GENERALIZED LITTLE $q$-JACOBI POLYNOMIALS AS EIGENSOLUTIONS OF HIGHER-ORDER $q$-DIFFERENCE OPERATORS

LUC VINET AND ALEXEI ZHEDANOV

(Communicated by Hal L. Smith)

Abstract. We consider the polynomials $p_n(x; a, b; M)$ obtained from the little $q$-Jacobi polynomials $p_n(x; a, b)$ by inserting a discrete mass $M$ at $x = 0$ in the orthogonality measure. We show that for $a = q^j, j = 0, 1, 2, \ldots$, the polynomials $p_n(x; a, b; M)$ are eigensolutions of a linear $q$-difference operator of order $2j + 4$ with polynomial coefficients. This provides a $q$-analog of results recently obtained for the Krall polynomials.

1. $q$-DERIVATIVE OPERATORS AND THEIR REPRESENTATION COEFFICIENTS

Let $T$ be the $q$-shift operator that acts on functions according to

\begin{equation}
TF(x) = F(qx),
\end{equation}

with $0 < q < 1$ a real number. Obviously

\begin{equation}
T^n F(x) = F(q^n x), \quad n = 0, \pm 1, \pm 2, \ldots.
\end{equation}

Introduce the $q$-derivative operator (see, e.g., [2])

\begin{equation}
D_q F(x) = (x(1 - q))^{-1} (1 - T).
\end{equation}

It is called the $q$-derivative because its action on monomials is

\begin{equation}
D_q x^n = [n] x^{n-1},
\end{equation}

with

\begin{equation}
[n] = (q^n - 1)/(q - 1)
\end{equation}

the so-called $q$-number. Moreover $\lim_{q \to 1} D_q = D$, where $D$ is the ordinary derivative operator with respect to $x$: $DF(x) = F'(x)$. 

Received by the editors December 11, 1998.

2000 Mathematics Subject Classification. Primary 33D45.

Key words and phrases. Krall's polynomials, little $q$-Jacobi polynomials.

The work of the first author was supported in part through funds provided by NSERC (Canada) and FCAR (Quebec). The work of the second author was supported in part through funds provided by SCST (Ukraine) Project #2.4/197, INTAS-96-0700 grant and project 96-01-00281 supported by RFBR (Russia). The second author thanks Centre de recherches mathématiques of the Université de Montréal for hospitality.

©2001 American Mathematical Society
Consider operators of the form
\[ L_q = \sum_{k=0}^{2N} a_k(x) D_q^k, \]
where \( N \) is a fixed positive integer and \( a_k(x) \) are polynomials in \( x \) of degrees not exceeding \( k \):
\[ a_k(x) = \sum_{s=0}^{k} \alpha_{ks} x^s, \quad k = 0, 1, \ldots, 2N. \]

Let us introduce also the related operators
\[ L_q = T^{-N} \sum_{k=0}^{2N} a_k(q^{-N} x) T^{-N} D_q^k. \]
The operator \( L_q \) is a linear combination of the operators \( T^{2N}, T^{2N-1}, \ldots, T, T^0 = I \), while the operator \( \mathcal{L}_q \) is a linear combination of the operators \( T^N, T^{N-1}, \ldots, T^{1-N}, T^{-N} \).

For \( q \to 1 \) both operators \( L_q \) and \( \mathcal{L}_q \) become \( 2N \)-order differential operators with polynomial coefficients:
\[ \lim_{q \to 1} L_q = \lim_{q \to 1} \mathcal{L}_q = \sum_{k=0}^{2N} a_k^{(0)}(x) D^k, \]
where \( a_k^{(0)}(x) = \lim_{q \to 1} a_k(x) \).

The operator \( \mathcal{L}_q \) will prove more practical in searching for orthogonal polynomials \( P_n(x) \) satisfying eigenvalue equations of the kind
\[ \mathcal{L}_q P_n(x) = \lambda_n P_n(x). \]

It is known that for \( N = 1 \), the little \( q \)-Jacobi polynomials satisfy an equation of the form (1.10) with \( N = 1 \) [9]. We wish to determine other systems of orthogonal polynomials satisfying equation (1.10) with \( N > 1 \).

