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# STURMIAN SEQUENCES AND THE LEXICOGRAPHIC WORLD

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Abstract. In this paper, we give a complete description for the lexicographic world $\mathcal{L} = \{(x, y) \in \Sigma \times \Sigma : \Sigma_{xy} \neq \emptyset\} = \{(x, y) : y \geq \phi(x)\}$, where $\Sigma = \{0,1\}^{\mathbf{N}}$, $\Sigma_{ab} = \{x \in \Sigma : a \leq \sigma^i(x) \leq b, \text{for all } i \geq 0\}$, $\phi : \Sigma \to \Sigma$ is defined by $\phi(a) = \inf\{b : \Sigma_{ab} \neq \emptyset\}$ and the order $\leq$ is the lexicographic order on $\Sigma$. The main result is that $b = \phi(a)$ for some $a = 0x$ if and only if $b$ is the Sturmian sequence $b$ such that $\mathrm{Orb}(b) \subset [0x, 1x]$ and $\sigma^i(b) \leq b$ for all $i \geq 0$. At the same time, a new description of Sturmian minimal sets is given. A minimal set $M$ is a Sturmian minimal set if and only if, for some $x \in \Sigma$, $M \subset [0x, 1x]$. Moreover, for any $x \in \Sigma$, there exists a unique Sturmian minimal set in $[0x, 1x]$.

## 1. Introduction and the definition of lexicographic world

Let $\Sigma = \{0,1\}^{\mathbf{N}}$ and denote by $\sigma$ the left (one-sided) shift on $\Sigma$.

First, let us define the lexicographic order on $\Sigma$. For any $x, y \in \Sigma$, $x < y$ iff $x \neq y$ and for some $n \in \mathbf{N}$, $x_i = y_i$ for $i < n$ and $x_n = 0$, $y_n = 1$. Note that $\Sigma$ is well-ordered and the order topology given by the above order is the same as the usual topology on $\Sigma$.

For any $a, b \in \Sigma$ define $\Sigma_{ab}$ as

$$\begin{aligned} \Sigma_{ab} &= \{x \in \Sigma : a \leq \sigma^i(x) \leq b, \text{ for all } i \geq 0\}, \\ \mathcal{L} &= \{(x, y) \in \Sigma \times \Sigma : \Sigma_{xy} \neq \emptyset\}. \end{aligned} \tag{1}$$

For any $a, b \in \Sigma$ denote $\{x : a \leq x \leq b\}$ by $[a, b]$, which will be called a (closed) interval. And for any interval $I = [a, b]$, denote $\Sigma_{ab}$ by $\Sigma_I$. If $\Sigma_I \neq \emptyset$, $I$ will be called an $\mathcal{L}$-interval. Now we define a map $\phi : \Sigma \to \Sigma$ as

$$\phi(a) = \inf\{b \in \Sigma : \Sigma_{a,b} \neq \emptyset\}. \tag{2}$$

Then $\mathcal{L} = \{(x, y) : y \geq \phi(x)\}$ (Lemma 2.1). $\mathcal{L}$ is called the lexicographic world and it is closely related to the bifurcation of a Lorenz-like map (see [2]). In a talk, Labarca raised the question of studying $\phi$. In this paper, we give a complete description for $\phi$. The main result is

**Theorem 1.1.** *$b = \phi(a)$ for some $a = 0x$ if and only if $b$ is the Sturmian sequence $b$ such that* $\mathrm{Orb}(b) \subset [0x, 1x]$ *and $\sigma^i(b) \leq b$ for all $i \geq 0$.*

At the same time, we obtain a new description for Sturmian minimal sets.

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**Theorem 1.2.** *A minimal set $M$ is a Sturmian minimal set if and only if, for some $x \in \Sigma$, $M \subset [0x, 1x]$. Moreover, for any $x \in \Sigma$, there exists a unique Sturmian minimal set in $[0x, 1x]$.*

This paper is organized as following. In §2, some basic properties of $\phi$ are given. And then we introduce the concept of order minimality in §3. It is shown that $b = \phi(a)$ if and only if $b$ is order minimal and $\sigma^i(b) \leq b$ for $i \geq 0$. In §4, we show that $b = \phi(a)$ is a Sturmian sequence. The main result and the characterization of Sturmian minimal sets are given in §5.

## 2. Some basic properties of $\phi$

**Lemma 2.1.** $\Sigma_{a\phi(a)} \neq \emptyset$. *Therefore,* $\mathcal{L} = \{(x, y) \in \Sigma \times \Sigma : y \geq \phi(x)\}$.

