GENERALIZED LITTLE $q$-JACOBI POLYNOMIALS AS EIGENSOLUTIONS OF HIGHER-ORDER $q$-DIFFERENCE OPERATORS

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ABSTRACT. We consider the polynomials $p_n(x; a, b; M)$ obtained from the little $q$-Jacobi polynomials $p_n(x; a, b)$ by inserting a discrete mass $M$ at $x = 0$ in the orthogonality measure. We show that for $a = q^j, j = 0, 1, 2, \ldots$, the polynomials $p_n(x; a, b, M)$ are eigensolutions of a linear $q$-difference operator of order $2j + 4$ with polynomial coefficients. This provides a $q$-analog of results recently obtained for the Krall polynomials.

1. $q$-DERIVATIVE OPERATORS AND THEIR REPRESENTATION COEFFICIENTS

Let $T$ be the $q$-shift operator that acts on functions according to

$$T F(x) = F(qx),$$

with $0 < q < 1$ a real number. Obviously

$$T^n F(x) = F(q^n x), \quad n = 0, \pm 1, \pm 2, \ldots.$$  

Introduce the $q$-derivative operator (see, e.g., [2])

$$D_q F(x) = (x(1-q))^{-1} (1 - T).$$

It is called the $q$-derivative because its action on monomials is

$$D_q x^n = [n] x^{n-1},$$

with

$$[n] = (q^n - 1)/(q - 1)$$

the so-called $q$-number. Moreover $\lim_{q \to 1} D_q = D$, where $D$ is the ordinary derivative operator with respect to $x$: $DF(x) = F'(x)$. 

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Consider operators of the form

\[ L_q = \sum_{k=0}^{2N} a_k(x) D_q^k, \]

where \( N \) is a fixed positive integer and \( a_k(x) \) are polynomials in \( x \) of degrees not exceeding \( k \):

\[ a_k(x) = \sum_{s=0}^{k} \alpha_{k,s} x^s, \quad k = 0, 1, \ldots, 2N. \]

Let us introduce also the related operators

\[ L_q = T^{-N} L_q = \sum_{k=0}^{2N} a_k(q^{-N} x) T^{-N} D_q^k. \]

The operator \( L_q \) is a linear combination of the operators \( T^{2N}, T^{2N-1}, \ldots, T, T^0 = I \), while the operator \( L_q \) is a linear combination of the operators \( T^N, T^{N-1}, \ldots, T^{1-N}, T^{-N} \).

For \( q \to 1 \) both operators \( L_q \) and \( L_q \) become \( 2N \)-order differential operators with polynomial coefficients:

\[ \lim_{q \to 1} L_q = \lim_{q \to 1} L_q = \sum_{k=0}^{2N} a^{(0)}_k(x) D^k, \]

where \( a^{(0)}_k(x) = \lim_{q \to 1} a_k(x) \).

The operator \( L_q \) will prove more practical in searching for orthogonal polynomials \( P_n(x) \) satisfying eigenvalue equations of the kind

\[ L_q P_n(x) = \lambda_n P_n(x). \]

It is known that for \( N = 1 \), the little \( q \)-Jacobi polynomials satisfy an equation of the form (1.10) with \( N = 1 \) [9]. We wish to determine other systems of orthogonal polynomials satisfying equation (1.10) with \( N > 1 \).

To this end, we shall extend to \( q \)-difference operators the method proposed in [12].

