ON ABSORBING EXTENSIONS

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ABSTRACT. Building on the work of Kasparov we show that there always exists a trivial absorbing extension of \( A \) by \( B \otimes \mathcal{K} \), provided only that \( A \) and \( B \) are separable. If \( A \) is unital there is a unital trivial extension which is unitally absorbing.

1. Introduction

Absorbing trivial extensions play an important role in the theory of extensions of \( C^* \)-algebras; cf. 15.12 in [B]. Recently the interest in such extensions has been renewed because of the way \( KK \)-theory comes into the classification program. In this connection, as well as in the proper theory of \( C^* \)-extensions, it is slightly disturbing that the existence of an absorbing trivial extension has only been established in the case where at least one of the \( C^* \)-algebras involved is nuclear; cf. Theorem 5 of [K]. The purpose of the present note is to show that such extensions always exist when both \( C^* \)-algebras are separable. The argument for this is a modification of Kasparov’s approach from [K]. The absorbing trivial extensions were constructed, in [K] as well as before Kasparov’s work, by taking the infinite direct sum of the same copy of a faithful unital representation of the separable \( C^* \)-algebra \( A \) (for the moment assumed to be unital) which plays the role of the quotient in the extensions. The resulting representation \( \pi : A \to \mathcal{B}(\mathcal{H}) \) was then composed with the natural imbedding \( \mathcal{B}(\mathcal{H}) \subseteq \mathcal{M}(B \otimes \mathcal{K}) \), where \( B \otimes \mathcal{K} \) is the stable \( C^* \)-algebra which features as the ideal in the extensions. So in practice this means that the absorbing extension was constructed by taking a weak* dense sequence of states of \( A \), repeating all states in the sequence infinitely often, and then adding the corresponding GNS-representations. This procedure has nothing to do with the \( C^* \)-algebra \( B \), and it is a highly non-trivial task to show that it often results in an absorbing extension when prolonged to a map \( A \to \mathcal{M}(B \otimes \mathcal{K}) \); cf. [K]. The observation we offer here is that if one instead takes a sequence \( s_n : A \to B \otimes \mathcal{K} \) of completely positive contractions which is dense for the topology of pointwise norm-convergence among all completely positive contractions (such a sequence exists when both \( A \) and \( B \) are separable), repeats each \( s_n \) infinitely often and add up the unital representations

\[
\pi_n : A \to \mathcal{M}(B \otimes \mathcal{K}) , \ n \in \mathbb{N} ,
\]
coming from the Kasparov-Stinespring decompositions
\[ s_n(\cdot) = W_n^* \pi_n(\cdot) W_n, \]
the resulting representation \( A \to \mathcal{M}(B \otimes K) \) will be a unitaly absorbing trivial extension. The general trivial absorbing extensions can then be obtained (for a not necessarily unital \( C^* \)-algebra \( A \)) by taking a unitaly absorbing representation \( \pi : A^+ \to \mathcal{M}(B \otimes K) \) and restricting it to \( A \).

In order to illustrate how the absorbing \( * \)-homomorphisms constructed here can be used to extend known results we prove a general version of the Paschke-Valette-Skandalis duality which realizes the group \( KK(A, B) \) as the \( K_1 \)-group of a \( C^* \)-algebra \( D_n \) built out of \( A \) and \( B \) by using an absorbing \( * \)-homomorphism \( \pi : A \to \mathcal{M}(B) \); cf. \( [P, V, S, H] \).

2. Absorbing \( * \)-Homomorphisms

Given Hilbert \( B \)-modules \( E \) and \( F \), we let \( \mathcal{L}_B(E, F) \) denote the Banach space of adjoinable operators from \( E \) to \( F \). The ideal of ‘compact’ operators from \( E \) to \( F \) is denoted by \( \mathcal{K}_B(E, F) \). When \( E = F \) we write \( \mathcal{L}_B(E) \) and \( \mathcal{K}_B(E) \) instead of \( \mathcal{L}_B(E, E) \) and \( \mathcal{K}_B(E, E) \), respectively. In the special case where \( E = B \) there are well-known identifications \( \mathcal{L}_B(B) = \mathcal{M}(B) = \) the multiplier algebra of \( B \) and \( \mathcal{K}_B(B) = B \), which we shall use freely.

**Theorem 2.1.** Let \( A \) and \( B \) be separable \( C^* \)-algebras with \( A \) unital and \( B \) stable. Let \( \pi : A \to \mathcal{M}(B) \) be a unital \( * \)-homomorphism. Then the following conditions are equivalent:

1) For any completely positive contraction \( \varphi : A \to B \) there is a sequence \( \{W_n\} \subseteq \mathcal{M}(B) \) such that
   a) \( \lim_{n \to \infty} \|\varphi(a) - W_n^* \pi(a) W_n\| = 0 \) for all \( a \in A \),
   b) \( \lim_{n \to \infty} \|W_n^*\| = 0 \) for all \( b \in B \).