To this end, we shall extend to \( q \)-difference operators the method proposed in [12].

The main idea of the method is the following. Consider the action of the operator \( \mathcal{L}_q \) upon the monomials \( x^n \). From (1.6) and (1.7) we get
\[ \mathcal{L}_q x^n = \sum_{s=0}^{2N} A_n^{(s)} x^{n-s}, \]
where
\[ A_n^{(s)} = q^{N(s-n)} [n][n-1] \ldots [n-s+1] \pi_s(q^n), \]
and
\[ \pi_s(q^n) = \alpha_{s0} + \sum_{i=1}^{2N-s} \alpha_{s+i,i} [n-s][n-s-1] \ldots [n-s-i+1] \]
are polynomials in \( z = q^n \) of degrees not exceeding \( 2N-s \). It is clear that, moreover,
\[ A_n^{(s)} = 0, \quad s > 2N. \]
The coefficients \( A_n^{(s)} \) completely characterize the operator \( \mathcal{L}_q \). We will call \( A_n^{(s)} \) the representation coefficients of the operator \( \mathcal{L}_q \).
Proposition 1.1. Assume that there are coefficients $A_n^{(s)}$ expressible as in (1.12), where $\pi_s(q^n)$ are arbitrary polynomials in $q^n$ of degrees not exceeding $2N - s$. Assume also that $A_n^{(s)} = 0$, $s > 2N$ (i.e. $\pi_s(q^n) = 0$ for $s > 2N$). Then there exists a unique operator $L_q$ of the form (1.8) such that $A_n^{(s)}$ are its representation coefficients.

Proof. Any polynomial $\pi_s(q^n)$ of degree not exceeding $2N - s$ can be presented in form (1.13) with some coefficients $\alpha s$. These coefficients are determined using Newton’s interpolation formula

$$\alpha_{s+i} = \frac{(q - 1)^{q^{s+i+(i-1)/2}}}{[i]!} D_q \pi_s(x) \bigg|_{x=q^s}, \quad i = 0, 1, \ldots, 2N - s,$$

where $[i]! = [1][2] \ldots [i]$ is the $q$-factorial. Clearly, the coefficients $\alpha s$ are determined uniquely by (1.13) from the given polynomials $\pi_s(q^n)$. Hence the operator $L_q$ is defined uniquely.

2. Basic relations for polynomials satisfying eigenvalue equations

In this section we consider the basic relations between the representation coefficients $A_n^{(s)}$ and the expansion coefficients of polynomials $P_n(x)$ satisfying eigenvalue equations. Assume that

$$P_n(x) = \sum_{s=0}^{n} B_n^{(s)} x^{n-s},$$

where $B_n^{(s)}$ are expansion coefficients. In what follows we will assume that the polynomials $P_n(x)$ are monic, i.e. that

$$B_n^{(0)} = 1.$$

Substituting (2.1) into the eigenvalue equation (1.10) we arrive at the following set of algebraic relations:

$$\sum_{i=0}^{s} B_n^{(s-i)} A_n^{(i)} A_{n-s+i} = \lambda_n B_n^{(s)}, \quad s = 0, 1, 2, \ldots, n.$$

These will be central in our analysis.

For $s = 0$, (2.3) gives

$$\lambda_n = A_n^{(0)}.$$

Thus from (1.12) we find that the eigenvalues $\lambda_n$ have the expression

$$\lambda_n = q^{-Nn} \pi_0(q^n),$$

where $\pi_0(q^n)$ is a polynomial in $q^n$ of degree not exceeding $2N$.