The main idea of the method is the following. Consider the action of the operator \( L_q \) upon the monomials \( x^n \). From (1.6) and (1.7) we get

\[ \mathcal{L}_q x^n = \sum_{s=0}^{2N} A_n^{(s)} x^{n-s}, \]

where

\[ A_n^{(s)} = q^{N(s-n)} [n][n-1] \ldots [n-s+1] \pi_s(q^n), \]

and

\[ \pi_s(q^n) = \alpha_0 + \sum_{i=1}^{2N-s} \alpha_{s+i,i} [n-s][n-s-1] \ldots [n-s-i+1] \]

are polynomials in \( z = q^n \) of degrees not exceeding \( 2N-s \). It is clear that, moreover,

\[ A_n^{(s)} = 0, \quad s > 2N. \]

The coefficients \( A_n^{(s)} \) completely characterize the operator \( \mathcal{L}_q \). We will call \( A_n^{(s)} \) the representation coefficients of the operator \( \mathcal{L}_q \).
Proposition 1.1. Assume that there are coefficients $A_n^{(s)}$ expressible as in (1.12), where $\pi_s(q^n)$ are arbitrary polynomials in $q^n$ of degrees not exceeding $2N + s$. Assume also that $A_0^{(s)} = 0$, $s > 2N$ (i.e. $\pi_s(q^n) = 0$ for $s > 2N$). Then there exists a unique operator $L_q$ of the form (1.8) such that $A_n^{(s)}$ are its representation coefficients.

Proof. Any polynomial $\pi_s(q^n)$ of degree not exceeding $2N + s$ can be presented in form (1.13) with some coefficients $\alpha_{i,k}$. These coefficients are determined using Newton’s interpolation formula

$$\alpha_{s+i,i} = \frac{(q - 1)^i q^{s+i+i(i-1)/2}}{[i]!} D_q^{(i)} \pi_s(x) \bigg|_{x=q^n}, \quad i = 0, 1, \ldots, 2N - s,$$

where $[i]! = [1][2] \ldots [i]$ is the $q$-factorial. Clearly, the coefficients $\alpha_{i,k}$ are determined uniquely by (1.15) from the given polynomials $\pi_s(q^n)$. Hence the operator $L_q$ is defined uniquely.

2. Basic relations for polynomials satisfying eigenvalue equations

In this section we consider the basic relations between the representation coefficients $A_n^{(s)}$ and the expansion coefficients of polynomials $P_n(x)$ satisfying eigenvalue equations. Assume that

$$P_n(x) = \sum_{s=0}^{n} B_n^{(s)} x^{n-s},$$

where $B_n^{(s)}$ are expansion coefficients. In what follows we will assume that the polynomials $P_n(x)$ are monic, i.e. that

$$B_n^{(0)} = 1.$$

Substituting (2.1) into the eigenvalue equation (1.10) we arrive at the following set of algebraic relations:

$$\sum_{i=0}^{s} B_n^{(s-i)} A_{n-s+i}^{(i)} = \lambda_n B_n^{(s)}, \quad s = 0, 1, 2, \ldots, n.$$

These will be central in our analysis.

For $s = 0$, (2.3) gives

$$\lambda_n = A_n^{(0)}.$$

Thus from (1.12) we find that the eigenvalues $\lambda_n$ have the expression

$$\lambda_n = q^{-Nn} \pi_0(q^n),$$

where $\pi_0(q^n)$ is a polynomial in $q^n$ of degree not exceeding $2N$.

Similarly, for $s = 1$, (2.3) yields

$$A_n^{(1)} = \Omega_n B_n^{(1)},$$

where $\Omega_n = \lambda_n - \lambda_{n-1}$. From this relation we find that

$$B_n^{(1)} = [n] \frac{\pi_1(q^n)}{q^{-Nn} \pi_0(q^n) - \pi_0(q^{n-1})}.$$
Relation (2.8) can be rewritten in the form
\[(\lambda_n - \lambda_{n-s}) B_n^{(s)} = B_n^{(s-1)} A_{n-s+1}^{(1)} + B_n^{(s-2)} A_{n-s+2}^{(2)} + \cdots + A_n^{(s)}.\]
From this relation we can conclude, by induction, that the coefficients $B_n^{(s)}$ are rational functions in $q^n$, namely that
\[(2.9) \quad B_n^{(s)} = [n][n-1] \cdots [n-s+1] \frac{Q_{1,s}(q^n)}{Q_{2,s}(q^n)},\]
where $Q_{1,s}(q^n)$ is a polynomial of degree not exceeding $2Ns - s$, whereas the degree of the polynomial
\[(2.10) \quad Q_{2,s}(q^n) = \prod_{s=1}^{s} (q^{-iN} \pi_0(q^n) - \pi_0(q^{n-i}))\]
do not exceed $2Ns$.

