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GENERALIZED LITTLE q-JACOBI POLYNOMIALS AS EIGENSOLUTIONS OF HIGHER-ORDER q-DIFFERENCE OPERATORS

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Abstract. We consider the polynomials $p_n(x; a, b; M)$ obtained from the little q-Jacobi polynomials $p_n(x; a, b)$ by inserting a discrete mass M at $x = 0$ in the orthogonality measure. We show that for $a = q^j$, $j = 0, 1, 2, \dots$, the polynomials $p_n(x; a, b; M)$ are eigensolutions of a linear q-difference operator of order $2j+4$ with polynomial coefficients. This provides a q-analog of results recently obtained for the Krall polynomials.

1. q-derivative operators and their representation coefficients

Let T be the q-shift operator that acts on functions according to

$$T\,F(x) = F(qx), \tag{1.1}$$

with $0 < q < 1$ a real number. Obviously

$$T^n\,F(x) = F(q^n x), \quad n = 0, \pm 1, \pm 2, \dots. \tag{1.2}$$

Introduce the q-derivative operator (see, e.g., [2])

$$\mathcal{D}_q F(x) = (x(1-q))^{-1}\,(1-T). \tag{1.3}$$

It is called the q-derivative because its action on monomials is

$$\mathcal{D}_q x^n = [n]\,x^{n-1}, \tag{1.4}$$

with

$$[n] = (q^n - 1)/(q-1) \tag{1.5}$$

the so-called q-number. Moreover $\lim_{q\to 1}\mathcal{D}_q = D$, where D is the ordinary derivative operator with respect to x: $DF(x) = F'(x)$.

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Consider operators of the form

$$L_q = \sum_{k=0}^{2N} a_k(x)\, \mathcal{D}_q^k, \tag{1.6}$$

where N is a fixed positive integer and $a_k(x)$ are polynomials in x of degrees not exceeding k:

$$a_k(x) = \sum_{s=0}^{k} \alpha_{ks} x^s, \quad k = 0, 1, \ldots, 2N. \tag{1.7}$$

Let us introduce also the related operators

$$\mathcal{L}_q = T^{-N}\, L_q = \sum_{k=0}^{2N} a_k(q^{-N}\, x)\, T^{-N}\, \mathcal{D}_q^k. \tag{1.8}$$

The operator L_q is a linear combination of the operators $T^{2N}, T^{2N-1}, \ldots, T, T^0 = I$, while the operator $\mathcal{L}_q$ is a linear combination of the operators $T^N, T^{N-1}, \ldots, T^{1-N}$, T^{-N}.

For $q \to 1$ both operators L_q and $\mathcal{L}_q$ become $2N$-order differential operators with polynomial coefficients:

$$\lim_{q\to 1} L_q = \lim_{q\to 1} \mathcal{L}_q = \sum_{k=0}^{2N} a_k^{(0)}(x)\, D^k, \tag{1.9}$$

where $a_k^{(0)}(x) = \lim_{q\to 1} a(x)$.

The operator $\mathcal{L}_q$ will prove more practical in searching for orthogonal polynomials $P_n(x)$ satisfying eigenvalue equations of the kind

$$\mathcal{L}_q P_n(x) = \lambda_n P_n(x). \tag{1.10}$$

It is known that for $N = 1$, the little q-Jacobi polynomials satisfy an equation of the form (1.10) with $N = 1$ [9]. We wish to determine other systems of orthogonal polynomials satisfying equation (1.10) with $N > 1$.

To this end, we shall extend to q-difference operators the method proposed in [12].

The main idea of the method is the following. Consider the action of the operator $\mathcal{L}_q$ upon the monomials x^n. From (1.6) and (1.7) we get

$$\mathcal{L}_q x^n = \sum_{s=0} A_n^{(s)} x^{n-s}, \tag{1.11}$$

where

$$A_n^{(s)} = q^{N(s-n)}\, [n][n-1]\ldots[n-s+1]\, \pi_s(q^n), \tag{1.12}$$

and

$$\pi_s(q^n) = \alpha_{s0} + \sum_{i=1}^{2N-s} \alpha_{s+i,i}[n-s][n-s-1]\ldots[n-s-i+1] \tag{1.13}$$

are polynomials in $z = q^n$ of degrees not exceeding $2N-s$. It is clear that, moreover,

$$A_n^{(s)} = 0, \quad s > 2N. \tag{1.14}$$

The coefficients $A_n^{(s)}$ completely characterize the operator $\mathcal{L}_q$. We will call $A_n^{(s)}$ the representation coefficients of the operator $\mathcal{L}_q$.