Similarly, for $s = 1$, (2.3) yields

$$A_n^{(1)} = \Omega_n B_n^{(1)},$$

where $\Omega_n = \lambda_n - \lambda_{n-1}$. From this relation we find that

$$B_n^{(1)} = [n] \frac{\pi_1(q^n)}{q^{-N} \pi_0(q^n) - \pi_0(q^{n-1})}.$$
Relation (2.3) can be rewritten in the form

\[(\lambda_n - \lambda_{n-s}) B_n^{(s)} = B_n^{(s-1)} A_{n-s+1}^{(1)} + B_n^{(s-2)} A_{n-s+2}^{(2)} + \cdots + A_n^{(s)}.\]

From this relation we can conclude, by induction, that the coefficients $B_n^{(s)}$ are rational functions in $q^n$, namely that

\[B_n^{(s)} = [n][n-1]\ldots[n-s+1] \frac{Q_{1,s}(q^n)}{Q_{2,s}(q^n)},\]

where $Q_{1,s}(q^n)$ is a polynomial of degree not exceeding $2Ns - s$, whereas the degree of the polynomial

\[Q_{2,s}(q^n) = \prod_{s=1}^{n} (q^{-iN} \pi_0(q^n) - \pi_0(q^{n-i}))\]

does not exceed $2Ns$.

The problem considered up to this point of finding the expansion coefficients $B_n^{(s)}$ of the polynomials $P_n(x)$ when the representation coefficients $A_n^{(s)}$ are given always has a unique solution.

**Proposition 2.1.** Assume that the representation coefficients $A_n^{(s)}$ of the operator $L_q$ satisfy the requirement

\[A_n^{(0)} \neq A_m^{(0)}, \quad n \neq m.\]

Then there exists a unique set of monic polynomials $P_n(x)$, $n = 0, 1, \ldots$, satisfying equation (1.10).

**Proof.** We find $\lambda_m$ from (2.4); in view of condition (2.11), a unique $B_n^{(1)}$ is found from (2.5). Assuming that all $B_n^{(1)}, B_n^{(2)}, \ldots, B_n^{(s-1)}$ have thus been found recursively, $B_n^{(s)}$ is then determined in an unambiguous way owing to (2.11).

The polynomials $P_n(x)$ are not orthogonal in general. The requirement that they form an orthogonal set implies strong additional restrictions upon the coefficients $B_n^{(s)}$ and $A_n^{(s)}$. Indeed, orthogonal polynomials satisfy a three-term recurrence relation of the form [1]

\[P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = x P_n(x),\]

with $b_n$ and $u_n$ referred to as the recurrence parameters. From (2.12) and (2.1) we get the set of relations

\[B_n^{(s+1)} - B_n^{(s+1)} + u_n B_n^{(s-1)} + b_n B_n^{(s)} = 0, \quad s = 0, 1, \ldots, n,\]

where it is assumed that $B_n^{(-1)} = B_n^{(n+1)} = 0$. Putting $s = 0$ and $s = 1$ in (2.13), we find

\[b_n = B_n^{(1)} - B_{n+1}^{(1)},\]
\[u_n = B_n^{(2)} - B_{n+1}^{(2)} - b_n B_n^{(1)}.\]

Taking into account that the coefficients $B_n^{(s)}$ are rational functions in $q^n$ we arrive at the following proposition.

**Proposition 2.2.** If the orthogonal polynomials $P_n(x)$ are eigenfunctions of the operator (1.8), their recurrence coefficients $b_n, u_n$ are rational functions of the argument $q^n$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
The problem of reconstructing the representation coefficients $A_n^{(s)}$ when the expansion coefficients $B_n^{(s)}$ are given is more difficult. The coefficients $B_n^{(s)}$ must of course be rational functions in $q^n$, since otherwise the problem has no solutions. Assume therefore that

$$B_n^{(s)} = |n||n-1|\ldots|n-s+1| \frac{G_{1,s}(q^n)}{G_{2,s}(q^n)},$$

(2.15)

where the degree of the polynomial $G_{1,s}(q^n)$ does not exceed $2Ms - s$ (for some positive integer $M \leq N$) whereas the degree polynomial $G_{2,s}(q^n)$ does not exceed $2Ms$. We assume that the polynomials $G_{1,s}(x)$ and $G_{2,s}(x)$ have no common divisors. Note that in this case the polynomial $G_{2,s}(q^n)$ need not coincide with expression (2.10) because in the expression (2.9) polynomials $Q_1,s(x)$ and $Q_{2,s}(x)$ may have coinciding zeroes.