The problem considered up to this point of finding the expansion coefficients $B_n^{(s)}$ of the polynomials $P_n(x)$ when the representation coefficients $A_n^{(s)}$ are given always has a unique solution.

**Proposition 2.1.** Assume that the representation coefficients $A_n^{(s)}$ of the operator $L_q$ satisfy the requirement
\[(2.11) \quad A_n^{(0)} \neq A_m^{(0)}, \quad n \neq m.\]
Then there exists a unique set of monic polynomials $P_n(x)$, $n = 0, 1, \ldots$, satisfying equation (1.10).

**Proof.** We find $\lambda_m$ from (2.4); in view of condition (2.11), a unique $B_n^{(1)}$ is found from (2.6). Assuming that all $B_n^{(1)}, B_n^{(2)}, \ldots, B_n^{(s-1)}$ have thus been found recursively, $B_n^{(s)}$ is then determined in an unambiguous way owing to (2.11).

The polynomials $P_n(x)$ are not orthogonal in general. The requirement that they form an orthogonal set implies strong additional restrictions upon the coefficients $B_n^{(s)}$ and $A_n^{(s)}$. Indeed, orthogonal polynomials satisfy a three-term recurrence relation of the form (1)
\[(2.12) \quad P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = xP_n(x),\]
with $b_n$ and $u_n$ referred to as the recurrence parameters. From (2.12) and (2.1) we get the set of relations
\[(2.13) \quad B_{n+1}^{(s+1)} - B_{n}^{(s+1)} + u_n B_{n-s}^{(s+1)} + b_n B_n^{(s)} = 0, \quad s = 0, 1, \ldots, n,\]
where it is assumed that $B_n^{(s+1)} = B_n^{(s)} = 0$. Putting $s = 0$ and $s = 1$ in (2.13), we find
\[(2.14) \quad b_n = B_n^{(1)} - B_{n+1}^{(1)}, \quad u_n = B_n^{(2)} - B_{n+1}^{(2)} - b_n B_n^{(1)}.\]
Taking into account that the coefficients $B_n^{(s)}$ are rational functions in $q^n$ we arrive at the following proposition.

**Proposition 2.2.** If the orthogonal polynomials $P_n(x)$ are eigenfunctions of the operator (1.8), their recurrence coefficients $b_n, u_n$ are rational functions of the argument $q^n$. 

The problem of reconstructing the representation coefficients $A_n^{(s)}$ when the expansion coefficients $B_n^{(s)}$ are given is more difficult. The coefficients $B_n^{(s)}$ must of course be rational functions in $q^n$, since otherwise the problem has no solutions. Assume therefore that
\begin{equation}
B_n^{(s)} = [n][n-1] \ldots [n-s+1] \frac{G_{1,s}(q^n)}{G_{2,s}(q^n)},
\end{equation}
where the degree of the polynomial $G_{1,s}(q^n)$ does not exceed $2Ms - s$ (for some positive integer $M \leq N$) whereas the degree polynomial $G_{2,s}(q^n)$ does not exceed $2Ms$. We assume that the polynomials $G_{1,s}(x)$ and $G_{2,s}(x)$ have no common divisors. Note that in this case the polynomial $G_{2,s}(q^n)$ need not coincide with expression (2.10) because in the expression (2.9) polynomials $Q_1,s(x)$ and $Q_{2,s}(x)$ may have coinciding zeroes.