Proposition 1.1. *Assume that there are coefficients $A_n^{(s)}$ expressible as in (1.12), where $\pi_s(q^n)$ are arbitrary polynomials in q^n of degrees not exceeding $2N - s$. Assume also that $A_n^{(s)} = 0,\ s > 2N$ (i.e. $\pi_s(q^n) = 0$ for $s > 2N$). Then there exists a unique operator $\mathcal{L}_q$ of the form (1.8) such that $A_n^{(s)}$ are its representation coefficients.*

Proof. Any polynomial $\pi_s(q^n)$ of degree not exceeding $2N - s$ can be presented in form (1.13) with some coefficients α_{ik}. These coefficients are determined using Newton's interpolation formula

$$\alpha_{s+i,i} = \frac{(q-1)^i q^{si+i(i-1)/2}}{[i]!} \mathcal{D}_q^i \pi_s(x)\Big|_{x=q^s}, \quad i = 0, 1, \ldots, 2N - s, \tag{1.15}$$

where $[i]! = [1][2]\ldots[i]$ is the q-factorial. Clearly, the coefficients α_{ik} are determined uniquely by (1.15) from the given polynomials $\pi_s(q^n)$. Hence the operator $\mathcal{L}_q$ is defined uniquely.

2. Basic relations for polynomials satisfying eigenvalue equations

In this section we consider the basic relations between the representation coefficients $A_n^{(s)}$ and the expansion coefficents of polynomials $P_n(x)$ satisfying eigenvalue equations. Assume that

$$P_n(x) = \sum_{s=0}^{n} B_n^{(s)} x^{n-s}, \tag{2.1}$$

where $B_n^{(s)}$ are expansion coefficients. In what follows we will assume that the polynomials $P_n(x)$ are monic, i.e. that

$$B_n^{(0)} = 1. \tag{2.2}$$

Substituting (2.1) into the eigenvalue equation (1.10) we arrive at the following set of algebraic relations:

$$\sum_{i=0}^{s} B_n^{(s-i)}\, A_{n-s+i}^{(i)} = \lambda_n\, B_n^{(s)}, \quad s = 0, 1, 2, \ldots, n. \tag{2.3}$$

These will be central in our analysis.

For $s = 0$, (2.3) gives

$$\lambda_n = A_n^{(0)}. \tag{2.4}$$

Thus from (1.12) we find that the eigenvalues λ_n have the expression

$$\lambda_n = q^{-Nn}\, \pi_0(q^n), \tag{2.5}$$

where $\pi_0(q^n)$ is a polynomial in q^n of degree not exceeding $2N$.

Similarly, for $s = 1$, (2.3) yields

$$A_n^{(1)} = \Omega_n\, B_n^{(1)}, \tag{2.6}$$

where $\Omega_n = \lambda_n - \lambda_{n-1}$. From this relation we find that

$$B_n^{(1)} = [n]\, \frac{\pi_1(q^n)}{q^{-N}\pi_0(q^n) - \pi_0(q^{n-1})}. \tag{2.7}$$

Relation (2.3) can be rewritten in the form

$$(\lambda_n - \lambda_{n-s})\, B_n^{(s)} = B_n^{(s-1)} A_{n-s+1}^{(1)} + B_n^{(s-2)} A_{n-s+2}^{(2)} + \cdots + A_n^{(s)}. \tag{2.8}$$

From this relation we can conclude, by induction, that the coefficients $B_n^{(s)}$ are rational functions in q^n, namely that

$$B_n^{(s)} = [n][n-1]\ldots[n-s+1]\frac{Q_{1,s}(q^n)}{Q_{2,s}(q^n)}, \tag{2.9}$$

where $Q_{1,s}(q^n)$ is a polynomial of degree not exceeding $2Ns-s$, whereas the degree of the polynomial

$$Q_{2,s}(q^n) = \prod_{i=1}^{s}\left(q^{-iN}\pi_0(q^n) - \pi_0(q^{n-i})\right) \tag{2.10}$$

does not exceed $2Ns$.