Consider relation (2.6) written in the form

$$A_n^{(1)} = \frac{G_{1,1}(q^n)}{G_{2,1}(q^n)}.$$  

(2.16)

Since both $q^{-N}A_n^{(1)}$ and $q^{-N}\Omega_n$ should be polynomials in $q^n$ of degrees not exceeding $2N$, we have

$$A_n^{(1)} = q^{N(1-n)} \rho_1(q^n) |n| G_{1,1}(q^n),$$

$$\Omega_n = q^{N(1-n)} \rho_1(q^n) G_{2,1}(q^n),$$

(2.17)

where $\rho_1(q^n)$ is some polynomial of degree not exceeding $2N - 2M$.

The relation (2.3), for $s = 2$, can then be rewritten in the form

$$A_n^{(2)} = (\lambda_n - \lambda_{n-2}) B_n^{(2)} - A_n^{(1)} B_n^{(1)} = (\Omega_n + \Omega_{n-1}) B_n^{(2)} - A_n^{(1)} B_n^{(1)}. $$

(2.18)

The representation coefficient $A_n^{(2)}$ is thus determined uniquely if $A_n^{(1)}$ and $\Omega_n$ are known.

Assume that the coefficients $A_n^{(2)}, A_n^{(3)}, \ldots, A_n^{(k-1)}$ have been determined iterating this process. With $s = k$, (2.3) can now be rewritten in the form

$$A_n^{(k)} = (\lambda_n - \lambda_{n-k}) B_n^{(k-1)} - B_n^{(k-1)} A_n^{(1)} - B_n^{(k-2)} A_n^{(2)} - \cdots - B_n^{(2)} A_n^{(k-2)}.$$  

(2.19)

Taking into account the fact that

$$\lambda_n - \lambda_{n-k} = \Omega_n + \Omega_{n-1} + \cdots + \Omega_{n-k+1},$$

we see that $A_n^{(k)}$ is completely determined from the coefficients $\Omega_n$, $A_n^{(1)}, \ldots, A_n^{(k-1)}$.

For the $A_n^{(s)}$ thus obtained to actually be representation coefficients of an operator $L_q$, they necessarily need to satisfy, in addition, the conditions of Proposition 1.1.

When this is so, the corresponding polynomials are eigenfunctions of the operator $L_q$.

### 3. Little $q$-Jacobi polynomials

The monic little $q$-Jacobi polynomials [9] are defined as

$$P_n(x; a, b) = (-1)^n a \frac{q^{n(n-1)/2}}{(aq^n+1; q)_n} \frac{2\phi_1}{aq \mid qx},$$

(3.1)

where $(a; q)_n = (1 - a)(1 -aq)\ldots(1 -aq^{n-1})$ is the $q$-shifted factorial and $2\phi_1$ denotes the $q$-hypergeometric function (see, e.g., [2]).
The orthogonality relation is
\begin{equation}
\sum_{k=0}^{\infty} w_k P_n(q^k; a, b) \ P_m(q^k; a, b) = h_n \delta_{nm},
\end{equation}
where $h_n$ are appropriate normalization constants, and the normalized weight function is
\begin{equation}
w_k = \frac{(aq; q)_k}{(abq^2; q)_k} \frac{(aq_k)}{(q; q)_k}.
\end{equation}
It is assumed that $0 < aq < 1$, $b < q^{-1}$. The expansion coefficients of the little $q$-Jacobi polynomials are
\begin{equation}
P_n(s) = b^{-s} \frac{(q^{-n}, a^{-1}q^{-n}; q)_s}{(q, a^{-1}b^{-1}q^{-2n}; q)_s}.
\end{equation}
It is easily verified that equations (2.8) have the following solutions:
\begin{equation}
A_n(0) = \lambda_n = [n](q^{1-n} - abq^2),
\end{equation}
\begin{equation}
A_n(1) = [n](aq - q^{1-n}),
\end{equation}
\begin{equation}
A_n(s) = 0, \quad s \geq 2.
\end{equation}
Hence, the little $q$-Jacobi polynomials satisfy a second-order $q$-difference equation, as is well known [9].