2) For any completely positive unital map \( \varphi : A \to \mathcal{M}(B) \) there is a sequence \( \{V_n\} \) of isometries in \( \mathcal{M}(B) \) such that
   a) \( V_n^* \pi(a) V_n - \varphi(a) \in B \), \( n \in \mathbb{N} \), \( a \in A \),
   b) \( \lim_{n \to \infty} \|V_n^* \pi(a) V_n - \varphi(a)\| = 0 \), \( a \in A \).

3) For any unital \( * \)-homomorphism \( \varphi : A \to \mathcal{M}(B) \) there is a sequence \( \{U_n\} \) of unitaries \( U_n \in \mathcal{L}_B(B \oplus B, B) \) such that
   a) \( U_n \begin{pmatrix} \pi(a) & 0 \\ 0 & \varphi(a) \end{pmatrix} U_n^* - \pi(a) \in B \), \( n \in \mathbb{N} \), \( a \in A \),
   b) \( \lim_{n \to \infty} \|U_n \begin{pmatrix} \pi(a) & 0 \\ 0 & \varphi(a) \end{pmatrix} U_n^* - \pi(a)\| = 0 \), \( a \in A \).

4) For any unital \( * \)-homomorphism \( \varphi : A \to \mathcal{M}(B) \) there is a sequence \( \{U_n\} \) of unitaries \( U_n \in \mathcal{L}_B(B \oplus B, B) \) such that
   \[ \lim_{n \to \infty} \|U_n \begin{pmatrix} \pi(a) & 0 \\ 0 & \varphi(a) \end{pmatrix} U_n^* - \pi(a)\| = 0 \), \( a \in A \).

**Proof.** 1) \( \Rightarrow \) 2): Let \( F \subseteq A \) be a finite set containing \( 1 \) and \( \epsilon > 0 \). Let \( \varphi : A \to \mathcal{M}(B) \) be a completely positive unital map. It suffices to find an element \( V \in \mathcal{M}(B) \) such that

\[ (2.1) \quad V^* \pi(a) V - \varphi(a) \in B \]
for all \( a \in A \) and
\[
\|V^* \pi(x) V - \varphi(x)\| < 3\epsilon
\]
for all \( x \in F \). If namely \( \epsilon \) is small enough this will imply that \( W = V[V^* V]^{-1} \) is an isometry close to \( V \) such that \( V - W \in B \), and we can then work with \( W \) instead of \( V \). We repeat Kasparov’s arguments: Let \( X \) be a compact subset of \( A \) containing \( F \) and with dense span in \( A \). By Lemma 10 of [K] there is a sequence \( \psi_k : A \to B, k \in \mathbb{N} \), of completely positive contractions such that \( \psi(a) = \sum_{k=1}^\infty \psi_k(a) \) converges in the strict topology, \( \varphi(a) - \psi(a) \in B \) for all \( a \in A \), and \( \|\varphi(x) - \psi(x)\| < \epsilon \) for all \( x \in X \). Let \( \{b_n\} \) be a countable approximate unit for \( B \). It follows from 1) that we can find a sequence \( \{m_i\} \subseteq B \) such that
1) \( \|\psi_i(x) - m^*_i \pi(x) m_i\| \leq \epsilon 2^{-i}, \ x \in X, \ i \in \mathbb{N}, \)
2) \( \|m^*_i \pi(x) m_i\| \leq \epsilon 2^{-i-j}, \ x \in X, \ i, j \in \mathbb{N}, \ i \neq j, \)
3) \( \sum_{i=1}^\infty \|m^*_i b_k\| < \infty \) for all \( k \in \mathbb{N} \).

The argument from the proof of Theorem 5 in [K] shows that \( \sum_{i=1}^\infty m_i \) converges in the strict topology to an element \( V \in \mathcal{M}(B) \) satisfying (2.1) and (2.2).

2) \( \Rightarrow \) 3): This follows from the arguments of Arveson given on pp. 338-339 of [A] by substituting Hilbert spaces with Hilbert \( B \)-modules. We leave this to the reader.

3) \( \Rightarrow \) 4) is trivial.