Consider relation (2.6) written in the form
\begin{equation}
\frac{A_n^{(1)}}{\Omega_n} = [n] \frac{G_{1,1}(q^n)}{G_{2,1}(q^n)},
\end{equation}
Since both $q^n N A_n^{(1)}$ and $q^n N \Omega_n$ should be polynomials in $q^n$ of degrees not exceeding $2N$, we have
\begin{equation}
A_n^{(1)} = q^{N(1-n)} \rho_1(q^n) [n] G_{1,1}(q^n),
\end{equation}
\begin{equation}
\Omega_n = q^{N(1-n)} \rho_1(q^n) G_{2,1}(q^n),
\end{equation}
where $\rho_1(q^n)$ is some polynomial of degree not exceeding $2N - 2M$.

The relation (2.3), for $s = 2$, can then be rewritten in the form
\begin{equation}
A_n^{(2)} = (\lambda_n - \lambda_{n-2}) B_n^{(2)} - A_n^{(1)} - B_n^{(1)} = (\Omega_n + \Omega_{n-1}) B_n^{(2)} - A_n^{(1)} B_n^{(1)}.
\end{equation}
The representation coefficient $A_n^{(2)}$ is thus determined uniquely if $A_n^{(1)}$ and $\Omega_n$ are known.

Assume that the coefficients $A_n^{(2)}, A_n^{(3)}, \ldots, A_n^{(k-1)}$ have been determined iterating this process. With $s = k$, (2.3) can now be rewritten in the form
\begin{equation}
A_n^{(k)} = (\lambda_n - \lambda_{n-k}) B_n^{(k)} - B_n^{(k-1)} A_n^{(k-1)} - B_n^{(k-2)} A_n^{(k-2)} - \cdots - B_n^{(1)} A_n^{(1)}.
\end{equation}
Taking into account the fact that
\begin{equation}
\lambda_n - \lambda_{n-k} = \Omega_n + \Omega_{n-1} + \cdots + \Omega_{n-k+1},
\end{equation}
we see that $A_n^{(k)}$ is completely determined from the coefficients $\Omega_n, A_n^{(1)}, \ldots, A_n^{(k-1)}$. For the $A_n^{(s)}$ thus obtained to actually be representation coefficients of an operator $L_q$, they necessarily need to satisfy, in addition, the conditions of Proposition 1.1. When this is so, the corresponding polynomials are eigenfunctions of the operator $L_q$.

3. Little $q$-Jacobi polynomials

The monic little $q$-Jacobi polynomials [9] are defined as
\begin{equation}
P_n(x; a, b) = (-1)^n \frac{q^{n(n-1)/2} (aq; q)_n}{(abq^{n+1}; q)_n} \phi_2 \left( \frac{q^{-n}, abq^{n+1}}{aq} \middle| qx \right),
\end{equation}
where $(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$ is the $q$-shifted factorial and $\phi_2$ denotes the $q$-hypergeometric function (see, e.g., [2]).
The orthogonality relation is
\[
\sum_{k=0}^{\infty} w_k P_n(q^k; a, b) P_m(q^k; a, b) = h_n \delta_{nm},
\]
where \( h_n \) are appropriate normalization constants, and the normalized weight function is
\[
w_k = \frac{(aq; q)_\infty}{(abq^2; q)_\infty} \frac{(bq; q)_k (aq)_k}{(q; q)_k}.
\]
It is assumed that \( 0 < aq < 1, b < q^{-1} \). The expansion coefficients of the little \( q \)-Jacobi polynomials are
\[
P^{(s)}_n = b^{-s} \frac{(q^{-n}, a^{-1} q^{-n}; q)_s}{(q, a^{-1} b^{-1} q^{-2n}; q)_s}.
\]
It is easily verified that equations (2.8) have the following solutions:
\[
A^{(0)}_n = [n](q^{1-n} - abq) = \lambda_n = [n](q^{1-n} - abq),
\]
\[
A^{(1)}_n = [n](aq - q^{1-n}),
\]
\[
A^{(s)}_n = 0, \quad s \geq 2.
\]
Hence, the little \( q \)-Jacobi polynomials satisfy a second-order \( q \)-difference equation, as is well known [9].