We also need the value of the function of second kind, $Q_n(z)$, at $z = 0$ (the point $z = 0$ is an accumulation point of the orthogonality measure):
\begin{equation}
Q_n(0; a, b) = - \sum_{k=0}^{\infty} \frac{P_n(q^k; a, b) \ w_k}{q^k}.
\end{equation}
This sum can be evaluated using the $q$-binomial theorem and the $q$-Saalschütz formula (see, e.g., [2]):
\begin{equation}
Q_n(0; a, b) = (-1)^{n+1} a^n q^{n(n-1)/2} \frac{1 - abq}{1 - a} \frac{(q; q)_n (aq; q)_n}{(abq; q)_n (abq^{n+1}; q)_n}.
\end{equation}
Taking into account that
\begin{equation}
P_n(0; a, b) = (-1)^n q^{n(n-1)/2} \frac{(aq; q)_n}{(abq^{n+1}; q)_n},
\end{equation}
we note that if $a = q^j$, $j = 1, 2, 3, \ldots$, then
\begin{equation}
\Phi_n = Q_n(0; q^j, b) + \beta P_n(0; q^j, b)
\end{equation}
\begin{equation}
= (-1)^n q^{n(n-1)/2} \frac{(q^{j+1}; q)_n}{(bq^{n+j+1}; q)_n} \frac{(1 - b q^{j+1}) (aq; q)_j (q; q)_j}{(1 - q^j) (q^{n+1}; q)_j (b q^{n+1}; q)_j}.
\end{equation}

4. **Transformed $q$-Jacobi Polynomials**

Let $P_n(x)$ be arbitrary orthogonal polynomials with measure localized on the interval $[a, b]$. The corresponding weight function $w(x)$ is assumed to be normalized to 1, i.e.
\[
\int_a^b w(x) \, dx = 1.
\]
Introduce the functions of second kind,
\begin{equation}
Q_n(z) = \int_a^b \frac{P_n(x) w(x)}{z-x} dx.
\end{equation}

Let \(c\) be a point beyond the orthogonality interval \([a, b]\) such that \(Q_n(c)\) exists.

Consider the polynomials
\begin{equation}
\tilde{P}_n(x) = G(c)\{P_n(x)\} = P_n(x) - \frac{\Phi_n}{\Phi_{n-1}} P_{n-1}(x), \quad n = 1, 2, \ldots, \quad \tilde{P}_0(x) = 1,
\end{equation}
where
\begin{equation}
\Phi_n = Q_n(c) + \beta P_n(c).
\end{equation}

The notation \(G(c)\{P_n(x)\}\) stands for the Geronimus transformation \([3, 4]\) of the polynomials \(P_n(x)\) at the point \(x = c\) (for details see, e.g., \([11]\)). The weight function \(\tilde{w}(x)\) of the polynomials \(G(c)\{P_n(x)\}\) is
\begin{equation}
\tilde{w}(x) = \kappa \left( \frac{w(x)}{x-c} - \beta \delta(x-c) \right),
\end{equation}
where \(\kappa\) is an appropriate normalization constant. The Geronimus transformation thus inserts a concentrated mass at the point \(x = c\). The value of this mass depends on the parameter \(\beta\).

Return to the case of the little \(q\)-Jacobi polynomials with \(a = q^j, \ j = 1, 2, 3, \ldots\).
Perform the Geronimus transformation \((4.2)\) with \(n\) given by \((3.9)\). (In this case \(c = 0\).)