4) \( \Rightarrow \) 1): Let \( \varphi : A \to B \) be a completely positive contraction. Let \( F \subseteq A \) and \( G \subseteq B \) be finite sets and \( \epsilon > 0 \). Since \( A \) and \( B \) are separable it suffices to find an element \( L \in \mathcal{M}(B) \) such that \( \|\varphi(a) - L^* \pi(a)L\| < \epsilon \), \( a \in F \), and \( \|Lb\| < \epsilon \) for all \( b \in B \). By Kasparov’s Stinespring theorem (Theorem 3 of [K]), there is a unital \( * \)-homomorphism \( \chi : A \to \mathcal{M}(B) \) and an element \( W \in \mathcal{M}(B) \) such that \( \varphi(\cdot) = W^* \chi(\cdot) W \). Let \( S_i, i = 1, 2, 3, \ldots \), be a sequence of isometries in \( \mathcal{M}(B) \) such that \( S^*_i S_i = 0 \), \( i \neq j \), and \( \sum_{i=1}^\infty S^*_i S_i = 1 \) in the strict topology, and set \( \chi^\infty(a) = \sum_{i=1}^\infty S_i \pi(a) S^*_i \). It follows from 4) that there is a sequence \( \{U_n\} \) of unitaries in \( \mathcal{L}_B(B \oplus B, B) \) such that
\[
\lim_{n \to \infty} \|U_n \begin{pmatrix} \pi(a) & 0 \\ 0 & \chi^\infty(a) \end{pmatrix} U^*_n - \pi(a)\| = 0, \quad a \in A.
\]
Define \( T_i : B \to B \oplus B \) by \( T_i b = (0, S_i b) \). Then \( \chi(a) = T^*_i \begin{pmatrix} \pi(a) & 0 \\ 0 & \chi^\infty(a) \end{pmatrix} T_i \) and
\[
\varphi(a) = W^* T^*_i \begin{pmatrix} \pi(a) & 0 \\ 0 & \chi^\infty(a) \end{pmatrix} T_i W \text{ for all } a \text{ and } i.
\]
Choose \( n \) so large that
\[
\| \begin{pmatrix} \pi(a) & 0 \\ 0 & \chi^\infty(a) \end{pmatrix} - U^*_n \pi(a) U_n \| < \frac{\epsilon}{1 + \|W\|^2}, \quad a \in F.
\]
Then
\[
\|\varphi(a) - W^* T^*_i U^*_n \pi(a) U_n T_i W\| < \epsilon, \quad a \in F.
\]
for all \( i \). Since \( \lim_{m \to \infty} \|T^*_i x\| = 0 \) for all \( x \in B \oplus B \), we can choose \( i \) so large that \( \|W^* T^*_i U^*_n b\| < \epsilon \) for all \( b \in G \). Set \( L = U_n T_i W \).

**Definition 2.2.** Let \( A \) and \( B \) be separable \( C^* \)-algebras with \( A \) unital and \( B \) stable. A unital \( * \)-homomorphism \( \pi : A \to \mathcal{M}(B) \) which satisfies the four equivalent conditions in Theorem [2.1] is called *unitaly absorbing* (for \( (A, B) \)).
The following lemma is surely known, but it is so crucial for us here that we include a proof.

**Lemma 2.3.** Let $A$ and $B$ be separable $C^*$-algebras. There is then a countable set $X$ of completely positive contractions $A \to B$ such that for any completely positive contraction $\mu : A \to B$, any finite set $F \subseteq A$ and any $\epsilon > 0$ there is an element $l \in X$ such that
\[
\|\mu(f) - l(f)\| \leq \epsilon, \quad f \in F.
\]

**Proof.** Let $\{a_1, a_2, a_3, \cdots\}$ be a dense sequence in the unit ball of $A$ and set $F_n = \text{span}\{a_1, a_2, \cdots, a_n\}$. Let $\omega$ be a faithful state of $A$ and let $(\pi_\omega, H_\omega)$ be the GNS-representation coming from $\omega$. We can then consider $A$ as a subspace of $H_\omega$. The orthogonal projection $P_n : H_\omega \to F_n$ gives us then by restriction a continuous idempotent map $P_n : A \to F_n$. Let $1 < m_1 < m_2 < m_3 < \cdots$ be a sequence of numbers such that $\|P_n\| \leq m_n$ for all $n$. We can then define a metric $d$ on the space $B(A,B)$ of continuous linear maps $L : A \to B$ by
\[
d(L_1, L_2) = \sum_{i=1}^\infty \frac{2^{-i}}{m_i} \|L_1(a_i) - L_2(a_i)\|.
\]

Choose a linear basis $\{x_1, x_2, \cdots, x_{n_0}\}$ for $F_{n_0}$. For each $n_0$-tuple $b = (b_1, b_2, \cdots, b_{n_0})$ in $B^{n_0}$ there is a linear map $L_b : F_{n_0} \to B$ such that $L_b(x_i) = b_i$, $i = 1, 2, \cdots, n_0$.

