skandalis duality which realizes the group \( KK(A, B) \) as the \( K_1 \)-group of a \( C^* \)-algebra \( D_n \) built out of \( A \) and \( B \) by using an absorbing \(*\)-homomorphism \( \pi : A \to \mathcal{M}(B) \); cf. \([P, V, S, H]\).

2. Absorbing \(*\)-Homomorphisms

Given Hilbert \( B \)-modules \( E \) and \( F \), we let \( \mathcal{L}_B(E, F) \) denote the Banach space of adjointable operators from \( E \) to \( F \). The ideal of ‘compact’ operators from \( E \) to \( F \) is denoted by \( \mathcal{K}_B(E, F) \). When \( E = F \) we write \( \mathcal{L}_B(E) \) and \( \mathcal{K}_B(E) \) instead of \( \mathcal{L}_B(E, E) \) and \( \mathcal{K}_B(E, E) \), respectively. In the special case where \( E = B \) there are well-known identifications \( \mathcal{L}_B(B) = \mathcal{M}(B) = \) the multiplier algebra of \( B \) and \( \mathcal{K}_B(B) = B \), which we shall use freely.

**Theorem 2.1.** Let \( A \) and \( B \) be separable \( C^* \)-algebras with \( A \) unital and \( B \) stable. Let \( \pi : A \to \mathcal{M}(B) \) be a unital \(*\)-homomorphism. Then the following conditions are equivalent:

1) For any completely positive contraction \( \varphi : A \to B \) there is a sequence \( \{W_n\} \subseteq \mathcal{M}(B) \) such that
   1a) \( \lim_{n \to \infty} \|\varphi(a) - W_n^* \pi(a) W_n\| = 0 \) for all \( a \in A \),
   1b) \( \lim_{n \to \infty} \|W_n^* b\| = 0 \) for all \( b \in B \).
2) For any completely positive unital map \( \varphi : A \to \mathcal{M}(B) \) there is a sequence \( \{V_n\} \) of isometries in \( \mathcal{M}(B) \) such that
   2a) \( V_n^* \pi(a) V_n - \varphi(a) \in B \), \( n \in \mathbb{N} \), \( a \in A \),
   2b) \( \lim_{n \to \infty} \|V_n^* \pi(a) V_n - \varphi(a)\| = 0 \), \( a \in A \).
3) For any unital \(*\)-homomorphism \( \varphi : A \to \mathcal{M}(B) \) there is a sequence \( \{U_n\} \) of unitaries \( U_n \in \mathcal{L}_B(B \oplus B, B) \) such that
   3a) \( U_n \begin{pmatrix} \pi(a) & 0 \\ 0 & \varphi(a) \end{pmatrix} U_n^* - \pi(a) \in B \), \( n \in \mathbb{N} \), \( a \in A \),
   3b) \( \lim_{n \to \infty} \|U_n \begin{pmatrix} \pi(a) & 0 \\ 0 & \varphi(a) \end{pmatrix} U_n^* - \pi(a)\| = 0 \), \( a \in A \).
4) For any unital \(*\)-homomorphism \( \varphi : A \to \mathcal{M}(B) \) there is a sequence \( \{U_n\} \) of unitaries \( U_n \in \mathcal{L}_B(B \oplus B, B) \) such that
   \( \lim_{n \to \infty} \|U_n \begin{pmatrix} \pi(a) & 0 \\ 0 & \varphi(a) \end{pmatrix} U_n^* - \pi(a)\| = 0 \), \( a \in A \).

**Proof.** 1) \( \Rightarrow \) 2): Let \( F \subseteq A \) be a finite set containing 1 and \( \epsilon > 0 \). Let \( \varphi : A \to \mathcal{M}(B) \) be a completely positive unital map. It suffices to find an element \( V \in \mathcal{M}(B) \) such that

\[ V^* \pi(a) V - \varphi(a) \in B \]
for all $a \in A$ and

\begin{equation}
\|V^*\pi(x)V - \varphi(x)\| < 3\epsilon
\end{equation}

for all $x \in F$. If namely $\epsilon$ is small enough this will imply that $W = V[V^*V]^{-\frac{1}{2}}$ is an isometry close to $V$ such that $V - W \in B$, and we can then work with $W$ instead of $V$. We repeat Kasparov’s arguments: Let $X$ be a compact subset of $A$ containing $F$ and with dense span in $A$. By Lemma 10 of [K] there is a sequence $\psi_k : A \to B$, $k \in \mathbb{N}$, of completely positive contractions such that $\psi(a) = \sum_{k=1}^{\infty} \psi_k(a)$ converges in the strict topology, $\varphi(a) - \psi(a) \in B$ for all $a \in A$, and $\|\varphi(x) - \psi(x)\| < \epsilon$ for all $x \in X$. Let $\{b_i\}$ be a countable approximate unit for $B$. It follows from 1) that we can find a sequence $\{m_i\} \subseteq B$ such that