The problem considered up to this point of finding the expansion coefficients $B_n^{(s)}$ of the polynomials $P_n(x)$ when the representation coefficients $A_n^{(s)}$ are given always has a unique solution.

Proposition 2.1. *Assume that the representation coefficients $A_n^{(s)}$ of the operator $\mathcal{L}_q$ satisfy the requirement*

$$A_n^{(0)} \neq A_m^{(0)}, \quad n \neq m. \tag{2.11}$$

Then there exists a unique set of monic polynomials $P_n(x)$, $n = 0, 1, \ldots$, satisfying equation (1.10).

Proof. We find λ_m from (2.4); in view of condition (2.11), a unique $B_n^{(1)}$ is found from (2.6). Assuming that all $B_n^{(1)}, B_n^{(2)}, \ldots, B_n^{(s-1)}$ have thus been found recursively, $B_n^{(s)}$ is then determined in an unambigous way owing to (2.11).

The polynomials $P_n(x)$ are not orthogonal in general. The requirement that they form an orthogonal set implies strong additional restrictions upon the coefficents $B_n^{(s)}$ and $A_n^{(s)}$. Indeed, orthogonal polynomials satisfy a three-term recurrence relation of the form [1]

$$P_{n+1}(x) + b_n\, P_n(x) + u_n\, P_{n-1}(x) = xP_n(x), \tag{2.12}$$

with b_n and u_n referred to as the recurrence parameters. From (2.12) and (2.1) we get the set of relations

$$B_{n+1}^{(s+1)} - B_n^{(s+1)} + u_n\, B_{n-1}^{(s-1)} + b_n\, B_n^{(s)} = 0, \quad s = 0, 1, \ldots, n, \tag{2.13}$$

where it is assumed that $B_n^{(-1)} = B_n^{(n+1)} = 0$. Putting $s = 0$ and $s = 1$ in (2.13), we find

$$\begin{aligned} b_n &= B_n^{(1)} - B_{n+1}^{(1)}, \\ u_n &= B_n^{(2)} - B_{n+1}^{(2)} - b_n\, B_n^{(1)}. \end{aligned} \tag{2.14}$$

Taking into account that the coefficients $B_n^{(s)}$ are rational functions in q^n we arrive at the following proposition.

Proposition 2.2. *If the orthogonal polynomials $P_n(x)$ are eigenfunctions of the operator* (1.8)*, their recurrence coefficients b_n, u_n are rational functions of the argument q^n.*

The problem of reconstructing the representation coefficients $A_n^{(s)}$ when the expansion coefficients $B_n^{(s)}$ are given is more difficult. The coefficients $B_n^{(s)}$ must of course be rational functions in q^n, since otherwise the problem has no solutions. Assume therefore that

$$B_n^{(s)} = [n][n-1]\dots[n-s+1]\frac{G_{1,s}(q^n)}{G_{2,s}(q^n)}, \tag{2.15}$$

where the degree of the polynomial $G_{1,s}(q^n)$ does not exceed $2Ms - s$ (for some positive integer $M \leq N$) whereas the degree polynomial $G_{2,s}(q^n)$ does not exceed $2Ms$. We assume that the polynomials $G_{1,s}(x)$ and $G_{2,s}(x)$ have no common divisors. Note that in this case the polynomial $G_{2,s}(q^n)$ need not coincide with expression (2.10) because in the expression (2.9) polynomials $Q_{1,s}(x)$ and $Q_{2,s}(x)$ may have coinciding zeroes.

Consider relation (2.6) written in the form

$$\frac{A_n^{(1)}}{\Omega_n} = [n]\,\frac{G_{1,1}(q^n)}{G_{2,1}(q^n)}. \tag{2.16}$$

Since both $q^{nN}A_n^{(1)}$ and $q^{nN}\Omega_n$ should be polynomials in q^n of degrees not exceeding $2N$, we have

$$\begin{aligned} A_n^{(1)} &= q^{N(1-n)}\,\rho_1(q^n)\,[n]\,G_{1,1}(q^n),\\ \Omega_n &= q^{N(1-n)}\,\rho_1(q^n)\,G_{2,1}(q^n), \end{aligned} \tag{2.17}$$

where $\rho_1(q^n)$ is some polynomial of degree not exceeding $2N - 2M$.