By using that $B^{n_0}$ is separable this construction gives us a countable set $M$ of linear maps $F_n \to B$ which is dense in the strong topology of $B(F_n, B)$. Now let $0 < \epsilon < 1$ and let a finite set $D \subseteq F_n$ be given. Let $\mu \in B(F_n, B)$ be a contraction. There is then a countable set $G$ of $F_n$ such that every $x \in F_n$ with $\|x\| \leq 1 - \epsilon$ is a convex combination of elements from $G$. Choose $l \in M$ such that
\[
(2.3) \quad \|\mu(z) - l(z)\| < \epsilon, \quad z \in D \cup G.
\]

Then $\|\mu(x) - l(x)\| \leq \epsilon$ for all $x \in F_n$ with $\|x\| \leq 1 - \epsilon$, and hence $\|l\| \leq \frac{1 + \epsilon}{1 - \epsilon}$. Let $q$ be a positive rational number in $\left|\frac{1 - \epsilon}{1 + \epsilon}, \frac{1 + \epsilon}{1 - \epsilon}\right|$. Then $ql \in \mathbb{Q}_+M$ is a contraction and we find that
\[
\begin{align*}
\|\mu(z) - ql(z)\| &\leq \|\mu(z) - l(z)\| + \|l(z) - ql(z)\| \\
&\leq \epsilon + |1 - q||l| \sup \{\|z\| : z \in D\} \\
&< \frac{2\epsilon + 2\epsilon^2}{1 - \epsilon^2} \sup \{\|z\| : z \in D\} + \epsilon
\end{align*}
\]
for all $z \in D$. It follows that we can find a countable set $\mathcal{Y}_n \subseteq \mathbb{Q}_+M$ of linear contractions which is strongly dense among all contractions in $B(F_n, B)$. Set
\[
\mathcal{Y} = \bigcup_{n=1}^\infty \{l \circ P_n : l \in \mathcal{Y}_n\}.
\]

Let $\mu : A \to B$ be a linear contraction and let $\epsilon > 0$. Choose $n$ so large that $2 \sum_{i \geq n+1} 2^{-i} < \frac{\epsilon}{2}$. From what we have just proved there is an element $l \in \mathcal{Y}_n$ such that
\[
\|\mu(a_i) - l(a_i)\| < \frac{\epsilon}{2}, \quad i = 1, 2, \cdots, n.
\]
Then \( l \circ P_n \in \mathcal{Y} \) and

\[
d(\mu, l \circ P_n) \leq \sum_{i=1}^{n} \frac{2^{-i} \epsilon}{m_i} + \sum_{i=n+1}^{\infty} \frac{2^{-i}}{m_i} (1 + \|P_n\|)
\]

\[
\leq \frac{\epsilon}{2} + \sum_{i=n+1}^{\infty} \frac{2^{-i}}{m_i} (1 + m_i) \leq \epsilon.
\]

It follows that \( \mathcal{Y} \) is a countable set in \( B(A, B) \) with the property that for any linear contraction \( \mu : A \to B \) and any \( \epsilon > 0 \) there is an element \( l \in \mathcal{Y} \) such that \( d(\mu, l) < \epsilon \). For each \( l \in \mathcal{Y} \) choose a completely positive contraction \( l' : A \to B \) such that

\[
d(l, l') \leq 2 \inf \{d(l, L) : L \in B(A, B) \text{ is a completely positive contraction}\}.
\]

Then \( \mathcal{Y}' = \{l' : l \in \mathcal{Y}\} \) is a countable set of completely positive contractions in \( B(A, B) \) with the property that for any completely positive linear contraction \( \mu : A \to B \) and any \( \epsilon > 0 \) there is an element \( l \in \mathcal{Y}' \) such that \( d(\mu, l) < \epsilon \). \( \square \)

**Theorem 2.4.** Let \( A \) and \( B \) be separable \( C^* \)-algebras. Assume that \( B \) is stable and \( A \) unital. Then there exists a unitaly absorbing \(*\)-homomorphism \( \pi : A \to \mathcal{M}(B) \) for \((A, B)\).

**Proof.** By Lemma 2.3 there is a dense sequence \( \{s_n\} \) in the set of completely positive contractions from \( A \) to \( B \). We may assume that each \( s_n \) is repeated infinitely often in this sequence. By Kasparov’s Stinespring Theorem (Theorem 3 of [K]), there are elements \( V_n \in \mathcal{M}(B) \) and unital \(*\)-homomorphisms \( \pi_n : A \to \mathcal{M}(B) \) such that

\[
s_n(\cdot) = V_n^* \pi_n(\cdot) V_n
\]

for all \( n \). Note that \( \|V_n\|^2 = \|V_n^* V_n\| = \|s_n(1)\| \leq 1 \) for all \( n \). Define a unital \(*\)-homomorphism \( \pi_\infty : A \to \mathcal{L}_B(l_2(B)) \) by

\[
\pi_\infty(a)(b_1, b_2, b_3, \cdots) = (\pi_1(a)b_1, \pi_2(a)b_2, \pi_3(a)b_3, \cdots).
\]

Define \( L_n \in \mathcal{L}_B(B, l_2(B)) \) by

\[
L_n b = (0, 0, \cdots, 0, V_n b, 0, 0, \cdots),
\]

where the non-trivial entry occurs at the \( n \)'th coordinate. Since we repeated the \( s_n \)'s infinitely often there is, for each \( n \), a sequence \( k_1 < k_2 < k_3 < \cdots \) in \( \mathbb{N} \) such that

\[
s_n(a) = L_{k_i}^* \pi_\infty(a)L_{k_i}
\]

for all \( a \in A, i \in \mathbb{N} \), and

\[
\lim_{i \to \infty} \|L_{k_i}^* \psi\| = 0, \quad \psi \in l_2(B).
\]