1) $\|\psi_i(x) - m_i^*\pi(x)m_i\| \leq \epsilon 2^{-i}$, $x \in X$, $i \in \mathbb{N}$,

2) $\|m_i^*\pi(x)m_j\| \leq \epsilon 2^{-i-j}$, $x \in X$, $i, j \in \mathbb{N}$, $i \neq j$,

3) $\sum_{i=1}^{\infty} \|m_i^*b_k\| < \infty$ for all $k \in \mathbb{N}$.

The argument from the proof of Theorem 5 in [K] shows that $\sum_{i=1}^{\infty} m_i$ converges in the strict topology to an element $V \in \mathcal{M}(B)$ satisfying (2.1) and (2.2).

2) $\Rightarrow$ 3): This follows from the arguments of Arveson given on pp. 338-339 of [A] by substituting Hilbert spaces with Hilbert $B$-modules. We leave this to the reader.

3) $\Rightarrow$ 4) is trivial.

4) $\Rightarrow$ 1): Let $\varphi : A \to B$ be a completely positive contraction. Let $F \subseteq A$ and $G \subseteq B$ be finite sets and $\epsilon > 0$. Since $A$ and $B$ are separable it suffices to find an element $L \in \mathcal{M}(B)$ such that $\|\varphi(a) - L^*\pi(a)L\| < \epsilon$, $a \in F$, and $\|Lb\| < \epsilon$ for all $b \in B$. By Kasparov’s Stinespring theorem (Theorem 3 of [K]), there is a unital $*$-homomorphism $\chi : A \to \mathcal{M}(B)$ and an element $W \in \mathcal{M}(B)$ such that $\varphi(\cdot) = W^*\chi(\cdot)W$. Let $S_i$, $i = 1, 2, 3, \cdots$, be a sequence of isometries in $\mathcal{M}(B)$ such that $S_iS_i^* = 0$, $i \neq j$, and $\sum_{i=1}^{\infty} S_iS_i^* = 1$ in the strict topology, and set $\chi^\infty(a) = \sum_{i=1}^{\infty} S_i\chi(a)S_i^*$. It follows from 4) that there is a sequence $\{U_n\}$ of unitaries in $\mathcal{L}(B \oplus B, B)$ such that

$$\lim_{n \to \infty} \|U_n \begin{pmatrix} \pi(a) & 0 \\ 0 & \chi^\infty(a) \end{pmatrix} U_n^* - \pi(a)\| = 0$$

for all $a \in A$.

Define $T_i : B \to B \oplus B$ by $T_i b = (0, S_ib)$. Then $\chi(a) = T_i^* \begin{pmatrix} \pi(a) & 0 \\ 0 & \chi^\infty(a) \end{pmatrix} T_i$ and

$$\varphi(a) = W^*T_i^* \begin{pmatrix} \pi(a) & 0 \\ 0 & \chi^\infty(a) \end{pmatrix} T_i W$$

for all $a$ and $i$. Choose $n$ so large that

$$\| \begin{pmatrix} \pi(a) & 0 \\ 0 & \chi^\infty(a) \end{pmatrix} - U_n^*\pi(a)U_n\| < \frac{\epsilon}{1 + \|W\|^2}, \ a \in F.$$ 

Then

$$\|\varphi(a) - W^*T_i^*U_n^*\pi(a)U_nT_iW\| < \epsilon, \ a \in F,$$

for all $i$. Since $\lim_{n \to \infty} \|T_i^*x\| = 0$ for all $x \in B \oplus B$, we can choose $i$ so large that $\|W^*T_i^*U_n^*b\| < \epsilon$ for all $b \in G$. Set $L = U_nT_iW$.

**Definition 2.2.** Let $A$ and $B$ be separable $C^*$-algebras with $A$ unital and $B$ stable. A unital $*$-homomorphism $\pi : A \to \mathcal{M}(B)$ which satisfies the four equivalent conditions in Theorem 2.1 is called *unitally absorbing* (for $(A, B)$).
The following lemma is surely known, but it is so crucial for us here that we include a proof.