*Proof.* Let $b^i \in \{b \in \Sigma : \Sigma_{a,b} \neq \emptyset\}$ and $b^i \searrow \phi(a)$. This means $\exists x^i \in \Sigma_{ab^i}$, i.e.,

$$a \leq \sigma^n(x^i) \leq b^i, \quad \forall i, n.$$

We may assume $x^i \to x$. Letting $i$ tend to $\infty$ we get

$$a \leq \sigma^n(x) \leq \phi(a).$$

This means that $x \in \Sigma_{a\phi(a)}$. □

**Lemma 2.2.** *$\phi$ is non-decreasing.*

*Proof.* If $a \leq b \in \Sigma$, then $\Sigma_{b\phi(b)} \neq \emptyset$ implies $\Sigma_{a\phi(b)} \neq \emptyset$. So $\phi(a) \leq \phi(b)$. □

**Lemma 2.3.** *$\phi$ is continuous from the left, i.e., if $a^i$ tends to $a$ monotonically increasingly, then $\phi(a^i)$ tends to $\phi(a)$.*

*Proof.* Let $\phi(a^i) \nearrow y$ and $x^i \in \Sigma_{a^i\phi(a^i)}, x^i \to x$. Then

$$a^i \leq \sigma^n(x^i) \leq \phi(a^i), \quad \forall i, n.$$

Letting $i \to \infty$,

$$a \leq \sigma^n(x) \leq y.$$

So $\Sigma_{ay} \neq \emptyset$. This implies $\phi(a) \leq y$. But obviously $\phi(a) \geq y$ since $a \geq a^i$ and $\phi(a) \geq \phi(a^i)$. This proves $\phi(a) = y$. □

**Lemma 2.4.** 1. $\phi(0^\infty) = 0^\infty$.
2. $\phi(01^\infty) = 1^\infty$.
3. $\phi(1x) = 1^\infty$, *for any* $x \in \Sigma$.

*Proof.* 1. Since $0^\infty \in \Sigma_{0^\infty, 0^\infty}$, $\phi(0^\infty) = 0^\infty$.

2. Assume $01^\infty \leq \sigma^n(x) \leq 1^\infty, \forall n$ and $x = 0^{n_1}1^{l_1}0^{n_2}1^{l_2}\cdots$. If $n_1 > 0$, then $n_1 = 1$ and $n_i = 0$ for $i > 1$. If $n_1 = 0 (l_1 > 0)$, then either $n_i = 0$ for all $i$ or $n_1 = n_2 = \cdots = n_k = 0$ and $n_{k+1} > 0$, which implies $n_{k+i} > 0$ for all $i > 1$.

So for some $m$ such that $\sigma^m(x) = 1^\infty$. This means $\phi(01^\infty) = 1^\infty$.

3. This is a corollary of 2. and Lemma 2.2. □

**Lemma 2.5.** *Let $a = w01^\infty$ and $a' = w10^\infty$. Then $\phi(a) = \phi(a')$. Moreover if $b = w10^m1v$ and $m > |w|$, then $\phi(b) = \phi(a)$.*

*Proof.* We only prove the second conclusion. The first is just a consequence of the second.

Take $x \in \Sigma_{a\phi(a)}$. If $\sigma^l(x) \geq b$ for all $l \geq 0$, then $x \in \Sigma_{b\phi(a)}$. Hence $\phi(b) = \phi(a)$. We will show that if there is $l \geq 0$ such that $a \leq \sigma^l(x) < b$, then with $k = |w|$ and $n = l + 2(k+1)$ $\sigma^n(x) = 1^\infty$. This implies $\phi(a) = 1^\infty$ and so $\phi(a) = \phi(b)$.

If $\sigma^l(x) = a$, then $\sigma^n(x) = 1^\infty$. On the other hand if $\sigma^l(x) > a$, then $\sigma^l(x) = w10^m u$ $\sigma^{l+k+1}(x) = 0^m u \geq a = w01^\infty$. Since $m > k$, we have $m = k+1$ and $u = 1^\infty$ and hence $\sigma^n(x) = u = 1^\infty$. □

**Corollary 2.6.** *$\phi$ is continuous at $a = w01^\infty$ and $a' = w10^\infty$.*

## 3. Ordering minimality and the image of $\phi$

For any $x \in \Sigma$ denote $I(x) = [i(x), s(x)] = \{y \in \Sigma : i(x) \leq y \leq s(x)\}$, where $i(x) = \inf \overline{\mathrm{Orb}(x)}$ and $s(x) = \sup \overline{\mathrm{Orb}(x)}$ and the infimum and supremum are taken with respect to the order $\leq$. Obviously, $I(x)$ is an $\mathcal{L}$-interval. A point $x \in \Sigma$ is called order minimal if $I(x) \supset I(y)$ then $I(x) = I(y)$. An $\mathcal{L}$-interval is called minimal if it has no proper $\mathcal{L}$-subinterval. Note that if $I$ is a minimal $\mathcal{L}$-interval, then every point in $\Sigma_I$ is order minimal.