By Lemma 1.3.2 of [KJT] there is an isomorphism \( S : l_2(B) \to B \) of Hilbert \( B \)-modules. Set \( T_n = S L_n \in \mathcal{M}(B) \) and \( \pi(\cdot) = S \pi_\infty(\cdot) S^* \). We assert that \( \pi \) satisfies condition 1) of Theorem 2.1 and to prove it we let \( \varphi : A \to B \) be a completely positive contraction. In order to construct a sequence \( \{W_n\} \) in \( \mathcal{M}(B) \) such that 1a) and 1b) of Theorem 2.1 hold it suffices, because \( A \) and \( B \) are separable, to pick \( \epsilon > 0 \) and finite subsets \( F_1 \subseteq A \) and \( F_2 \subseteq B \) and find an element \( W \in \mathcal{M}(B) \) such that

\[
\|\varphi(a) - W^* \pi(a)W\| < \epsilon, \quad a \in F_1, \quad \text{and} \quad \|W^* b\| < \epsilon, \quad b \in F_2.
\]

Choose first an \( n \in \mathbb{N} \) such that \( \|\varphi(a) - s_n(a)\| < \epsilon, \quad a \in F_1 \). If we then choose \( k_1 < k_2 < k_3 < \cdots \)
such that (2.3) and (2.5) hold we have that $T^*_k \pi(a) T_k = s_n(a)$ for all $a \in F_1$ and $\|T^*_k b\| < \varepsilon$ for all $b \in F_2$, provided only that $i$ is large enough. We can then set $W = T_k$, for such an $i$.

We now turn to the case of a not necessarily unital $C^*$-algebra $A$ and the general notion of absorbing $*$-homomorphisms. Given a $C^*$-algebra $A$ we denote in the following by $A^+$ the $C^*$-algebra obtained by adding a unit to $A$. Let $B$ be another $C^*$-algebra. Any linear completely positive contraction $\varphi : A \to \mathcal{M}(B)$ admits a unique linear extension $\varphi^+ : A^+ \to \mathcal{M}(B)$ such that $\varphi^+(1) = 1$. $\varphi^+$ is automatically a completely positive contraction (cf. e.g. Lemma 3.2.8 of [KJT]), and is automatically a $*$-homomorphism when $\varphi$ is. The following theorem is therefore an immediate consequence of Theorem 2.4.

**Theorem 2.5.** Let $A$ and $B$ be separable $C^*$-algebras with $B$ stable. Let $\pi : A \to \mathcal{M}(B)$ be a $*$-homomorphism. Then the following conditions are equivalent:

1) $\pi^+ : A^+ \to \mathcal{M}(B)$ is unitally absorbing for $(A^+, B)$.

2) For any completely positive contraction $\varphi : A \to \mathcal{M}(B)$ there is a sequence $\{V_n\}$ of isometries in $\mathcal{M}(B)$ such that
   2a) $V^*_n \pi(a) V_n - \varphi(a) \in B$, $n \in \mathbb{N}$, $a \in A$,
   2b) $\lim_{n \to \infty} \|V^*_n \pi(a) V_n - \varphi(a)\| = 0$, $a \in A$.

3) For any $*$-homomorphism $\varphi : A \to \mathcal{M}(B)$ there is a sequence $\{U_n\}$ of unitaries $U_n \in \mathcal{L}_B(B \oplus B, B)$ such that
   3a) $U_n \begin{pmatrix} \pi(a) & 0 \\ 0 & \varphi(a) \end{pmatrix} U^*_n - \pi(a) \in B$, $n \in \mathbb{N}$, $a \in A$,
   3b) $\lim_{n \to \infty} \|U_n \begin{pmatrix} \pi(a) & 0 \\ 0 & \varphi(a) \end{pmatrix} U^*_n - \pi(a)\| = 0$, $a \in A$.

4) For any $*$-homomorphism $\varphi : A \to \mathcal{M}(B)$ there is a sequence $\{U_n\}$ of unitaries $U_n \in \mathcal{L}_B(B \oplus B, B)$ such that
   $\lim_{n \to \infty} \|U_n \begin{pmatrix} \pi(a) & 0 \\ 0 & \varphi(a) \end{pmatrix} U^*_n - \pi(a)\| = 0$, $a \in A$.

**Definition 2.6.** Let $A$ and $B$ be separable $C^*$-algebras with $B$ stable. A $*$-homomorphism $\pi : A \to \mathcal{M}(B)$ is absorbing (for $(A, B)$) when it satisfies the four equivalent conditions of Theorem 2.5.