We also need the value of the function of second kind, \( Q_n(z) \), at \( z = 0 \) (the point \( z = 0 \) is an accumulation point of the orthogonality measure):
\[
Q_n(0; a, b) = - \sum_{k=0}^{\infty} \frac{P_n(q^k; a, b) w_k}{q^k}.
\]
This sum can be evaluated using the \( q \)-binomial theorem and the \( q \)-Saalschütz formula (see, e.g., [2]):
\[
Q_n(0; a, b) = (-1)^{n+1} a^n q^{n(n-1)/2} \frac{1 - abq}{1 - a} \frac{(q; q)_n (bq; q)_n}{(abq; q)_n (abq^{n+1}; q)_n}.
\]
Taking into account that
\[
P_n(0; a, b) = (-1)^n a^n q^{n(n-1)/2} \frac{(aq; q)_n}{(abq^{n+1}; q)_n},
\]
we note that if \( a = q^j, \ j = 1, 2, 3, \ldots \), then
\[
\Phi_n = Q_n(0; q^j, b) + \beta P_n(0; q^j, b)
\]
\[
= (-1)^n a^{n(n-1)/2} \frac{(q^{j+1}; q)_n}{(bq^{n+j+1}; q)_n} \left[ \beta - q^{n-j} \frac{(1 - bq^{j+1})(bq; q)_j (q; q)_j}{(1 - q^j)(q^{n+1}; q)_j (bq^{n+1}; q)_j} \right].
\]

4. Transformed \( q \)-Jacobi Polynomials

Let \( P_n(x) \) be arbitrary orthogonal polynomials with measure localized on the interval \([a, b]\). The corresponding weight function \( w(x) \) is assumed to be normalized to 1, i.e.
\[
\int_a^b w(x) \, dx = 1.
\]
Introduce the functions of second kind,
\begin{equation}
Q_n(z) = \int_a^b \frac{P_n(x) w(x)}{z-x} dx.
\end{equation}
Let \( c \) be a point beyond the orthogonality interval \([a, b]\) such that \( Q_n(c) \) exists.
Consider the polynomials
\begin{equation}
\tilde{P}_n(x) = G(c) \{ P_n(x) \} = P_n(x) - \frac{\Phi_n}{\Phi_{n-1}} P_{n-1}(x), \quad n = 1, 2, \ldots, \quad \tilde{P}_0(x) = 1,
\end{equation}
where
\begin{equation}
\Phi_n = Q_n(c) + \beta P_n(c).
\end{equation}
The notation \( G(c) \{ P_n(x) \} \) stands for the Geronimus transformation [3], [4] of the polynomials \( P_n(x) \) at the point \( x = c \) (for details see, e.g., [11]). The weight function \( \tilde{w}(x) \) of the polynomials \( G(c) \{ P_n(x) \} \) is
\begin{equation}
\tilde{w}(x) = \kappa \left( \frac{w(x)}{x-c} - \beta \delta(x-c) \right),
\end{equation}
where \( \kappa \) is an appropriate normalization constant. The Geronimus transformation thus inserts a concentrated mass at the point \( x = c \). The value of this mass depends on the parameter \( \beta \).