The weight function \(\tilde{w}(x)\) for the polynomials \(G(c)\{P_n(x)\}\) is
\begin{equation}
\tilde{w}(x) = \kappa \left( \sum_{k=0}^{\infty} \tilde{w}_k \delta(x - q^k) - \beta \delta(x) \right),
\end{equation}
where
\begin{equation}
\tilde{w}_k = \frac{(q^j+1; q)_{\infty}}{(q; q)_k} \frac{(bq; q)_k q^{jk}}{q^{jk+1}}.
\end{equation}
The weight function \((4.5)\) can be rewritten in the form
\begin{equation}
\tilde{w}(x) = \kappa_1 \left( w(x; j - 1) + M \delta(x) \right),
\end{equation}
where
\begin{equation}
M = -\beta \frac{1 - q^j}{1 - bq^{j+1}}, \quad \kappa_1 = \kappa \frac{1 - bq^{j+1}}{1 - q^j},
\end{equation}
and \(w(x; j - 1)\) is the weight function corresponding to the little \(q\)-Jacobi polynomials with the parameter \(a = q^j\) replaced with \(a = q^{j-1}\), i.e.
\begin{equation}
w(x; j - 1) = \sum_{k=0}^{\infty} w_k(j - 1) \delta(x - q^k),
\end{equation}
and
\begin{equation}
w_k(j - 1) = \frac{(q^j; q)_{\infty}}{(bq^{j+1}; q)_{\infty}} \frac{(bq; q)_k q^{jk}}{(q; q)_k}.
\end{equation}
Thus the weight function $\tilde{w}(x; j)$ for the polynomials $G(0)\{P_n(x)\}$ is obtained from the weight function $w(x; j - 1)$ through the addition of an arbitrary mass $M$ at the point $x = 0$.

In the expansion
\[
G(0)\{P_n(x; q^j, b)\} = \sum_{s=0}^{n} B_n^{(s)} x^{n-s},
\]
the coefficients $B_n^{(s)}$ are found from (3.1) and (3.2): 
\[
B_n^{(s)} = b^{-s} \frac{(q^{-n}; q)_s (q^{-n-j}; q)_s}{(q; q)_s (b^{-1}q^{-j-2n}; q)_s} \times \left(1 - q^{n+j-s}(1 - bq^n)(1 - q^s)\frac{Y_j(n)}{(1 - q^{n+j})(1 - b^q q^{n+2n-s}) Y_j(n-1)}\right),
\]
with
\[
Y_j(n) = \beta q^{-jn}(q^n+1; q)_j bq^{n+1}(q)_j - (q; q)_j^{-1}(bq; q)_j+1.
\]

5. Construction of the coefficients $A_n^{(s)}$

In this section we construct the coefficients $A_n^{(s)}$ for the $q$-difference operator $L_q$ that has the polynomials $P_n(x)$ as eigenfunctions.

We start from the relation
\[
A_n^{(1)} = \Omega_n B_n^{(1)},
\]
where
\[
\Omega_n = \lambda_n - \lambda_{n-1} = A_n^{(0)} - A_{n-1}^{(0)}.
\]
Choose $\Omega_n$ proportional to the denominator of $B_n^{(1)}$:
\[
\Omega_n = bq^{n+j-1}(q - 1)(1 - b^{-1}q^{-j-2n})
\times \left(\beta q^{-jn-1}(q^n; q)_j(bq^n; q)_j - (bq; q)_j+1(q; q)_j^{-1}\right).
\]

Then from (5.1) we get for $A_n^{(1)}$ the expression
\[
A_n^{(1)} = (1 - q^{-n}) \left(\beta q^j(1-n)(q^{n+1}; q)_j bq^{n}(q; q)_j (1 - q^{n-1})
\right.
\]
\[
- (q; q)_j^{-1}(bq; q)_j+1(1 - q^{n+j-1})\right).
\]

Using (5.2) it is not difficult to show that 
\[
A_n^{(0)} = \frac{\beta (q - 1)q^{-n(j+1)-1}(q^n; q)_j+1(bq^n; q)_j+1}{1 - q^{-j-1}}
- (q^{-n} - 1)(1 - bq^{n+j})(bq; q)_j+1(q; q)_j^{-1}.
\]
(Note that $A_0^{(0)} = 0$.)

Now using the explicit expressions for $B_n^{(1,2)}$ and $A_n^{(0,1)}$, we find
\[
A_n^{(2)} = \beta(q - 1)q^{(2-n)(j+1)-1}(1 - q^{-j})(q^{n-2}; q)_j+1(bq^n; q)_j+1 \frac{(bq; q)_j+1}{(q; q)_2}.
\]
Repeating this procedure for \( s = 3, 4, \ldots \) one can guess the expression

\[
A_n^{(s)} = \beta(q - 1)q^{(s-n)(j+1) - 1} \frac{(q^{-j}; q)_{s-1}(q^{n-s}; q)_{s+j+1}(bq^n; q)_{j-s+1}}{(q; q)_s} + \xi_n\delta_{s,1} + \eta_n\delta_{s,0},
\]

where

\[
\xi_n = (q^{-n} - 1)(q; q)_{j-1}(bq; q)_{j+1}(1 - q^{j+n-1}),
\]

\[
\eta_n = (1 - q^{-n})(1 - bq^{n+j})(bq; q)_{j+1}(q; q)_{j-1},
\]

and \( s = 0, 1, 2, \ldots \).