The relation (2.3), for $s = 2$, can then be rewritten in the form

$$A_n^{(2)} = (\lambda_n - \lambda_{n-2})\,B_n^{(2)} - A_{n-1}^{(1)}\,B_n^{(1)} = (\Omega_n + \Omega_{n-1})\,B_n^{(2)} - A_{n-1}^{(1)}\,B_n^{(1)}. \tag{2.18}$$

The representation coefficient $A_n^{(2)}$ is thus determined uniquely if $A_n^{(1)}$ and Ω_n are known.

Assume that the coefficients $A_n^{(2)}, A_n^{(3)}, \dots, A_n^{(k-1)}$ have been determined iterating this process. With $s = k$, (2.3) can now be rewritten in the form

$$A_n^{(k)} = (\lambda_n - \lambda_{n-k})B_n^{(k)} - B_n^{(k-1)}A_{n-k+1}^{(1)} - B_n^{(k-2)}A_{n-k+2}^{(2)} - \dots - B_n^{(1)}A_{n-1}^{(k-1)}. \tag{2.19}$$

Taking into account the fact that

$$\lambda_n - \lambda_{n-k} = \Omega_n + \Omega_{n-1} + \dots + \Omega_{n-k+1},$$

we see that $A_n^{(k)}$ is completely determined from the coefficients $\Omega_n, A_n^{(1)}, \dots, A_n^{(k-1)}$. For the $A_n^{(s)}$ thus obtained to actually be representation coefficients of an operator $\mathcal{L}_q$, they necessarily need to satisfy, in addition, the conditions of Proposition 1.1. When this is so, the corresponding polynomials are eigenfunctions of the operator $\mathcal{L}_q$.

3. Little q-Jacobi polynomials

The monic little q-Jacobi polynomials [9] are defined as

$$P_n(x;a,b) = (-1)^n\,\frac{q^{n(n-1)/2}\,(aq;q)_n}{(abq^{n+1};q)_n}\;{}_2\phi_1\left(\left.\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix}\,\right|qx\right), \tag{3.1}$$

where $(a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1})$ is the q-shifted factorial and ${}_2\phi_1$ denotes the q-hypergeometric function (see, e.g., [2]).

The orthogonality relation is

$$\sum_{k=0}^{\infty} w_k P_n(q^k; a, b)\, P_m(q^k; a, b) = h_n \delta_{nm}, \tag{3.2}$$

where h_n are appropriate normalization constants, and the normalized weight function is

$$w_k = \frac{(aq; q)_\infty}{(abq^2; q)_\infty} \, \frac{(bq; q)_k\, (aq)^k}{(q; q)_k}. \tag{3.3}$$

It is assumed that $0 < aq < 1$, $b < q^{-1}$. The expansion coefficients of the little q-Jacobi polynomials are

$$B_n^{(s)} = b^{-s} \, \frac{(q^{-n}, a^{-1}q^{-n}; q)_s}{(q, a^{-1}b^{-1}q^{-2n}; q)_s}. \tag{3.4}$$

It is easily verified that equations (2.8) have the following solutions:

$$\begin{aligned} A_n^{(0)} &= \lambda_n = [n](q^{1-n} - abq^2), \\ A_n^{(1)} &= [n](aq - q^{1-n}), \\ A_n^{(s)} &= 0, \quad s \geq 2. \end{aligned} \tag{3.5}$$

Hence, the little q-Jacobi polynomials satisfy a second-order q-difference equation, as is well known [9].

We also need the value of the function of second kind, $Q_n(z)$, at $z = 0$ (the point $z = 0$ is an accumulation point of the orthogonality measure):

$$Q_n(0; a, b) = -\sum_{k=0}^{\infty} \frac{P_n(q^k; a, b)\, w_k}{q^k}. \tag{3.6}$$

This sum can be evaluated using the q-binomial theorem and the q-Saalschütz formula (see, e.g., [2]):

$$Q_n(0; a, b) = (-1)^{n+1} a^n q^{n(n-1)/2} \, \frac{1 - abq}{1 - a} \, \frac{(q; q)_n (bq; q)_n}{(abq; q)_n (abq^{n+1}; q)_n}. \tag{3.7}$$