**Lemma 2.3.** Let $A$ and $B$ be separable $C^*$-algebras. There is then a countable set $X$ of completely positive contractions $A \to B$ such that for any completely positive contraction $\mu : A \to B$, any finite set $F \subseteq A$ and any $\epsilon > 0$ there is an element $l \in X$ such that

$$
||\mu(f) - l(f)|| \leq \epsilon, \, f \in F.
$$

**Proof.** Let $\{a_1, a_2, a_3, \cdots \}$ be a dense sequence in the unit ball of $A$ and set $F_n = \text{span}\{a_1, a_2, \cdots, a_n\}$. Let $\omega$ be a faithful state of $A$ and let $(\pi_\omega, H_\omega)$ be the GNS-representation coming from $\omega$. We can then consider $A$ as a subspace of $H_\omega$. The orthogonal projection $P_n : H_\omega \to F_n$ gives us then by restriction a continuous idempotent map $P_n : A \to F_n$. Let $1 < m_1 < m_2 < m_3 < \cdots$ be a sequence of numbers such that $\|P_n\| \leq m_n$ for all $n$. We can then define a metric $d$ on the space $B(A, B)$ of continuous linear maps $L : A \to B$ by

$$
d(L_1, L_2) = \sum_{i=1}^{\infty} \frac{2^{-i}}{m_i} \|L_1(a_i) - L_2(a_i)\|.
$$

Choose a linear basis $\{x_1, x_2, \cdots, x_n\}$ for $F_n$. For each $n_0$-tuple $b = (b_1, b_2, \cdots, b_{n_0}) \in B^{n_0}$ there is a linear map $L_b : F_n \to B$ such that $L_b(x_i) = b_i, \, i = 1, 2, \cdots, n$. By using that $B^{n_0}$ is separable this construction gives us a countable set $M$ of linear maps $F_n \to B$ which is dense in the strong topology of $B(F_n, B)$. Now let $0 < \epsilon < 1$ and let a finite set $D \subseteq F_n$ be given. Let $\mu \in B(F_n, B)$ be a contraction. There is a finite subset $G$ of $F_n$ such that every $x \in F_n$ with $\|x\| \leq 1 - \epsilon$ is a convex combination of elements from $G$. Choose $l \in M$ such that

$$
\|\mu(z) - l(z)\| < \epsilon, \, z \in D \cup G.
$$

Then $\|\mu(x) - l(x)\| \leq \epsilon$ for all $x \in F_n$ with $\|x\| \leq 1 - \epsilon$, and hence $\|l\| \leq \frac{1+\epsilon}{1-\epsilon}$. Let $q$ be a positive rational number in $[\frac{1-\epsilon}{1+\epsilon}, \frac{1+\epsilon}{1-\epsilon}]$. Then $ql \in \mathbb{Q}_+M$ is a contraction and we find that

$$
\|\mu(z) - ql(z)\| \leq \|\mu(z) - l(z)\| + \|l(z) - ql(z)\| \\
\leq \epsilon + |1-q||\|l\||\sup\{\|z\| : z \in D\} \\
< \frac{2\epsilon + 2q^2}{1-\epsilon^2} \sup\{\|z\| : z \in D\} + \epsilon
$$

for all $z \in D$. It follows that we can find a countable set $Y_n \subseteq \mathbb{Q}_+M$ of linear contractions which is strongly dense among all contractions in $B(F_n, B)$. Set

$$
Y = \bigcup_{n=1}^{\infty} \{l \circ P_n : l \in Y_n\}.
$$

Let $\mu : A \to B$ be a linear contraction and let $\epsilon > 0$. Choose $n$ so large that $\sum_{i \geq n+1} 2^{-i} < \frac{\epsilon}{2}$. From what we have just proved there is an element $l \in Y_n$ such that

$$
\|\mu(a_i) - l(a_i)\| < \frac{\epsilon}{2}, \, i = 1, 2, \cdots, n.
$$
Then \( l \circ P_n \in \mathcal{Y} \) and
\[
\begin{align*}
d(\mu, l \circ P_n) &\leq \sum_{i=1}^{n} \frac{2^{-i}}{m_i} \epsilon + \sum_{i \geq n+1} \frac{2^{-i}}{m_i} (1 + \|P_n\|) \\
\quad &\leq \epsilon + \sum_{i \geq n+1} \frac{2^{-i}}{m_i} (1 + m_i) \leq \epsilon.
\end{align*}
\]