**Lemma 3.1.** *Every $\mathcal{L}$-interval contains a minimal $\mathcal{L}$-interval. Therefore, for any $x \in \Sigma$ there is an order minimal $y \in \Sigma$ such that $I(y) \subset I(x)$.*

*Proof.* The inclusion relation on subsets gives a natural partial order on the set of all $\mathcal{L}$-intervals. For any totally ordering $\mathcal{L}$-intervals $\{I_\alpha\}$ let $I = \bigcap_\alpha I_\alpha$.

$$\begin{aligned} \Sigma_I &= \bigcap_l \sigma^{-l} I = \bigcap_l \sigma^{-l} \bigcap_\alpha I_\alpha \\ &= \bigcap_\alpha \bigcap_l \sigma^{-l} I_\alpha = \bigcap_\alpha \Sigma_{I_\alpha} \neq \emptyset. \end{aligned}$$

So $I$ is also an $\mathcal{L}$-interval.

According to Zorn's Lemma, the first conclusion of the lemma is correct.

Since $I(x)$ is an $\mathcal{L}$-interval, $I(x)$ contains a minimal $\mathcal{L}$-interval $I$. Then every point $y \in \Sigma_I$ is order minimal such that $I(y) = I \subset I(x)$. □

**Lemma 3.2.** *If $b = \phi(a)$, then $\sigma^j(b) \leq b, \forall j$.*

*Proof.* Since $b = \phi(a)$, there exists $x \in \Sigma_{ab}$. Let $I(x) = [c, d]$. Then obviously, $c, d \in \Sigma_{ab}$. If $d < b$, then $\phi(a) \leq d < b$. So $b = d \in \Sigma_{ab}$. □

**Lemma 3.3.** *If $b = \phi(a)$, then $\overline{\mathrm{Orb}(b)}$ is a minimal set.*

*Proof.* For any $x \in \overline{\mathrm{Orb}(b)}$, write $I(x) = [c, d]$. According to the definition of $\phi$, $b = d$. This means that $b \in \overline{\mathrm{Orb}(x)}$. And hence $\overline{\mathrm{Orb}(b)} \subset \overline{\mathrm{Orb}(x)}$. This proves the result. □

**Theorem 3.4.** *$b = \phi(a)$ if and only if $\sigma^j(b) \leq b, j \geq 0$ and $b$ is order minimal.*

*Proof.* "$\Rightarrow$". We only need to show $b$ is order minimal. Let $I(b) = [c, b]$. Assume $I(b) \supset I(h) = [d, e]$ and $I(b) \neq I(h)$. Since $\overline{\mathrm{Orb}(b)}$ is a minimal set, $c < d$ and $b > e$. This is contradictory to $\phi(a) = b$.

"$\Leftarrow$". Let $I(b) = [c, b]$. Then $\phi(c) \leq b$. If $\phi(c) = d < b$, then $I(d) \subset I(b)$ but $I(d) \neq I(b)$. This is contradictory to the order minimality of $b$. □

Let $X \subset \Sigma$ be a subshift, i.e., a closed and invariant subset of $\Sigma$. Denote by

$$B_n(X) = \{w \in \{0,1\}^n : w \text{ occurs in some element of } X\}.$$

Similarly, for any $x \in \Sigma$, we can define

$$B_n(x) = \{w \in \{0,1\}^n : w \text{ occurs in } x\}.$$

**Lemma 3.5.** *Assume that $X$ is a minimal set. If there exists $n > 0$ such that $\#B_n = \#B_{n+1}$, then $X$ is trivial, i.e., $X$ is a periodic orbit.*