Taking into account that

$$P_n(0; a, b) = (-1)^n q^{n(n-1)/2} \, \frac{(aq; q)_n}{(abq^{n+1}; q)_n}, \tag{3.8}$$

we note that if $a = q^j$, $j = 1, 2, 3, \ldots$, then

$$\begin{aligned} \Phi_n &= Q_n(0; q^j, b) + \beta P_n(0; q^j, b) \\ &= (-1)^n q^{n(n-1)/2} \frac{(q^{j+1}; q)_n}{(bq^{n+j+1}; q)_n} \left(\beta - q^{nj} \, \frac{(1 - bq^{j+1})\, (bq; q)_j\, (q; q)_j}{(1 - q^j)\, (q^{n+1}; q)_j\, (bq^{n+1}; q)_j} \right). \end{aligned} \tag{3.9}$$

4. Transformed q-Jacobi polynomials

Let $P_n(x)$ be arbitrary orthogonal polynomials with measure localized on the interval $[a, b]$. The corresponding weight function $w(x)$ is assumed to be normalized to 1, i.e.

$$\int_a^b w(x)\, dx = 1.$$

Introduce the functions of second kind,

$$Q_n(z) = \int_a^b \frac{P_n(x)\, w(x)}{z-x} dx. \tag{4.1}$$

Let c be a point beyond the orthogonality interval $[a, b]$ such that $Q_n(c)$ exists.

Consider the polynomials

(4.2)

$$\tilde{P}_n(x) = \mathcal{G}(c)\{P_n(x)\} = P_n(x) - \frac{\Phi_n}{\Phi_{n-1}}\, P_{n-1}(x), \quad n = 1, 2, \ldots, \quad \tilde{P}_0(x) = 1,$$

where

$$\Phi_n = Q_n(c) + \beta\, P_n(c). \tag{4.3}$$

The notation $\mathcal{G}(c)\{P_n(x)\}$ stands for the Geronimus transformation [3], [4] of the polynomials $P_n(x)$ at the point $x = c$ (for details see, e.g., [11]). The weight function $\tilde{w}(x)$ of the polynomials $\mathcal{G}(c)\{P_n(x)\}$ is

$$\tilde{w}(x) = \kappa \left(\frac{w(x)}{x-c} - \beta\, \delta(x-c) \right), \tag{4.4}$$

where κ is an appropriate normalization constant. The Geronimus transformation thus inserts a concentrated mass at the point $x = c$. The value of this mass depends on the parameter β.

Return to the case of the little q-Jacobi polynomials with $a = q^j,\ j = 1, 2, 3, \ldots.$ Perform the Geronimus transformation (4.2) with Φ_n given by (3.9). (In this case $c = 0$.)

The weight function $\tilde{w}(x)$ for the polynomials $\mathcal{G}(c)\{P_n(x)\}$ is

$$\tilde{w}(x) = \kappa \left(\sum_{k=0}^{\infty} \tilde{w}_k \delta(x - q^k) - \beta\, \delta(x) \right), \tag{4.5}$$

where

$$\tilde{w}_k = \frac{(q^{j+1}; q)_\infty}{(bq^{j+2}; q)_\infty}\, \frac{(bq; q)_k\, q^{jk}}{(q; q)_k}. \tag{4.6}$$

The weight function (4.5) can be rewritten in the form

$$\tilde{w}(x) = \kappa_1 \left(w(x; j-1) + M\, \delta(x) \right), \tag{4.7}$$

where

$$M = -\beta\, \frac{1 - q^j}{1 - bq^{j+1}}, \quad \kappa_1 = \kappa \frac{1 - bq^{j+1}}{1 - q^j},$$

and $w(x; j-1)$ is the weight function corresponding to the little q-Jacobi polynomials with the parameter $a = q^j$ replaced with $a = q^{j-1}$, i.e.

$$w(x; j-1) = \sum_{k=0}^{\infty} w_k(j-1) \delta(x - q^k), \tag{4.8}$$

and

$$w_k(j-1) = \frac{(q^j; q)_\infty}{(bq^{j+1}; q)_\infty} \frac{(bq; q)_k q^{jk}}{(q; q)_k}. \tag{4.9}$$

Thus the weight function $\tilde{w}(x;j)$ for the polynomials $\mathcal{G}(0)\{P_n(x)\}$ is obtained from the weight function $w(x;j-1)$ through the addition of an arbitrary mass M at the point $x=0$.