It follows that \( \mathcal{Y} \) is a countable set in \( \mathcal{B}(A, B) \) with the property that for any linear contraction \( \mu : A \to B \) and any \( \epsilon > 0 \) there is an element \( l \in \mathcal{Y} \) such that \( d(\mu, l) < \epsilon \). For each \( l \in \mathcal{Y} \) choose a completely positive contraction \( l' : A \to B \) such that
\[
d(l, l') \leq 2 \inf \{d(l, L) : L \in \mathcal{B}(A, B) \text{ is a completely positive contraction}\}.
\]

Then \( \mathcal{Y}' = \{l' : l \in \mathcal{Y}\} \) is a countable set of completely positive contractions in \( \mathcal{B}(A, B) \) with the property that for any completely positive linear contraction \( \mu : A \to B \) and any \( \epsilon > 0 \) there is an element \( l \in \mathcal{Y}' \) such that \( d(\mu, l) < \epsilon \). \( \square \)

**Theorem 2.4.** Let \( A \) and \( B \) be separable \( C^* \)-algebras. Assume that \( A \) is stable and \( A \) unital. Then there exists a unitaly absorbing \( \ast \)-homomorphism \( \pi : A \to \mathcal{M}(B) \) for \((A, B)\).

**Proof.** By Lemma 2.3 there is a dense sequence \( \{s_n\} \) in the set of completely positive contractions from \( A \) to \( B \). We may assume that each \( s_n \) is repeated infinitely often in this sequence. By Kasparov’s Stinespring Theorem (Theorem 3 of [K]), there are elements \( V_n \in \mathcal{M}(B) \) and unital \( \ast \)-homomorphisms \( \pi_n : A \to \mathcal{M}(B) \) such that
\[
s_n(\cdot) = V_n^* \pi_n(\cdot) V_n
\]
for all \( n \). Note that \( \|V_n\|^2 = \|V_n^* V_n\| = \|s_n(1)\| \leq 1 \) for all \( n \). Define a unital \( \ast \)-homomorphism \( \pi_\infty : A \to \mathcal{L}_B(l_2(B)) \) by
\[
\pi_\infty(a)(b_1, b_2, b_3, \cdots) = (\pi_1(a)b_1, \pi_2(a)b_2, \pi_3(a)b_3, \cdots).
\]
Define \( L_n \in \mathcal{L}_B(l_2(B)) \) by
\[
L_n b = (0, 0, \cdots, 0, V_n b, 0, 0, \cdots),
\]
where the non-trivial entry occurs at the \( n \)’th coordinate. Since we repeated the \( s_n \)’s infinitely often there is, for each \( n \), a sequence \( k_1 < k_2 < k_3 < \cdots \) in \( N \) such that
\[
(2.4) \quad s_n(a) = L_{k_n}^* \pi_\infty(a) L_{k_n}
\]
for all \( a \in A, i \in N \), and
\[
(2.5) \quad \lim_{i \to \infty} \|L_{k_i}^* \psi\| = 0, \quad \psi \in l_2(B).
\]

By Lemma 1.3.2 of [KJT] there is an isomorphism \( S : l_2(B) \to B \) of Hilbert \( B \)-modules. Set \( T_n = S L_n \in \mathcal{M}(B) \) and \( \pi(\cdot) = S \pi_\infty(\cdot) S^* \). We assert that \( \pi \) satisfies condition 1) of Theorem 2.1 and to prove it we let \( \varphi : A \to B \) be a completely positive contraction. In order to construct a sequence \( \{W_n\} \) in \( \mathcal{M}(B) \) such that 1a and 1b) of Theorem 2.1 hold it suffices, because \( A \) and \( B \) are separable, to pick \( \epsilon > 0 \) and finite subsets \( F_1 \subseteq A \) and \( F_2 \subseteq B \) and find an element \( W \in \mathcal{M}(B) \) such that \( \|\varphi(a) - W^* \pi(a) W\| < \epsilon, a \in F_1 \), and \( \|W^* b\| < \epsilon, b \in F_2 \). Choose first an \( n \in N \) such that \( \|\varphi(a) - s_n(a)\| < \epsilon, a \in F_1 \). If we then choose \( k_1 < k_2 < k_3 < \cdots \)
such that (2.3) and (2.5) hold we have that $T_k^*\pi(a)T_k = s_n(a)$ for all $a \in F_1$ and $\|T_k^*b\| < \epsilon$ for all $b \in F_2$, provided only that $i$ is large enough. We can then set $W = T_k$, for such an $i$.