**Proposition 5.1.** The coefficients \( A_n^{(s)} \) given by (5.7) satisfy the basic relations

\[
(5.8) \quad \sum_{i=0}^{s} B_n^{(s-i)} A_n^{(i)} = A_n^{(0)} B_n^{(s)}.
\]

**Proof.** Using the explicit expressions for \( B_n^{(s)} \) and \( A_n(s) \) we can rewrite the lhs of (5.8) in the form

\[
(5.9) \quad \sum_{i=0}^{s} B_n^{(s-i)} A_n^{(i)} = \eta_{n-s} B_n^{(s)} + \xi_{n-s+1} B_n^{(s-1)} + \kappa_n (S_1 + \nu_n S_2),
\]

where

\[
\kappa_n = \beta(q - 1)b^{-s} q^{(j+1)(s-n) - 1}(q^{-n}; q)_s(q^{-j-n}; q)_s(q^{n-s}; q)_{j+1}(bq^n; q)_{j+1} \frac{(q; q)_s(b^{-1}q^{-2n}; q)_s(1 - q^{-j-1})}{(q; q)_s(b^{-1}q^{-2n}; q)_s(1 - q^{-j-1})},
\]

\[
\nu_n = \frac{q^{n+1}(1 - bq^n)(1 - q^{-s})Y_j(n)}{(1 - q^{n+j})(1 - bq^{2n})Y_j(n - 1)},
\]

and \( S_1, S_2 \) are the sums

\[
S_1 = \sum_{i=0}^{s} q^i(q^{-s}; q)_i(bq^{j+2n-1-s}; q)_i(q^{-j-1}; q)_i(q^n; q)_i(q^{n+1-s}; q)_i(bq^n; q)_i(q^{n+1-s}; q)_i(q^{n-s}; q)_i(bq^n; q)_i,
\]

\[
S_2 = \sum_{i=0}^{s} q^i(q^{1-s}; q)_i(bq^{j+2n-1-s}; q)_i(q^{-j-1}; q)_i(q^n; q)_i(q^{n-s}; q)_i(bq^n; q)_i(q^n; q)_i(q^{n-s}; q)_i(bq^n; q)_i.
\]

These sums can be evaluated using the \( q \)-analog of the Saalschütz formula [2]:

\[
S_1 = q^{(j+1)n} \frac{(b^{-1}q^{-j-n}, q^{-1-j-n}; q)_s}{(b^{-1}q^{-n}, q^{-n}; q)_s},
\]

\[
S_2 = q^{(j+1)(s-1)} \frac{(b^{-1}q^{-j-n}, q^{-j-n}; q)_{s-1}}{(b^{-1}q^{-n}, q^{-n}; q)_{s-1}}.
\]

Relation (5.8) now becomes

\[
(5.10) \quad (\eta_{n-s} - \lambda_n) B_n^{(s)} + \xi_{n-s+1} B_n^{(s-1)} + \kappa_n (S_1 + \nu_n S_2) = 0
\]

and is seen to be identically satisfied. This proves the proposition.

From expression (5.7) it follows that

\[
A_n^{(s)} = 0, \quad \text{if} \quad s \geq j + 2.
\]

Moreover, for \( s < j + 2 \) the coefficients \( A_n^{(s)} \) have the form \( A_n^{(s)} = q^{-(j+1)n} Q_{2j+2}(q^n; s) \), where \( Q_{2j+2}(q^n; s) \) is a polynomial in \( q^n \) of degree \( 2j + 2 \).
Hence we have

**Proposition 5.2.** The polynomials \( \mathcal{G}(0) \{ P_n(x; q^j, b) \} \) are the eigenfunctions of a \( q \)-difference operator \( L_q \) of order \( 2N = 2j + 2 \).