In the expansion

$$\mathcal{G}(0)\{P_n(x;q^j,b)\} = \sum_{s=0}^{n} B_n^{(s)}\, x^{n-s}, \tag{4.10}$$

the coefficients $B_n^{(s)}$ are found from (3.1) and (3.9):

$$\begin{aligned} B_n^{(s)} = b^{-s}\, &\frac{(q^{-n};q)_s\,(q^{-n-j};q)_s}{(q;q)_s\,(b^{-1}q^{-j-2n};q)_s} \\ &\times \left(1 - q^{n+j-s}\,\frac{(1-bq^n)(1-q^s)}{(1-q^{n+j})(1-bq^{j+2n-s})}\,\frac{Y_j(n)}{Y_j(n-1)}\right), \end{aligned} \tag{4.11}$$

with

$$Y_j(n) = \beta q^{-jn}(q^{n+1};q)_j(bq^{n+1};q)_j - (q;q)_{j-1}(bq;q)_{j+1}. \tag{4.12}$$

5. Construction of the coefficients $A_n^{(s)}$

In this section we construct the coefficients $A_n^{(s)}$ for the q-difference operator $\mathcal{L}_q$ that has the polynomials $\tilde{P}_n(x)$ as eigenfunctions.

We start from the relation

$$A_n^{(1)} = \Omega_n\, B_n^{(1)}, \tag{5.1}$$

where

$$\Omega_n = \lambda_n - \lambda_{n-1} = A_n^{(0)} - A_{n-1}^{(0)}. \tag{5.2}$$

Choose Ω_n proportional to the denominator of $B_n^{(1)}$:

$$\begin{aligned} \Omega_n = bq^{n+j-1}(q-1)&(1-b^{-1}q^{1-j-2n}) \\ &\times \left(\beta q^{-j(n-1)}(q^n;q)_j(bq^n;q)_j - (bq;q)_{j+1}(q;q)_{j-1}\right). \end{aligned} \tag{5.3}$$

Then from (5.1) we get for $A_n^{(1)}$ the expression

$$\begin{aligned} A_n^{(1)} = (1-q^{-n})\Big(&\beta q^{j(1-n)}(q^{n+1};q)_j(bq^n;q)_j\,(1-q^{n-1}) \\ &- (q;q)_{j-1}(bq;q)_{j+1}(1-q^{n+j-1})\Big). \end{aligned} \tag{5.4}$$

Using (5.2) it is not difficult to show that

$$\begin{aligned} A_n^{(0)} = \lambda_n = &\frac{\beta\,(q-1)q^{-n(j+1)-1}(q^n;q)_{j+1}(bq^n;q)_{j+1}}{1-q^{-j-1}} \\ &- (q^{-n}-1)(1-bq^{n+j})(bq;q)_{j+1}(q;q)_{j-1}. \end{aligned} \tag{5.5}$$

(Note that $A_0^{(0)} = 0$.)

Now using the explicit expressions for $B_n^{(1,2)}$ and $A_n^{(0,1)}$, we find

$$A_n^{(2)} = \beta(q-1)q^{(2-n)(j+1)-1}\,\frac{(1-q^{-j})(q^{n-2};q)_{j+3}(bq^n;q)_{j-1}}{(q;q)_2}. \tag{5.6}$$

Repeating this procedure for $s = 3, 4, \dots$ one can guess the expression

$$A_n^{(s)} = \beta(q-1)q^{(s-n)(j+1)-1}\,\frac{(q^{-j};q)_{s-1}(q^{n-s};q)_{s+j+1}(bq^n;q)_{j-s+1}}{(q;q)_s} + \xi_n\delta_{s,1} + \eta_n\delta_{s,0}, \tag{5.7}$$

where

$$\xi_n = (q^{-n}-1)(q;q)_{j-1}(bq;q)_{j+1}(1-q^{j+n-1}),$$
$$\eta_n = (1-q^{-n})(1-bq^{n+j})(bq;q)_{j+1}(q;q)_{j-1},$$

and $s = 0, 1, 2, \dots$.