We now turn to the case of a not necessarily unital $C^*$-algebra $A$ and the general notion of absorbing $*$-homomorphisms. Given a $C^*$-algebra $A$ we denote in the following by $A^+$ the $C^*$-algebra obtained by adding a unit to $A$. Let $B$ be another $C^*$-algebra. Any linear completely positive contraction $\varphi : A \to \mathcal{M}(B)$ admits a unique linear extension $\varphi^+ : A^+ \to \mathcal{M}(B)$ such that $\varphi^+(1) = 1$. $\varphi^+$ is automatically a completely positive contraction (cf. e.g. Lemma 3.2.8 of [KJT]), and is automatically a $*$-homomorphism when $\varphi$ is. The following theorem is therefore an immediate consequence of Theorem 2.1

**Theorem 2.5.** Let $A$ and $B$ be separable $C^*$-algebras with $B$ stable. Let $\pi : A \to \mathcal{M}(B)$ be a $*$-homomorphism. Then the following conditions are equivalent:

1) $\pi^+ : A^+ \to \mathcal{M}(B)$ is unitally absorbing for $(A^+, B)$.

2) For any completely positive contraction $\varphi : A \to \mathcal{M}(B)$ there is a sequence $\{V_n\}$ of isometries in $\mathcal{M}(B)$ such that

   2a) $V_n^*\pi(a)V_n - \varphi(a) \in B$, $n \in \mathbb{N}$, $a \in A$,

   2b) $\lim_{n \to \infty} \|V_n^*\pi(a)V_n - \varphi(a)\| = 0$, $a \in A$.

3) For any $*$-homomorphism $\varphi : A \to \mathcal{M}(B)$ there is a sequence $\{U_n\}$ of unitaries $U_n \in L_B(B \oplus B, B)$ such that

   3a) $U_n \begin{pmatrix} \pi(a) & 0 \\ 0 & \varphi(a) \end{pmatrix} U_n^* - \pi(a) \in B$, $n \in \mathbb{N}$, $a \in A$,

   3b) $\lim_{n \to \infty} \|U_n \begin{pmatrix} \pi(a) & 0 \\ 0 & \varphi(a) \end{pmatrix} U_n^* - \pi(a)\| = 0$, $a \in A$.

4) For any $*$-homomorphism $\varphi : A \to \mathcal{M}(B)$ there is a sequence $\{U_n\}$ of unitaries $U_n \in L_B(B \oplus B, B)$ such that

   \[ \lim_{n \to \infty} \|U_n \begin{pmatrix} \pi(a) & 0 \\ 0 & \varphi(a) \end{pmatrix} U_n^* - \pi(a)\| = 0, \; a \in A. \]

**Definition 2.6.** Let $A$ and $B$ be separable $C^*$-algebras with $B$ stable. A $*$-homomorphism $\pi : A \to \mathcal{M}(B)$ is absorbing (for $(A, B)$) when it satisfies the four equivalent conditions of Theorem 2.5

**Theorem 2.7.** Let $A$ and $B$ be separable $C^*$-algebras with $B$ stable. There exists an absorbing $*$-homomorphism $\pi : A \to \mathcal{M}(B)$ for $(A, B)$.

**Proof.** Combine Theorem 2.5 and Theorem 2.4

An absorbing $*$-homomorphism is clearly unique in the following sense: Given two absorbing $*$-homomorphisms $\pi_1, \pi_2 : A \to \mathcal{M}(B)$ there is a sequence $\{U_n\} \subseteq \mathcal{M}(B)$ of unitaries such that $U_n\pi_1(a)U_n^* - \pi_2(a) \in B$, $a \in A$, $n \in \mathbb{N}$, and

\[ \lim_{n \to \infty} U_n\pi_1(a)U_n^* - \pi_2(a) = 0, \; a \in A. \]