*Proof.* See [1], Theorem 2.11. □

**Proposition 3.6.** *Assume that $X$ is a minimal set. If $X$ is nontrivial, then $\sigma : X \to X$ is not* 1-1.

*Proof.* Since $X$ is nontrivial, $\#B_n \to \infty$ as $n$ tends to $\infty$. According the above lemma we have $\#B_{n+1} > \#B_n$. Hence there exists $w_n \in B_n$ such that $0w_n, 1w_n \in B_{n+1}$. So let $x_n = 0w_nu_n$ and $y_n = 1w_nv_n$ be in $X$. We may assume that $\lim x_n = x$ and $\lim y_n = y$. Then $x \neq y$ but $\sigma(x) = \sigma(y)$. □

## 4. The image of $\phi$ and Sturmian sequences

**Lemma 4.1.** *For any subsets $A, B \subset \Sigma$ we have $\sigma(A \cap \sigma^{-1}(B)) = \sigma(A) \cap B$.*

*Proof.* Trivial. □

**Lemma 4.2.** *For any $x \in \Sigma$, $I = [0x, 1x]$ is an $\mathcal{L}$-interval.*

*Proof.* We only have to show that $\Lambda_n = \bigcap_{i\geq 0}^{n} \sigma^{-i} I$ is nonempty. By induction we may assume that $\Lambda_n \neq \emptyset$. Now let us show $\Lambda_{n+1} \neq \emptyset$.

$$\begin{aligned} \Lambda_{n+1} &= \bigcap_{i\geq 0}^{n+1} \sigma^{-i} I \\ &= I \cap \sigma^{-1}(\bigcap_{i=0}^{n} \sigma^{-i} I) \\ &= I \cap \sigma^{-1}\Lambda_n. \end{aligned}$$

So, by the above lemma,

$$\begin{aligned} \sigma(\Lambda_{n+1}) &= \sigma(I) \cap \Lambda_n \\ &= \sigma([0x, 01^\infty] \cup [10^\infty, 1x]) \cap \Lambda_n \\ &= ([x, 1^\infty] \cup [0^\infty, x]) \cap \Lambda_n \\ &= \Sigma \cap \Lambda_n = \Lambda_n \\ &\neq \emptyset. \end{aligned}$$

□

**Corollary 4.3.** *If $x$ is order minimal, then*
(BC) *for any $n$ and $B \in \{0,1\}^n$, $0B0$ and $1B1$ cannot both occur in $b$.*
*In particular, only one of* 00 *and* 11 *can occur in $b$.*

The condition (BC) in the above corollary will be called the Block Condition. Note that according to [1], Lemma 3.06, (BC) is equivalent to the Sturmian block condition there. We have the following simple property for the BC.

**Lemma 4.4.** *$b \in \Sigma$ satisfies the BC if and only if there exists $z \in \Sigma$ such that* $\mathrm{Orb}(b) \subset [0z, 1z]$.

*Proof.* Assume that $\mathrm{Orb}(b) \subset [0z, 1z]$ for some $z \in \Sigma$. If both $0B0$ and $1B1$ occur in $b$ for some $B \in \{0,1\}^n$, then there exist $x, y \in \Sigma$ such that $0B0x, 1B1y \in \mathrm{Orb}(b) \subset [0z, 1z]$. Hence, $B0x \geq z \geq B1y$, which is absurd.

Now, assume that $b$ satisfies the BC. Suppose that $I(b) = [0x, 1y]$ and $x < y$. Let $i$ be the smallest integer such that $x_i = 0, y_i = 1$. Let $B = x_1 \cdots x_{i-1} = y_1 \cdots y_{i-1}$. Then both $0B0$ and $1B1$ occur in $b$, which is contradictory to the BC. □

Then let us recall the definition of Sturmian sequence.

For any $\alpha \in (0,1)$, let $J_\alpha = [0, \alpha)$ and $\bar{J}_\alpha = (0, \alpha]$. For any $x \in [0,1)$ define $x^\alpha, \bar{x}^\alpha \in \Sigma$ as follows:

$$x^\alpha(i) = \begin{cases} 1, & x + (i-1)\alpha \in J_\alpha (\mathrm{mod}\, 1), \\ 0, & \text{otherwise}; \end{cases}$$
$$\bar{x}^\alpha(i) = \begin{cases} 1, & x + (i-1)\alpha \in \bar{J}_\alpha (\mathrm{mod}\, 1), \\ 0, & \text{otherwise}. \end{cases}$$

As usual let $M_\alpha = \{x^\alpha, \bar{x}^\alpha : x \in [0,1)\}$, and then let $M_0 = \{0^\infty\}$ and $M_1 = \{1^\infty\}$. For any $\alpha \in [0,1]$, a sequence in $M_\alpha$ is called a Sturmian sequence.