Return to the case of the little \( q \)-Jacobi polynomials with \( a = q^j \), \( j = 1, 2, 3, \ldots \).
Perform the Geronimus transformation (4.2) with \( n \) given by (3.9). (In this case \( c = 0 \).)
The weight function \( \tilde{w}(x) \) for the polynomials \( G(c) \{ P_n(x) \} \) is
\begin{equation}
\tilde{w}(x) = \kappa \left( \sum_{k=0}^{\infty} \tilde{w}_k \delta(x-q^k) - \beta \delta(x) \right),
\end{equation}
where
\begin{equation}
\tilde{w}_k = \frac{(q^{j+1}; q)_\infty (bq; q)_k q^{jk}}{(bq^{j+2}; q)_\infty (q; q)_k}.
\end{equation}
The weight function (4.5) can be rewritten in the form
\begin{equation}
\tilde{w}(x) = \kappa_1 (w(x; j-1) + M \delta(x)),
\end{equation}
where
\begin{equation}
M = -\beta \frac{1-q^j}{1-bq^{j+1}}, \quad \kappa_1 = \kappa \frac{1-bq^{j+1}}{1-q^j},
\end{equation}
and \( w(x; j-1) \) is the weight function corresponding to the little \( q \)-Jacobi polynomials with the parameter \( a = q^j \) replaced with \( a = q^{j-1} \), i.e.
\begin{equation}
w(x; j-1) = \sum_{k=0}^{\infty} w_k(j-1) \delta(x-q^k),
\end{equation}
and
\begin{equation}
w_k(j-1) = \frac{(q^j; q)_\infty (bq; q)_k q^{jk}}{(bq^{j+1}; q)_\infty (q; q)_k}.
Thus the weight function $\tilde{w}(x; j)$ for the polynomials $G(0)\{P_n(x)\}$ is obtained from the weight function $w(x; j - 1)$ through the addition of an arbitrary mass $M$ at the point $x = 0$.

In the expansion

$$G(0)\{P_n(x; q^j, b)\} = \sum_{s=0}^{n} B_n^{(s)} x^{n-s},$$

the coefficients $B_n^{(s)}$ are found from (5.1) and (5.2):

$$B_n^{(s)} = b^{-s} \frac{\left(q^{-n}; q\right)_s (q^{-n-j}; q)_s}{\left(q; q\right)_s (b^{-1} q^{-j-2n}; q)_s} \times \left(1 - q^{n+j-s} \frac{(1 - bq^n)(1 - q^s)}{(1 - q^{n+j})(1 - bq^{j+2n-s})} \frac{Y_j(n)}{Y_j(n-1)}\right),$$

with

$$Y_j(n) = \beta q^{-j} (q^{n+1}; q)_j (bq^{n+1}; q)_j - (q; q)_{j-1} (bq; q)_{j+1}.$$

5. Construction of the coefficients $A_n^{(s)}$

In this section we construct the coefficients $A_n^{(s)}$ for the $q$-difference operator $L_q$ that has the polynomials $P_n(x)$ as eigenfunctions.

We start from the relation

$$A_n^{(1)} = \Omega_n B_n^{(1)},$$

where

$$\Omega_n = \lambda_n - \lambda_n - 1 = A_n^{(0)} - A_{n-1}^{(0)}.$$

Choose $\Omega_n$ proportional to the denominator of $B_n^{(1)}$:

$$\Omega_n = bq^{n+j-1}(q - 1)(1 - b^{-1}q^{1-j-2n})$$

$$\times \left(\beta q^{-j(n-1)} (q^n; q)_j (bq^n; q)_j - (bq; q)_{j+1}(q; q)_{j-1}\right).$$

Then from (5.1) we get for $A_n^{(1)}$ the expression

$$A_n^{(1)} = (1 - q^{-n}) \left(\beta q^{j(1-n)} (q^{n+1}; q)_j (bq^n; q)_j (1 - q^{n-1}) \right.$$

$$\left. - (q; q)_{j-1} (bq; q)_{j+1}(1 - q^{n+j-1})\right).$$

Using (5.2) it is not difficult to show that

$$A_n^{(0)} = \lambda_n = \frac{\beta (q - 1)q^{-n(j+1)-1}(q^n; q)_{j+1}(bq^n; q)_{j+1}}{1 - q^{-j-1}}$$

$$- (q^{-n} - 1)(1 - bq^{n+j})(bq; q)_{j+1}(q; q)_{j-1}.$$

(Note that $A_n^{(0)} = 0$.)