3. Duality in $KK$-theory

Throughout this section $A$ and $B$ will be separable $C^*$-algebras and $B$ will be stable. A $*$-homomorphism $\pi : A \to \mathcal{M}(B)$ is of infinite multiplicity when $\pi$ is unitarily equivalent to $\pi^\infty$, where $\pi^\infty : A \to \mathcal{M}(B)$ is the $*$-homomorphism given by $\pi^\infty(a) = \sum_{i=1}^{\infty} S_i \pi(a) S_i^*$, for some sequence $S_i$, $i \in \mathbb{N}$, of isometries in $\mathcal{M}(B)$ such that $S_i^* S_j = 0$, $i \neq j$, and $\sum_{i=1}^{\infty} S_i S_i^* = 1$ in the strict topology.
Lemma 3.1. Let \( \pi : A \to \mathcal{M}(B) \) be a \( * \)-homomorphism of infinite multiplicity and set
\[
E = \{ m \in \mathcal{M}(B) : m\pi(a) = \pi(a)m \quad \forall a \in A \}.
\]
Then \( K_s(E) = \{0\} \).

Proof. Since \( \pi \) has infinite multiplicity,
\[
E \cong \{ m \in \mathcal{L}_B(l_2(B)) : m\mu(a) = \mu(a)m \quad \forall a \in A \}
\]
where \( \mu : A \to \mathcal{L}_B(l_2(B)) \) is given by
\[
\mu(a)(b_1, b_2, b_3, \cdots) = (\pi(a)b_1, \pi(a)b_2, \pi(a)b_3, \cdots).
\]
The usual proof that \( K_s(\mathcal{L}_B(l_2(B))) = 0 \) works to show that \( K_s(E) = 0 \); cf. e.g. Proposition 12.2.1 of [Bl]. \( \square \)

Given an absorbing \( * \)-homomorphism \( \pi : A \to \mathcal{M}(B) \) we set
\[
C_\pi = \{ x \in \mathcal{M}(B) : x\pi(a) = \pi(a)x \quad \forall a \in A \}
\]
and
\[
A_\pi = \{ x \in C_\pi : x\pi(A) \subseteq B \}.
\]
Then \( A_\pi \) is a closed two-sided ideal in \( C_\pi \) and we set \( D_\pi = C_\pi / A_\pi \). The quotient map \( C_\pi \to D_\pi \) will be denoted by \( q \). If \( \tau : A \to \mathcal{M}(B) \) is another absorbing \( * \)-homomorphism there is a unitary \( w \in \mathcal{M}(B) \) such that \( \text{Ad} w \circ \pi(a) - \tau(a) \in B \) for all \( a \in A \) and then \( x \mapsto wxw^* \) defines a \( * \)-isomorphism of \( C_\pi \) onto \( C_\tau \) which takes \( A_\pi \) onto \( A_\tau \). In particular, \( D_\pi \simeq D_\tau \).

Let \( u \) be a unitary in \( M_n(D_\pi) \). Choose \( v \in M_n(C_\pi) \) such that \( \text{id}_{M_n} \otimes q(v) = u \). Define \( \pi_n : A \to \mathcal{L}_B(B^n) \) by \( \pi_n(a)(b_1, b_2, \cdots, b_n) = (\pi(a)b_1, \pi(a)b_2, \cdots, \pi(a)b_n) \). Let \( B^n \oplus B^n \) be graded by \( (x, y) \mapsto (x, -y) \). Then
\[
(B^n \oplus B^n, \ (\pi_n), \ (\pi^*, \ *) )
\]
is a Kasparov \( A-B \)-module. We leave it to the reader to check that the class of this module in \( KK(A, B) \) only depends on the class of \( u \) in \( K_1(D_\pi) \), and that the construction gives rise to a group homomorphism \( \Theta : K_1(D_\pi) \to KK(A, B) \).

Theorem 3.2. Assume that \( \pi : A \to \mathcal{M}(B) \) is an absorbing \( * \)-homomorphism. Then \( \Theta : K_1(D_\pi) \to KK(A, B) \) is an isomorphism.