We know that the polynomials \( \mathcal{G}(0) \{ P_n(x; q^j, b) \} \) coincide with the polynomials \( P_n(x; q^{j-1}, b; M) \) obtained from the little \( q \)-Jacobi polynomials by adding to the orthogonality measure a mass \( M \) at \( x = 0 \). We thus have equivalently the following

**Proposition 5.3.** The polynomials \( P_n(x; q^j, b; M) \) obtained from the little \( q \)-Jacobi polynomials by inserting a concentrated mass at \( x = 0 \) in the orthogonality measure are the eigenfunctions of a \( q \)-difference operator of order \( 2N = 2j + 4 \).

This proposition is a \( q \)-analogue of the corresponding proposition for the ordinary Jacobi polynomials [7], [11].

Note that the first explicit example of the generalized little \( q \)-Jacobi polynomials satisfying a fourth-order \( q \)-differential equation was found in [5].

**Remark.** As the referee pointed out, when \( a \neq q^j \), \( j = 0, 1, 2, \ldots \), then the coefficients \( A_n^{(s)} \) (given by the expression (5.7)) do not vanish for all \( s \). In this case one can expect that the corresponding polynomials are eigenfunctions of a \( q \)-difference operator of infinite order. When \( q = 1 \), corresponding differential operators of infinite order were found e.g. in [9], [7].

6. The Case of Little \( q \)-Laguerre Polynomials

The monic little \( q \)-Laguerre polynomials [9]

\[
(6.1) \quad P_n(x; a) = (-1)^n q^{n(n-1)/2} (aq; q)_n 2\phi_1 \left( \frac{q^{-n}, 0}{aq}; qx \right)
\]

are obtained from the little \( q \)-Jacobi polynomials by setting \( b = 0 \). Hence, these polynomials also satisfy a \( q \)-difference equation.

Consider the polynomials \( \mathcal{G}(0) \{ P_n(x; q^j) \} \) obtained from the little \( q \)-Laguerre polynomials by the Geronimus transformation at \( x = 0 \). All formulas for these polynomials are obtained from those for little \( q \)-Jacobi polynomials by putting \( b = 0 \).

In particular, their coefficients \( A_n^{(s)} \) are easily obtained from (5.7).

We thus have

**Proposition 6.1.** The polynomials \( \mathcal{G}(0) \{ P_n(x; q^j) \} \) are the eigenfunctions of a \( q \)-difference operator of order \( 2N = 2j + 2 \).

In this case, the polynomials \( \mathcal{G}(0) \{ P_n(x; q^j) \} \) coincide with polynomials \( P_n(x; q^{j-1}; M) \) obtained from the little \( q \)-Laguerre polynomials by adding to the orthogonality measure a mass \( M \) at \( x = 0 \). Hence

**Proposition 6.2.** The polynomials \( P_n(x; q^j; M) \) are the eigenfunctions of a \( q \)-difference operator of order \( 2N = 2j + 4 \).

When \( q \to 1 \) we get Koornwinder’s generalized Laguerre polynomials \( L_n^{(j;M)}(x) \) [11] whose measure differs from that of the ordinary Laguerre polynomials \( L_n^{(j)}(x) \) by inserting a concentrated mass \( M \) at the endpoint \( x = 0 \) of the orthogonality interval \((0, \infty)\). These polynomials are known to satisfy a differential equation of order \( 2j + 4 \) [5], [9].
REFERENCES


Department of Mathematics and Statistics and Department of Physics, McGill University, 845 Sherbrooke St. W., Montréal, Québec, Canada H3A 2T5 – Centre de Recherches Mathématiques, Université de Montréal, C.P. 6128, succursale Centre-ville, Montréal, Québec, Canada H3C 3J7
E-mail address: vinet@crm.umontreal.ca

Donetsk Institute for Physics and Technology, Donetsk 340114, Ukraine
E-mail address: zhedanov@kinetic.ac.donetsk.ua

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use