Now using the explicit expressions for $B_n^{(1,2)}$ and $A_n^{(0,1)}$, we find

$$A_n^{(2)} = \beta(q - 1)q^{2-n(j+1)-1} \frac{(1 - q^{-j})(q^{n-2}; q)_j+3(bq^n; q)_{j-1}}{(q; q)_{j+1}}.$$
Repeating this procedure for \( s = 3, 4, \ldots \) one can guess the expression
\[
(5.7) \quad A_n^{(s)} = \beta(q - 1)q^{(s-n)(j+1)-1} \frac{(q^{-j};q)s_{s-1}(q^{n-s};q)s_{s+j+1}(bq^n;q)_{j-s+1}}{(q;q)_s} + \xi_n\delta_{s,1} + \eta_n\delta_{s,0},
\]
where
\[
\xi_n = (q^{-n} - 1)(q; q)_{j-1}(bq; q)_{j+1}(1 - q^{j+n-1}),
\]
\[
\eta_n = (1 - q^{-n})(1 - bq^{n+j})(bq; q)_{j+1}(q; q)_{j-1},
\]
and \( s = 0, 1, 2, \ldots \).

**Proposition 5.1.** The coefficients \( A_n^{(s)} \) given by (5.7) satisfy the basic relations
\[
(5.8) \quad \sum_{i=0}^{s} B_n^{(s-i)} A_{n-s+i}^{(i)} = A_n^{(0)} B_n^{(s)}.
\]

**Proof.** Using the explicit expressions for \( B_n^{(s)} \) and \( A_n^{(s)} \) we can rewrite the lhs of (5.8) in the form
\[
(5.9) \quad \sum_{i=0}^{s} B_n^{(s-i)} A_{n-s+i}^{(i)} = \eta_{n-s} B_n^{(s)} + \xi_{n-s+1} B_n^{(s-1)} + \kappa_n (S_1 + \nu_n S_2),
\]
where
\[
\kappa_n = \beta(q - 1)b^{-s} q^{(j+1)(s-n)-1} \frac{(q^{-n};q)_{s}(q^{-j-n};q)_{s}(q^{n-s};q)_{j+1}(bq^n;q)_{j+1}}{(q;q)_{s}(b^{-1}q^{-j+2n};q)_{s}(1 - q^{-j-1})},
\]
\[
\nu_n = \frac{q^{n+j}(1 - bq^n)(1 - q^{-s})Y_j(n)}{(1 - q^{n+j})(1 - bq^{2n-s})Y_j(n - 1)}.
\]
and \( S_1, S_2 \) are the sums
\[
S_1 = \sum_{i=0}^{s} \frac{q^i(q^{-s};q)_i(bq^{j+2n+1-s};q)_i(q^{-j};q)_i}{(q;q)_i(q^{n+1-s};q)_i(bq^n;q)_i},
\]
\[
S_2 = \sum_{i=0}^{s} \frac{q^i(q^{-1-s};q)_i(bq^{j+2n-s};q)_i(q^{-j-1};q)_i}{(q;q)_i(q^{n+1-s};q)_i(bq^n;q)_i}.
\]
These sums can be evaluated using the \( q \)-analog of the Saalschütz formula [2]:
\[
S_1 = q^{(j+1)s} \frac{(b^{-1}q^{-j-n}, q^{-1-j-n};q)_s}{(b^{-1}q^{1-n}, q^{1-n};q)_s},
\]
\[
S_2 = q^{(j+1)(s-1)} \frac{(b^{-1}q^{-j-n}, q^{-j-n};q)_{s-1}}{(b^{-1}q^{2-n}, q^{1-n};q)_{s-1}}.
\]
Relation (5.8) now becomes
\[
(5.10) \quad (\eta_{n-s} - \lambda_n) B_n^{(s)} + \xi_{n-s+1} B_n^{(s-1)} + \kappa_n (S_1 + \nu_n S_2) = 0
\]
and is seen to be identically satisfied. This proves the proposition.