Proof. When \( \tau \) is another absorbing \( * \)-homomorphism there is a commuting diagram
\[
\begin{array}{ccc}
K_1(D_\pi) & \xrightarrow{\Theta} & KK(A, B) \\
\downarrow & & \\
K_1(D_\tau) & & \\
\end{array}
\]
where \( K_1(D_\pi) \to K_1(D_\tau) \) is induced by the isomorphism \( D_\pi \to D_\tau \) described above, and \( K_1(D_\tau) \to KK(A, B) \) is the map obtained by using \( \tau \) instead of \( \pi \) in the definition of \( \Theta \). Indeed if one considers a specific unitary in \( M_n(D_\pi) \), the Kasparov \( A-B \)-module which results by going down and up in the diagram differs from the one which arises by going across by an isomorphism and a compact perturbation. Thus if we prove that \( \Theta : K_1(A_\pi) \to KK(A, B) \) is an isomorphism for one absorbing
*-homomorphism \( \pi \) it will follow that it is an isomorphism for any other. Hence by working with \( \pi^n \) instead of \( \pi \) we may assume that \( \pi \) is of infinite multiplicity.

\( \Theta \) is injective: Let \( u \in M_n(D_n) \) be a unitary and choose \( v \in M_n(C_\pi) \) such that \( \text{id}_{M_n} \otimes q(v) = u \). Assume that \( [B^n \oplus B^n, (\pi^n, \pi^n), (v^n, v^n)] = 0 \) in \( KK(A, B) \). This means that there are degenerate Kasparov \( A \rightarrow B \)-modules \( D_1 \) and \( D_2 \) such that

\[
\text{Hom}_D(B^n \oplus B^n, (\pi^n, \pi^n), (v^n, v^n)) \oplus D_1 \text{ is operator homotopic to } (B^n \oplus B^n, (\pi^n, \pi^n), (1, 1)) \oplus D_2.
\]

Since \( D_1 \) and \( D_2 \) are degenerate we can define a new degenerate Kasparov \( A \rightarrow B \)-module \( D \) by \( D = D_1 \oplus D_2 \oplus D_1 \oplus D_2 \oplus \ldots \). Then \( D_1 \oplus D \) and \( D_2 \oplus D \) are both isomorphic to \( D \) and hence \( (B^n \oplus B^n, (\pi^n, \pi^n), (v^n, v^n)) \oplus D \) is operator homotopic to \( (B^n \oplus B^n, (\pi^n, \pi^n), (1, 1)) \oplus D \). By combining Kasparov's stabilization theorem (Theorem 2.12 of [KJT]) with Lemma 1.3.2 of [KJT] we may assume that \( a = w \) and \( b = w^* \) for some unitary \( w \in \mathcal{M}(B) \). Finally, by applying the unitary of the Hilbert \( B \)-module \( B \oplus B \) given by \((x, y) \mapsto (x, wy)\), we see that we can assume that \( w = 1 \). So all in all we have that

\[
(B^n \oplus B^n, (\pi^n, \pi^n), (v^n, v^n)) \oplus (B \oplus B, (\lambda_+ \lambda_-, (1, 1)))
\]

is operator homotopic to

\[
(B^n \oplus B^n, (\pi^n, \pi^n), (1, 1)) \oplus (B \oplus B, (\lambda_+ \lambda_-, (1, 1))).
\]

Note that \( \lambda_+ = \lambda_- \) since \( (B \oplus B, (\lambda_+ \lambda_-), (1, 1)) \) is degenerate. Finally, by adding on an infinite number of copies of \( (B \oplus B, (\lambda_+ \lambda_-), (1, 1)) \) we find that there is a \( * \)-homomorphism of infinite multiplicity \( \lambda: A \rightarrow \mathcal{M}(B) \) such that

\[
(B^n \oplus B^n, (\pi^n, \pi^n), (v^n, v^n)) \oplus (B \oplus B, (\lambda \lambda), (1, 1))
\]

is operator homotopic to

\[
(B^n \oplus B^n, (\pi^n, \pi^n), (1, 1)) \oplus (B \oplus B, (\lambda \lambda), (1, 1)).
\]

Furthermore, by adding on \( (B \oplus B, (\pi^n, \pi^n), (1, 1)) \) we may assume that there is a unitary \( w \in \mathcal{M}(B) \) such that \( w\lambda(a)w^* = \pi(a) \in B, a \in A \). The operator homotopy consists of an isomorphism of Kasparov \( A \rightarrow B \) modules and a norm-continuous path of operators. The isomorphism gives us a unitary \( S \in M_{n+1}(\mathcal{M}(B)) \) such that

\[
S(\pi^n(a)\lambda(a)_{lambda}) = \pi^n(a)\lambda(a)_{lambda} S
\]

for all \( a \in A \), and in addition we have a norm-continuous path \( F_t, t \in [0, 1], \) in \( M_{n+1}(\mathcal{M}(B)) \) such that \( F_0 = S, F_1 = (v^n, v^n) \), and