**Theorem 2.7.** Let $A$ and $B$ be separable $C^*$-algebras with $B$ stable. There exists an absorbing $*$-homomorphism $\pi : A \to \mathcal{M}(B)$ for $(A, B)$.

**Proof.** Combine Theorem 2.5 and Theorem 2.4.

An absorbing $*$-homomorphism is clearly unique in the following sense: Given two absorbing $*$-homomorphisms $\pi_1, \pi_2 : A \to \mathcal{M}(B)$ there is a sequence $\{U_n\} \subseteq \mathcal{M}(B)$ of unitaries such that $U_n \pi_1(a) U^*_n - \pi_2(a) \in B$, $a \in A$, $n \in \mathbb{N}$, and $\lim_{n \to \infty} U_n \pi_1(a) U^*_n - \pi_2(a) = 0$, $a \in A$.

3. Duality in $KK$-theory

Throughout this section $A$ and $B$ will be separable $C^*$-algebras and $B$ will be stable. A $*$-homomorphism $\pi : A \to \mathcal{M}(B)$ is of infinite multiplicity when $\pi$ is unitarily equivalent to $\pi^\infty$, where $\pi^\infty : A \to \mathcal{M}(B)$ is the $*$-homomorphism given by $\pi^\infty(a) = \sum_{i=1}^\infty S_i \pi(a) S_i^*$, for some sequence $S_i$, $i \in \mathbb{N}$, of isometries in $\mathcal{M}(B)$ such that $S_i^* S_j = 0$, $i \neq j$, and $\sum_{i=1}^\infty S_i S_i^* = 1$ in the strict topology.
Lemma 3.1. Let \( \pi : A \to \mathcal{M}(B) \) be a \(*\)-homomorphism of infinite multiplicity and set
\[
E = \{ m \in \mathcal{M}(B) : m\pi(a) = \pi(a)m \ \forall a \in A \}.
\]
Then \( K_*(E) = \{0\} \).

Proof. Since \( \pi \) has infinite multiplicity,
\[
E \simeq \{ m \in \mathcal{L}_B(l_2(B)) : m\mu(a) = \mu(a)m \ \forall a \in A \}
\]
where \( \mu : A \to \mathcal{L}_B(l_2(B)) \) is given by
\[
\mu(a)(b_1, b_2, b_3, \cdots) = (\pi(a)b_1, \pi(a)b_2, \pi(a)b_3, \cdots).
\]
The usual proof that \( K_*(\mathcal{L}_B(l_2(B))) = 0 \) works to show that \( K_*(E) = 0 \); cf. e.g. Proposition 12.2.1 of [3].

Given an absorbing \(*\)-homomorphism \( \pi : A \to \mathcal{M}(B) \) we set
\[
C_\pi = \{ x \in \mathcal{M}(B) : x\pi(a) - \pi(a)x \in B, a \in A \}
\]
and
\[
A_\pi = \{ x \in C_\pi : x\pi(A) \subseteq B \}.
\]
Then \( A_\pi \) is a closed two-sided ideal in \( C_\pi \) and we set \( D_\pi = C_\pi/A_\pi \). The quotient map \( C_\pi \to D_\pi \) will be denoted by \( q \). If \( \tau : A \to \mathcal{M}(B) \) is another absorbing \(*\)-homomorphism there is a unitary \( w \in \mathcal{M}(B) \) such that \( \text{Ad} \, w \circ \pi(a) - \tau(a) \in B \) for all \( a \in A \) and then \( x \mapsto wxw^* \) defines a \(*\)-isomorphism of \( C_\pi \) onto \( C_\tau \) which takes \( A_\pi \) onto \( A_\tau \). In particular, \( D_\pi \simeq D_\tau \).

Let \( u \) be a unitary in \( M_n(D_\pi) \). Choose \( v \in M_n(C_\pi) \) such that \( \text{id}_{M_n} \otimes q(v) = u \). Define \( \pi^n : A \to \mathcal{L}_B(B^n) \) by \( \pi^n(a)(b_1, b_2, \cdots, b_n) = (\pi(a)b_1, \pi(a)b_2, \cdots, \pi(a)b_n) \). Let \( B^n \oplus B^n \) be graded by \( (x, y) \mapsto (x, -y) \). Then
\[
(B^n \oplus B^n, (\pi^n, (\pi^n)^*))
\]
is a Kasparov \( A - B \)-module. We leave it to the reader to check that the class of this module in \( KK(A, B) \) only depends on the class of \( u \) in \( K_1(D_\pi) \), and that the construction gives rise to a group homomorphism \( \Theta : K_1(D_\pi) \to KK(A, B) \).

Theorem 3.2. Assume that \( \pi : A \to \mathcal{M}(B) \) is an absorbing \(*\)-homomorphism. Then \( \Theta : K_1(D_\pi) \to KK(A, B) \) is an isomorphism.