Proposition 5.1. *The coefficients $A_n^{(s)}$ given by* (5.7) *satisfy the basic relations*

$$\sum_{i=0}^{s} B_n^{(s-i)}\, A_{n-s+i}^{(i)} = A_n^{(0)}\, B_n^{(s)}. \tag{5.8}$$

Proof. Using the explicit expressions for $B_n^{(s)}$ and $A_n(s)$ we can rewrite the lhs of (5.8) in the form

$$\sum_{i=0}^{s} B_n^{(s-i)}\, A_{n-s+i}^{(i)} = \eta_{n-s}B_n^{(s)} + \xi_{n-s+1}B_n^{(s-1)} + \kappa_n\left(S_1 + \nu_n\, S_2\right), \tag{5.9}$$

where

$$\kappa_n = \beta(q-1)b^{-s}\,q^{(j+1)(s-n)-1}\,\frac{(q^{-n};q)_s(q^{-j-n};q)_s(q^{n-s};q)_{j+1}(bq^{n-s};q)_{j+1}}{(q;q)_s(b^{-1}q^{-j-2n};q)_s(1-q^{-j-1})},$$
$$\nu_n = \frac{q^{n+j}(1-bq^n)(1-q^{-s})Y_j(n)}{(1-q^{n+j})(1-bq^{j+2n-s})Y_j(n-1)},$$

and S_1, S_2 are the sums

$$S_1 = \sum_{i=0}^{s}\frac{q^i(q^{-s};q)_i(bq^{j+2n+1-s};q)_i(q^{-j-1};q)_i}{(q;q)_i(q^{n+1-s};q)_i(bq^{n-s};q)_i},$$
$$S_2 = \sum_{i=0}^{s}\frac{q^i(q^{1-s};q)_i(bq^{j+2n-s};q)_i(q^{-j-1};q)_i}{(q;q)_i(q^{n+1-s};q)_i(bq^{n-s};q)_i}.$$

These sums can be evaluated using the q-analog of the Saalschütz formula [2]:

$$S_1 = q^{(j+1)s}\,\frac{(b^{-1}q^{-j-n},\ q^{-1-j-n};q)_s}{(b^{-1}q^{1-n},\ q^{-n};q)_s},$$
$$S_2 = q^{(j+1)(s-1)}\,\frac{(b^{-1}q^{1-j-n},\ q^{-j-n};q)_{s-1}}{(b^{-1}q^{2-n},\ q^{1-n};q)_{s-1}}.$$

Relation (5.8) now becomes

$$(\eta_{n-s}-\lambda_n)B_n^{(s)} + \xi_{n-s+1}B_n^{(s-1)} + \kappa_n\left(S_1+\nu_n\,S_2\right) = 0 \tag{5.10}$$

and is seen to be identically satisfied. This proves the proposition.

From expression (5.7) it follows that

$$A_n^{(s)} = 0, \quad \text{if} \quad s \ge j+2. \tag{5.11}$$

Moreover, for $s<j+2$ the coefficients $A_n^{(s)}$ have the form $A_n^{(s)} = q^{-(j+1)n}Q_{2j+2}(q^n;s)$, where $Q_{2j+2}(q^n;s)$ is a polynomial in q^n of degree $2j+2$.

Hence we have

Proposition 5.2. *The polynomials $\mathcal{G}(0)\{P_n(x;q^j,b)\}$ are the eigenfunctions of a q-difference operator $\mathcal{L}_q$ of order $2N=2j+2$.*

We know that the polynomials $\mathcal{G}(0)\{P_n(x;q^j,b)\}$ coincide with the polynomials $P_n(x;q^{j-1},b;M)$ obtained from the little q-Jacobi polynomials by adding to the orthogonality measure a mass M at $x=0$. We thus have equivalently the following

Proposition 5.3. *The polynomials $P_n(x;q^j,b;M)$ obtained from the little q-Jacobi polynomials by inserting a discrete mass at $x=0$ in the orthogonality measure are the eigenfunctions of a q-difference operator of order $2N=2j+4$.*

This proposition is a q-analogue of the corresponding proposition for the ordinary Jacobi polynomials [7], [11].

Note that the first explicit example of the generalized little q-Jacobi polynomials satisfying a fourth-order q-differential equation was found in [5].