\[
(F_tF_t^* - 1_{n+1})(\pi^n(a)\lambda(a)_{lambda}), (F_t^*F_t - 1_{n+1})(\pi^n(a)\lambda(a)_{lambda}), F_t(\pi^n(a)\lambda(a)) - (\pi^n(a)\lambda(a)) F_t \text{ are in } M_{n+1}(B)
\]

for all \( t \) and \( a \). Here and in the following we let \( 1_k \) denote the unit of \( M_k(\mathcal{M}(B)) \). Note that \( \nu = (\pi^n, \pi^n) \) is of infinity multiplicity, as a \( * \)-homomorphism \( A \rightarrow \mathcal{M}(M_{n+1}(B)) \), since \( \pi \) and \( \lambda \) both are of infinite multiplicity.

By Lemma 3.1 we can therefore find an \( m \in \mathbb{N} \) and a norm-continuous path of unitaries in \( \{ x \in M_{m(n+1)}(\mathcal{M}(B)) : x\nu^m(a) = \nu^m(a)x, a \in A \} \) connecting

\[
(\sum_{t=0}^{1_{(m+1)(n+1)}} \pi^n(a)_{lambda}(a)_{lambda}), (F_tF_t^* - 1_{m+1})(\pi^n(a)_{lambda}(a)_{lambda}), (F_t^*F_t - 1_{m+1})(\pi^n(a)_{lambda}(a)_{lambda}), F_t(\pi^n(a)_{lambda}(a)) - (\pi^n(a)_{lambda}(a)) F_t \text{ are in } M_{m+1}(B)
\]

for all \( t \) and \( a \). Set

\[
W = \text{diag}(1_{n}, w, 1_{n}, w, \ldots, 1_{n}, w) \in M_{m(n+1)}(\mathcal{M}(B))
\]

where \( \nu^m(a) \) denotes \( m \) times.
and $G_t = WH_tW^*$. Then $G_t$ is a norm-continuous path in $M_{m(n+1)}(\mathcal{M}(B))$ such that $G_0 = 1_{m(n+1)}$, $G_1 = (u, 1_{m(n+1)-n})$ and $(G_tG_t^* - 1_{m(n+1)})\pi_{m(n+1)}(a)$, $(G_tG_t^* - 1_{m(n+1)})\pi_{m(n+1)}(a) - \pi_{m(n+1)}(a)G_t$ are in $M_{m(n+1)}(B)$ for all $t$ and $a$. Thus $(id_{M_{m(n+1)}} \otimes q)(G_t)$ is a path of unitaries in $M_{m(n+1)}(\mathcal{D}_{\pi})$ connecting $(u, 1_{m(n+1)-n})$ to $1_{m(n+1)}$.

$\Theta$ is surjective: Let $(E, \psi, F)$ be a Kasparov $A - B$-module. The constructions on pages 125-126 of [KJT] show that $[E, \psi, F] \in KK(A, B)$ is also represented by a Kasparov $A - B$-module of the form $(B \oplus B, (\varphi_+, \varphi_-), (\psi, \psi))$ for some $\ast$-homomorphisms $\varphi_{\pm} : A \to \mathcal{M}(B)$ and some unitary $v \in \mathcal{M}(B)$. Using the trick from p. 354 of [H] we may assume that $\varphi_- = \varphi_+ = \varphi$. By adding on $(B \oplus B, (\pi, (1, 1)))$ and using that $\pi$ is absorbing we may assume that there is a unitary $u \in \mathcal{M}(B)$ such that $u\varphi(a)u^* - \pi(a) \in B$ for all $a \in A$. Then $(B \oplus B, (\varphi, (\varphi^*, \psi^*)))$ is isomorphic to

$$(B \oplus B, \left(\text{Ad}u\psi, \text{Ad}u\varphi\right), (\psi^*, u^*u\psi^*))$$

which in turn is a compact perturbation of $(B \oplus B, (\pi, (1, 1))), (\psi^*, u^*u\psi^*))$. Then $uu^*$ is a unitary $C_\pi$ such that $\Theta([q(uu^*)]) = [E, \psi, F]$ in $KK(A, B)$. $\square$

Of course there is also an isomorphism

$$K_0(\mathcal{D}_{\pi}) \cong \text{Ext}^{-1}(A, B)$$

which can be proved in basically the same way.

**References**