**Lemma 4.5** ([1]). *1) $M_\alpha$ is a minimal set for any $\alpha \in [0,1]$.*

*2) $x \in \Sigma$ is a Sturmian sequence if and only if $\overline{\mathrm{Orb}(x)}$ is a minimal set and $x$ satisfies the block condition.*

So it is clear that

**Theorem 4.6.** *$b \in \Sigma$ is order minimal and $\sigma^i b \leq b$ for $i \geq 0$. Then $b$ is a Sturmian sequence. Hence if $b = \phi(a)$, then $b$ is a Sturmian sequence.*

**Corollary 4.7.** *$\Lambda$ is a minimal set and contained in $[0x, 1x]$ for some $x \in \Sigma$ if and only if $\Lambda$ is a Sturmian minimal set.*

## 5. Sturmian sequences and the image of $\phi$ (continued)

In the last section we have shown that elements in the image of $\phi$ are all Sturmian sequences. In this section, we show the converse: for every $M_\alpha$, the largest element $\bar{\alpha}^\alpha$ is in the image of $\phi$. See Theorem 5.5.

For any $n \geq 1$ define

$$\Lambda_n^0 = \{x \in \Sigma : x = 10^{n_1}10^{n_2}1\cdots, n_i \in \{n, n+1\}\},$$
$$\Lambda_n^1 = \{x \in \Sigma : x = 1^{n_1}01^{n_2}0\cdots, n_i \in \{n, n+1\}\}.$$

And

$$\Lambda_n = \Lambda_n^0 \cup \Lambda_n^1, \quad \Lambda = \bigcup_{n \geq 1} \Lambda_n.$$

And define a natural mapping $f_n^i : \Lambda_n^i \to \Sigma$ as follows:

$$f_n^0(10^{n_1}10^{n_2}1\cdots) = (n+1-n_1)(n+1-n_2)\cdots,$$
$$f_n^1(1^{n_1}01^{n_2}0\cdots) = (n_1 - n)(n_2 - n)\cdots.$$

When no confusion will result, we may write $f_n^i$ as $f_n$ or simply $f$. Note that there is one possible confusion for $f$, i.e., $f_n(1^n01^n0\cdots) = 00\cdots$ but $f_{n-1}(1^n01^n0\cdots) = 11\cdots$.

*Remark* 5.1. $f_n^i$ is an order-preserving homeomorphism (1-1, onto and continuous) and maps periodic points to periodic points (but strictly decreases the periods).

**Lemma 5.2.** *If $b$ satisfies BC, $b \neq 0^\infty, 10^\infty, 1^\infty$ and $\sigma^i b \leq b$ for $i \geq 0$, then $b \in \Lambda$.*

*Proof.* We may assume that 00 does not occur in $b$. (The other case can be discussed similarly.)

Since $b \neq 1^\infty$, $\sigma^i b \leq b$ for all $i \geq 0$ and 00 does not occur in $b$, $b$ has the form $1^{n_1}01^{n_2}0\cdots$, $n_i > 0$. Obviously, $n_1 \geq n_i$ for all $i > 0$. Furthermore, we have $n_i \geq n_1 - 1$. In fact, if $n_i \leq n_1 - 2$, then $01^{n_i}0$ and $11^{n_i}1 \in B_{n_i+2}(b)$, which is impossible since $b$ satisfies the BC. □

We need the following lemma (see [3], Theorem 8.1).

**Lemma 5.3.** *$f$ maps Sturmian sequences to Sturmian sequences.*

**Lemma 5.4.** *For any $x \in \Sigma$, there exists a unique Sturmian minimal set $M_\alpha \subset [0x, 1x]$.*

*Proof.* The existence is a consequence of Lemma 3.1, Lemma 4.2 and Theorem 4.6.