Remark. As the referee pointed out, when $a \neq q^j,\ j=0,1,2,\dots,$ then the coefficients $A_n^{(s)}$ (given by the expression (5.7)) do not vanish for all s. In this case one can expect that the corresponding polynomials are eigenfunctions of a q-difference operator of infinite order. When $q=1$, corresponding differential operators of infinite order were found e.g. in [6], [7].

6. The case of little q-Laguerre polynomials

The monic little q-Laguerre polynomials [9]

$$P_n(x;a) = (-1)^n\, q^{n(n-1)/2}\, (aq;q)_n\, {}_2\phi_1\left(\begin{matrix} q^{-n},0 \\ aq \end{matrix}\middle|\, qx\right) \tag{6.1}$$

are obtained from the little q-Jacobi polynomials by setting $b=0$. Hence, these polynomials also satisfy a q-difference equation.

Consider the polynomials $\mathcal{G}(0)\{P_n(x;q^j)\}$ obtained from the little q-Laguerre polynomials by the Geronimus transformation at $x=0$. All formulas for these polynomials are obtained from those for little q-Jacobi polynomials by putting $b=0$.

In particular, their coefficients $A_n^{(s)}$ are easily obtained from (5.7).

We thus have

Proposition 6.1. *The polynomials $\mathcal{G}(0)\{P_n(x;q^j)\}$ are the eigenfunctions of a q-difference operator of order $2N=2j+2$.*

In this case, the polynomials $\mathcal{G}(0)\{P_n(x;q^j)\}$ coincide with polynomials $P_n(x;q^{j-1};M)$ obtained from the little q-Laguerre polynomials by adding to the orthogonality measure a mass M at $x=0$. Hence

Proposition 6.2. *The polynomials $P_n(x;q^j;M)$ are the eigenfunctions of a q-difference operator of order $2N=2j+4$.*

When $q\to 1$ we get Koornwinder's generalized Laguerre polynomials $L_n^{(j;M)}(x)$ [10] whose measure differs from that of the ordinary Laguerre polynomials $L_n^{(j)}(x)$ by inserting a concentrated mass M at the endpoint $x=0$ of the orthogonality interval $(0,\infty)$. These polynomials are known to satisfy a differential equation of order $2j+4$ [6], [8].

References

[1] T. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, NY, 1978. MR **58:**1979

[2] G. Gasper and M. Rahman, *Basic hypergeometric series*, Cambridge University Press, Cambridge, 1990. MR **91d:**33034

[3] Ya. L. Geronimus, *On the polynomials orthogonal with respect to a given number sequence* Zap. Mat. Otdel. Khar'kov. Univers. i NII Mat. i Mehan. **17** (1940), 3-18 (in Russian).

[4] Ya. L. Geronimus, *On the polynomials orthogonal with respect to a given number sequence and a theorem by W.Hahn*, Izv. Akad. Nauk SSSR **4** (1940), 215-228 (in Russian).

[5] F. Alberto Grünbaum and Luc Haine, *The q-version of a theorem of Bochner*, J. Comput. Appl. Math. **68** (1996), 103–114. MR **97m:**33005

[6] J. Koekoek and R. Koekoek, *On a differential equation for Koornwinder's generalized Laguerre polynomials*, Proc. Amer. Math. Soc. **112** (1991), 1045-1054. MR **91j:**33008

[7] J. Koekoek and R. Koekoek, *Differential equations for generalized Jacobi polynomials*, J. Comput. Appl. Math., to appear.

[8] J. Koekoek, R. Koekoek, and H. Bavinck, *On differential equations for Sobolev-type Laguerre polynomials*. Trans. Amer. Math. Soc. 350 (1998), no. 1, 347–393. MR **98d:**33003

[9] R. Koekoek and R.F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, Faculty of Technical Mathematics and Informatics, Report 98-17, Delft University of Technology.

[10] T. H. Koornwinder, *Orthogonal polynomials with weight function* $(1-x)^\alpha(1+x)^\beta + M\delta(x+1) + N\delta(x-1)$ Can. Math. Bull. **27** (1984), 205-214. MR **85i:**33011

[11] A. Zhedanov, *Rational spectral transformations and orthogonal polynomials*, J. Comput. Appl. Math. **85** (1997), 67-86. MR **98h:**42026

[12] A. Zhedanov, *A method of constructing Krall's polynomials*, J. Comput. Appl. Math. **107** (1999), no. 1, 1–20. CMP 99:15

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