Proof. When \( \tau \) is another absorbing \(*\)-homomorphism there is a commuting diagram
\[
\begin{array}{ccc}
K_1(D_\pi) & \xrightarrow{\Theta} & KK(A, B) \\
\downarrow & & \\
K_1(D_\tau) & &
\end{array}
\]
where \( K_1(D_\pi) \to K_1(D_\tau) \) is induced by the isomorphism \( D_\pi \to D_\tau \) described above, and \( K_1(D_\tau) \to KK(A, B) \) is the map obtained by using \( \tau \) instead of \( \pi \) in the definition of \( \Theta \). Indeed if one considers a specific unitary in \( M_n(D_\pi) \), the Kasparov \( A - B \)-module which results by going down and up in the diagram differs from the one which arises by going across by an isomorphism and a compact perturbation.

Thus if we prove that \( \Theta : K_1(A_\pi) \to KK(A, B) \) is an isomorphism for one absorbing
*-homomorphism \( \pi \) it will follow that it is an isomorphism for any other. Hence by working with \( \pi^{\infty} \) instead of \( \pi \) we may assume that \( \pi \) is of infinite multiplicity.

\( \Theta \) is injective: Let \( u \in M_n(D_{n}) \) be a unitary and choose \( v \in M_n(C_{\pi}) \) such that \( id_{M_n} \otimes q(v) = u \). Assume that \( [B^n \oplus B^n, (\pi^n, \pi^n), (\nu^n, \nu^n)] = 0 \) in KK(\( A, B \)). This means that there are degenerate Kasparov \( A - B \)-modules \( D_1 \) and \( D_2 \) such that \( (B^n \oplus B^n, (\pi^n, \pi^n), (\nu^n, \nu^n)) \oplus D_1 \) is operator homotopic to \( (B^n \oplus B^n, (\pi^n, \pi^n), (1, 1)) \oplus D_2 \). Since \( D_1 \) and \( D_2 \) are degenerate we can define a new degenerate Kasparov \( A - B \)-module \( D \) by \( D = D_1 \oplus D_2 \oplus D_1 \oplus D_2 \oplus D_2 \oplus \cdots \). Then \( D_1 \oplus D \) and \( D_2 \oplus D \) are both isomorphic to \( D \) and hence \( (B^n \oplus B^n, (\pi^n, \pi^n), (\nu^n, \nu^n)) \oplus D \) is operator homotopic to \( (B^n \oplus B^n, (\pi^n, \pi^n), (1, 1)) \oplus D \). By combining Kasparov’s stabilization theorem (Theorem 2.12 of [KJT]) with Lemma 1.3.2 of [KJT] we may assume that \( a = w \) and \( b = w^* \) for some unitary \( w \in \mathcal{M}(B) \). Finally, by applying the unitary of the Hilbert \( B \)-module \( A \oplus B \) given by \( (x, y) \mapsto (x, wy) \), we see that we can assume that \( w = 1 \).

So all in all we have that

\[
(B^n \oplus B^n, (\pi^n, \pi^n), (\nu^n, \nu^n)) \oplus (B \oplus B, (\lambda^+, \lambda^-), (1, 1))
\]

is operator homotopic to

\[
(B^n \oplus B^n, (\pi^n, \pi^n), (1, 1)) \oplus (B \oplus B, (\lambda^+, \lambda^-), (1, 1)).
\]

Note that \( \lambda^+ = \lambda^- \) since \( (B \oplus B, (\lambda^+, \lambda^-), (1, 1)) \) is degenerate. Finally, by adding on an infinite number of copies of \( (B \oplus B, (\lambda^+, \lambda^-), (1, 1)) \) we find that there is a *-homomorphism of infinite multiplicity \( \lambda : A \rightarrow \mathcal{M}(B) \) such that

\[
(B^n \oplus B^n, (\pi^n, \pi^n), (\nu^n, \nu^n)) \oplus (B \oplus B, (\lambda, \lambda), (1, 1))
\]

is operator homotopic to

\[
(B^n \oplus B^n, (\pi^n, \pi^n), (1, 1)) \oplus (B \oplus B, (\lambda, \lambda), (1, 1)).
\]