If there are two different Sturmian minimal sets $M_\alpha, M_\beta \subset [0x, 1x]$ with $\alpha \neq \beta$, let $a = \sup M_\alpha$,$b = \sup M_\beta$ so that $\sigma^i a \leq a$ and $\sigma^i b \leq b$ for $i \geq 0$. We may assume that $a < b$ (since $M_\alpha \cap M_\beta = \emptyset, a \neq b$). If $b = 1z$, then $z \leq x$ and so $0z \leq 0x$. So, without loss of generality, let $b = 1x$. Since $a < b$, $\overline{\mathrm{Orb}(a)} \subset [0x, 1x)$. So if $1y \in \overline{\mathrm{Orb}(a)}$, then $1y < 1x$ and hence $0y < 0x$ which implies $0y \notin \overline{\mathrm{Orb}(a)}$ and by Proposition 3.6 $\mathrm{Orb}(a)$ is a periodic orbit. Since $a < b$, $b \neq 1^\infty$. We may assume that 00 does not occur in $b$ (another case can be treated similarly). Let $b = 1^{n_1}01^{n_2}0\cdots$. If $n_1 = 1$, then 00 must occur in $a$ and since $a$ is periodic, this will be contradictory to $\sigma^i(a) \geq 0x$. So $n = n_1 - 1 > 0$. We claim that $a, b \in \Lambda_n^1$.

In fact, since $\sigma^j b \leq b$ for $j \geq 0, n_i \leq n+1$. And since $\sigma^j b \geq 0x$ for $j \geq 0, n_i \geq n$. So $b \in \Lambda_n^1$. Since $\sigma^i a \geq 0x$, 00 could not occur in $a$. Since $a$ is periodic, $a$ has the form $1^{m_1}01^{m_2}0\cdots$. Since $\sigma^i a \leq a$, $m_i \leq m_1$ for $i \geq 1$. And $a < b$ implies $m_1 \leq n+1$ and hence $m_i \leq n+1$ for $i \geq 1$. And $\sigma^i a > 0x$ implies $m_i \geq n$ for $i \geq 2$. And hence $m_1 \geq n$.

Let $f = f_n^1 : \Lambda_n^1 \to \Sigma$. Since $f$ maps Sturmian sequences to Sturmian sequences and periodic sequences to periodic sequences, $f(a), f(b)$ are Sturmian sequences and $f(a)$ is periodic.

For any $z = 1^{s_1}01^{s_2}0\cdots \in \Lambda_n^1$, denote by $l_i(z) = s_1 + s_2 + \cdots + s_i + i$ for $i \geq 1$ and $l_0(z) = 0$.

Let $f(b) = 1y$. Then $f(x) = 0y$. In the following we will show that for $i \geq 0, \sigma^i f(a), \sigma^i f(b) \in [0y, 1y]$.

Since $f$ is order-preserving, $\sigma^i f(b) = f(\sigma^{l_i(b)}) \leq f(b)$ for $i \geq 0$. Since $\sigma^{l_i(b)-1} b \geq 0x, \sigma^{l_i(b)} \geq x$, which implies $\sigma^i f(b) = f(\sigma^{l_i(b)} b) \geq f(x) = 0y$ for $i > 0$. But, obviously, we have $b > x (\in \Lambda_n^1)$ and hence $f(b) > f(x) = 0y$.

Similarly, we can show that $\sigma^i f(a) \leq 1y$ for $i \geq 0$ and $\sigma^i f(a) \geq 0y$ for $i > 0$. But since $f(a)$ is periodic, we have $f(a) \geq 0y$.

Since the period of $f(a)$ is strictly less than that of $a$, if we take the pair $(a, b)$ such that the period of $a$ is the smallest, this will be a contradiction. □

**Theorem 5.5.** *Let $b \in \Sigma$ be a Sturmian sequence. Then $b$ is order minimal.*

*Proof.* Since $b$ satisfies the Block Condition, $I(b) \subset [0z, 1z]$ for some $z \in \Sigma$. Assume that $b$ is not order minimal. Then there is an order minimal $c \in \overline{\mathrm{Orb}(b)}$ such that

$\sigma^i c \leq c$ for $i \geq 0$, $I(c) \subset I(b)$ and $I(c) \neq I(b)$. So $c$ is a Sturmian sequence according to Theorem 4.6. According to the above lemma, this is impossible. □

**Corollary 5.6.** *Assume that $b \in \Sigma$. The following conditions are equivalent.*

1) *$b = \phi(a)$ for some $a \in \Sigma$.*
2) *$b$ is order minimal and $\sigma^i b \leq b$ for all $i \geq 0$.*
3) *$b$ is Sturmian sequence and $\sigma^i b \leq b$ for all $i \geq 0$.*

So now the mapping $\phi$ can be calculated as following.

**Theorem 5.7.** *If $a = 1x \in \Sigma$, then $\phi(a) = 1^\infty$. If $a = 0x$, then $\phi(a)$ is equal to the Sturmian sequence $b$ such that $\mathrm{Orb}(b) \subset [0x, 1x]$ and $\sigma^i(b) \leq b$ for all $i \geq 0$.*


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