From expression (5.7) it follows that
\[
(5.11) \quad A_n^{(s)} = 0, \quad \text{if} \quad s \geq j + 2.
\]
Moreover, for \( s < j+2 \) the coefficients \( A_n^{(s)} \) have the form \( A_n^{(s)} = q^{-(j+1)n} Q_{2j+2}(q^n; s) \), where \( Q_{2j+2}(q^n; s) \) is a polynomial in \( q^n \) of degree \( 2j + 2 \).
Hence we have

**Proposition 5.2.** The polynomials \( \mathcal{G}(0) \{ P_n(x; q^j, b) \} \) are the eigenfunctions of a \( q \)-difference operator \( \mathcal{L}_q \) of order \( 2N = 2j + 2 \).

We know that the polynomials \( \mathcal{G}(0) \{ P_n(x; q^j, b) \} \) coincide with the polynomials \( P_n(x; q^{j-1}, b; M) \) obtained from the little \( q \)-Jacobi polynomials by adding to the orthogonality measure a mass \( M \) at \( x = 0 \). We thus have equivalently the following

**Proposition 5.3.** The polynomials \( P_n(x; q^j, b; M) \) obtained from the little \( q \)-Jacobi polynomials by inserting a discrete mass at \( x = 0 \) in the orthogonality measure are the eigenfunctions of a \( q \)-difference operator of order \( 2N = 2j + 4 \).

This proposition is a \( q \)-analogue of the corresponding proposition for the ordinary Jacobi polynomials [7], [11].

Note that the first explicit example of the generalized little \( q \)-Jacobi polynomials satisfying a fourth-order \( q \)-differential equation was found in [5].

**Remark.** As the referee pointed out, when \( a \neq q^j, j = 0, 1, 2, \ldots \), then the coefficients \( A_n^{(s)} \) (given by the expression (5.7)) do not vanish for all \( s \). In this case one can expect that the corresponding polynomials are eigenfunctions of a \( q \)-difference operator of infinite order. When \( q = 1 \), corresponding differential operators of infinite order were found e.g. in [6], [7].

6. THE CASE OF LITTLE \( q \)-LAGUERRE POLYNOMIALS

The monic little \( q \)-Laguerre polynomials [9]

\[
(6.1) \quad P_n(x; a) = (-1)^n \frac{q^{n(n-1)/2} \left( aq; q \right)_n \phi_1 \left( q^{-n} \right) \phi_1 \left( qx \right)}{aq} \n\]

are obtained from the little \( q \)-Jacobi polynomials by setting \( b = 0 \). Hence, these polynomials also satisfy a \( q \)-difference equation.

Consider the polynomials \( \mathcal{G}(0) \{ P_n(x; q^j) \} \) obtained from the little \( q \)-Laguerre polynomials by the Geronimus transformation at \( x = 0 \). All formulas for these polynomials are obtained from those for little \( q \)-Jacobi polynomials by putting \( b = 0 \).

In particular, their coefficients \( A_n^{(s)} \) are easily obtained from (5.7).

We thus have

**Proposition 6.1.** The polynomials \( \mathcal{G}(0) \{ P_n(x; q^j) \} \) are the eigenfunctions of a \( q \)-difference operator of order \( 2N = 2j + 2 \).

In this case, the polynomials \( \mathcal{G}(0) \{ P_n(x; q^j) \} \) coincide with polynomials \( P_n(x; q^{j-1}; M) \) obtained from the little \( q \)-Laguerre polynomials by adding to the orthogonality measure a mass \( M \) at \( x = 0 \). Hence

**Proposition 6.2.** The polynomials \( P_n(x; q^j; M) \) are the eigenfunctions of a \( q \)-difference operator of order \( 2N = 2j + 4 \).

When \( q \to 1 \) we get Koornwinder’s generalized Laguerre polynomials \( L_n^{(j; M)}(x) \) [10] whose measure differs from that of the ordinary Laguerre polynomials \( L_n^{(j)}(x) \) by inserting a concentrated mass \( M \) at the endpoint \( x = 0 \) of the orthogonality interval \((0, \infty)\). These polynomials are known to satisfy a differential equation of order \( 2j + 4 \) [5], [8].
REFERENCES


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