Furthermore, by adding on \( (B \oplus B, (\pi^n, (1, 1))) \) we may assume that there is a unitary \( w \in \mathcal{M}(B) \) such that \( w\lambda(a)w^* - \pi(a) \in B, a \in A \). The operator homotopy consists of an isomorphism of Kasparov \( A - B \) modules and a norm-continuous path of operators. The isomorphism gives us a unitary \( S \in M_{n+1}(\mathcal{M}(B)) \) such that \( S \left( \begin{array}{cc} \pi^n(a) & \lambda(a) \\ \lambda(a) & \pi^n(a) \end{array} \right) S \) for all \( a \in A \), and in addition we have a norm-continuous path \( F_t, t \in [0, 1] \), in \( M_{n+1}(\mathcal{M}(B)) \) such that \( F_0 = S, F_1 = (\nu, 1), \) and \( (F_t F_t^* - 1_{n+1}) \left( \begin{array}{cc} \pi^n(a) & \lambda(a) \\ \lambda(a) & \pi^n(a) \end{array} \right), (F_t^* F_t - 1_{n+1}) \left( \begin{array}{cc} \pi^n(a) & \lambda(a) \\ \lambda(a) & \pi^n(a) \end{array} \right), (F_t F_t^*) - \left( \begin{array}{cc} \pi^n(a) & \lambda(a) \\ \lambda(a) & \pi^n(a) \end{array} \right) \) are in \( M_{n+1}(B) \) for all \( t \) and \( a \). Here and in the following we let \( 1_k \) denote the unit of \( M_k(\mathcal{M}(B)) \). Note that \( \nu = (\pi^n, \lambda) \) is of infinite multiplicity, as a *-homomorphism \( A \rightarrow \mathcal{M}(M_{n+1}(B)) \), since \( \pi \) and \( \lambda \) both are of infinite multiplicity.

By Lemma 3.1 we can therefore find an \( m \in \mathbb{N} \) and a norm-continuous path of unitaries in \( \{x \in M_{m(n+1)}(\mathcal{M}(B)) : x \pi^m(a) = \pi^m(a)x, a \in A \} \) connecting

\[
\left( \begin{array}{cc} 1_{m(n+1)}(a) & 1_{m(n+1)}(a) \\ 1_{m(n+1)}(a) & 1_{m(n+1)}(a) \end{array} \right)
\]

to \( 1_{m(n+1)} \). In combination with \( F \) this gives us a norm-continuous path \( H_t, t \in [0, 1], \) in \( M_{m(n+1)}(\mathcal{M}(B)) \) such that \( H_0 = 1_{m(n+1)}, H_1 = (\nu_{m(n+1)}, 1_{m(n+1)}), (H_t H_t^* - 1_{m(n+1)}) \pi^m(a), (H_t^* H_t - 1_{m(n+1)}) \pi^m(a), H_t \pi^m(a) - \pi^m(a) \) are in \( M_{m(n+1)}(B) \) for all \( t \) and \( a \). Set

\[
W = \text{diag}(1, n, w, 1, n, w, \cdots, 1, n, w) \in M_{m(n+1)}(\mathcal{M}(B))
\]
and \( G_t = WH_tW^* \). Then \( G_t \) is a norm-continuous path in \( M_m(n+1)(\mathcal{M}(B)) \) such that \( G_0 = 1_m(n+1) \), \( G_1 = (\tau_1 1_m(n+1) - \tau_2) \) and \( (G_tG_t^* - 1_m(n+1))^{\pi(n+1)}(a), \), \( (G_tG_t^* - 1_m(n+1))^{\pi(n+1)}(a), \) \( G_t^{\pi(n+1)}(a) - \pi(n+1)(a)G_t \) are in \( M_m(n+1)(B) \) for all \( t \) and \( a \). Thus \( (\text{id}_{M_m(n+1)} \otimes q)(G_t) \) is a path of unitaries in \( M_m(n+1)(\mathcal{D}) \) connecting \( (\tau_1 1_m(n+1) - \tau_2) \) to \( 1_m(n+1) \).

\( \Theta \) is surjective: Let \( (E, \psi, F) \) be a Kasparov \( A-B \)-module. The constructions on pages 125-126 of [KJT] show that \([E, \psi, F] \in KK(A, B)\) is also represented by a Kasparov \( A-B \)-module of the form \((B \oplus B, (\varphi_+, \varphi_+), (\varphi, \varphi))\) for some \(+\)-homomorphisms \( \varphi_\pm : A \rightarrow \mathcal{M}(B) \) and some unitary \( \psi \in \mathcal{M}(B) \). Using the trick from p. 354 of [H] we may assume that \( \varphi_- = \varphi_+ = \varphi \). By adding on \((B \oplus B, (\pi, 1)), (\varphi, \varphi)\) and using that \( \pi \) is absorbing we may assume that there is a unitary \( u \in \mathcal{M}(B) \) such that \( u\varphi(a)u^* - \pi(a) \in B \) for all \( a \in A \). Then \((B \oplus B, (\pi, 1)), (\varphi, \varphi)\) is isomorphic to

\[
\begin{align*}
(B \oplus B, \left( \begin{array}{cc}
Ad u \varphi & \Ad u \varphi \\
Avu^* & uu^*
\end{array} \right), 
(uv^* u^* uv^*) \end{align*}
\]

which in turn is a compact perturbation of \((B \oplus B, (\pi, 1)), (uv^* u^* uv^*)\). Then \( uvu^* \) is a unitary \( C_\pi \) such that \( \Theta([q(\psi)] = [E, \psi, F] \in KK(A, B) \).

Of course there is also an isomorphism

\[
K_0(D_\pi) \simeq \text{Ext}^{-1}(A, B)
\]

which can be proved in basically the same way.

References


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